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A SIMPLE PROOF THAT THE SQUARE-FREE NUMBERS HAVE DENSITY $6/\pi^2$

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Abstract. In this note we give a simple proof of the well-known result that the square-free numbers have density $6/\pi^2$.

1. Introduction and preliminaries

In this note p_n denotes the n-th prime number and $\lfloor x \rfloor$ denotes the integer-part function. Note that

$$0 \leq x - \lfloor x \rfloor < 1. \tag{1.1}$$

A positive integer n is square-free if it is either a product of different primes or 1. It is well-known [1, Chapter XVIII, Theorem 333] that the square-free numbers have positive density $\frac{6}{\pi^2}$. That is, if $Q_1(x)$ is the number of square-free numbers not exceeding x we have

$$\lim_{x\to\infty} \frac{Q_1(x)}{x} = \frac{6}{\pi^2}.$$

In this note we give a simple proof of this fact.

Let $P_1(x)$ be the number of non square-free numbers not exceeding x. That is, the number of numbers not exceeding x such that in their prime factorization there exists a prime with exponent greater than 1. The density of this set will be $1 - \frac{6}{\pi^2}$. That is,

$$\lim_{x\to\infty} \frac{P_1(x)}{x} = 1 - \frac{6}{\pi^2},$$

since $P_1(x) = \lfloor x \rfloor - Q_1(x)$.

We need the following theorem, it is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

Theorem 1.1. *Let S be a set of N distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of S containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the*

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number of elements of $S_{ij\ldots l}$. Then the number K of elements of S not in any of $S_1, \ldots, S_r$ is

$$K = N - \sum_{1\leq i\leq r} N_i + \sum_{1\leq i<j\leq r} N_{ij} - \sum_{1\leq i<j<k\leq r} N_{ijk} + \cdots + (-1)^r N_{12\ldots r}. \tag{1.2}$$

Proof. See, for example, either [1, page 233] or [2, page 84]. □

We also need the following well-known limit [1, Chapter XVII, Theorem 280]

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^2}\right) = \frac{6}{\pi^2}. \tag{1.3}$$

2. Main results

Theorem 2.1. *The following limit holds*

$$\lim_{x\to\infty} \frac{Q_1(x)}{x} = \frac{6}{\pi^2}. \tag{2.1}$$

Proof. Let $A_{p_h}(x)$ be the number of positive integers not exceeding x such that p_h is their least prime factor with exponent greater than 1 in their prime factorization. We have (see equations (1.2) and (1.1))

$$\begin{aligned}
& A_{p_h}(x) \\
= & \left\lfloor \frac{x}{p_h^2} \right\rfloor - \sum_{1\leq i\leq h-1} \left\lfloor \frac{x}{p_h^2 p_i^2} \right\rfloor + \sum_{1\leq i<j\leq h-1} \left\lfloor \frac{x}{p_h^2 p_i^2 p_j^2} \right\rfloor - \sum_{1\leq i<j<k\leq h-1} \left\lfloor \frac{x}{p_h^2 p_i^2 p_j^2 p_k^2} \right\rfloor \\
+ & \cdots + (-1)^{h-1} \left\lfloor \frac{x}{p_h^2 p_1^2 \cdots p_{h-1}^2} \right\rfloor = \frac{x}{p_h^2} - \sum_{1\leq i\leq h-1} \frac{x}{p_h^2 p_i^2} + \sum_{1\leq i<j\leq h-1} \frac{x}{p_h^2 p_i^2 p_j^2} \\
- & \sum_{1\leq i<j<k\leq h-1} \frac{x}{p_h^2 p_i^2 p_j^2 p_k^2} + \cdots + (-1)^{h-1} \frac{x}{p_h^2 p_1^2 \cdots p_{h-1}^2} + O(1) \\
= & \frac{x}{p_h^2} \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i^2}\right) \right) + O(1).
\end{aligned} \tag{2.2}$$

Note that if $p_h^2 > x$, that is $p_h > \sqrt{x}$, then $Ap_h(x) = 0$. Therefore

$$P_1(x) = \sum_{p_h\leq\sqrt{x}} A_{p_h}(x). \tag{2.3}$$

On the other hand we have the inequality (see (2.2) and Theorem 1.1)

$$A_{p_h}(x) \leq \left\lfloor \frac{x}{p_h^2} \right\rfloor \leq \frac{x}{p_h^2}. \tag{2.4}$$

Now, we put

$$A = \sum_{j=1}^{\infty} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2}\right) \right), \tag{2.5}$$

$$B = \sum_{j=1}^{\infty} \frac{1}{p_j^2}, \tag{2.6}$$

and we choose m such that

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) < \epsilon, \tag{2.7}$$

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^2} < \epsilon. \tag{2.8}$$

We have (see (2.3), (2.2), (2.5) and (2.7))

$$\begin{aligned} P_1(x) = \sum_{p_j \le \sqrt{x}} A_{p_j}(x) &\ge \sum_{j=1}^{m} A_{p_j}(x) = x \sum_{j=1}^{m} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) + O(1) \\ = \; & xA - x \sum_{j=m+1}^{\infty} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) + O(1) \ge xA - \epsilon x + O(1) \\ \ge \; & x(A - 2\epsilon). \end{aligned} \tag{2.9}$$

On the other hand, we have (see (2.3), (2.4), (2.2), (2.5) and (2.8))

$$\begin{aligned} P_1(x) = \sum_{p_j \le \sqrt{x}} A_{p_j}(x) &\le \sum_{j=1}^{m} A_{p_j}(x) + \sum_{p_{m+1} \le p_j \le \sqrt{x}} \frac{x}{p_j^2} \\ = \; & x \sum_{j=1}^{m} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) + O(1) + \sum_{p_{m+1} \le p_j \le \sqrt{x}} \frac{x}{p_j^2} \\ \le \; & xA + O(1) + x \sum_{p_{m+1} \le p_j \le \sqrt{x}} \frac{1}{p_j^2} \\ \le \; & xA + O(1) + x\epsilon \le x(A + 2\epsilon). \end{aligned} \tag{2.10}$$

Equations (2.9) and (2.10) give

$$\lim_{x \to \infty} \frac{P_1(x)}{x} = A, \tag{2.11}$$

since ϵ is arbitrarily small.

Now, we have the following equality (use mathematical induction)

$$\sum_{j=1}^{n} \frac{1}{p_j^2} \left(\prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) = 1 - \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^2} \right). \tag{2.12}$$

Equations (2.5), (2.12) and (1.3) give

$$A = 1 - \frac{6}{\pi^2}. \tag{2.13}$$

On the other hand, we have

$$Q_1(x) = \lfloor x \rfloor - P_1(x). \tag{2.14}$$

Equations (2.11), (2.13) and (2.14) give equation (2.1). □

Let $Q_k(x)$ be the number of positive integers not exceeding x such that all prime in their prime factorization have exponent not exceeding k. In particular, if $k = 1$ we obtain the square-free numbers. We have the following well-known theorem.

Theorem 2.2. *The following limit holds*

$$\lim_{x\to\infty} \frac{Q_k(x)}{x} = \frac{1}{\zeta(k+1)}. \tag{2.15}$$

where $\zeta(z)$ is the Riemann Zeta-Function.

Proof. The proof is the same as Theorem 2.1 replacing 2 by $k+1$. In this case we need the more general limit [1, Chapter XVII, Theorem 280]

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^{k+1}}\right) = \frac{1}{\zeta(k+1)}.$$

□

In particular if $k = 1$ limit (2.15) becomes limit (2.1), since $\zeta(2) = \frac{\pi^2}{6}$.

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