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# ON THE LICHNEROWICZ-POISSON COHOMOLOGY COMPLEX AND QUASI-POISSON ALGEBRAS

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Abstract. This article explores $\mathbb{F}$-commutative algebra $(\mathcal{A}, \cdot)$ with Poisson structure defined by a non-Lie algebra bivector $Q$. We analyse the Lichnerowicz-Poisson coboundary $\delta_Q^*$ and its implications as quasi-Poisson structure, where the Schouten bracket does not vanishes, complicating cohomology computations. The examined open question is whether the deviating term is coboundary of Poisson cohomology. Our main objectives are to construct Poisson subalgebra on which twisted term $\delta_Q^2(Q)$ vanishes and, to study the existence of Poisson structures $P$ on Poisson ideal satisfying condition $\delta_P^2(\cdot) = \delta_Q^2(Q)$.

## 1. Introduction

The Poisson cohomology complex is a mathematical framework used to analyze the cohomology of Poisson structures, providing insights into their geometric and topological properties. It allows for the identification of non-trivial cohomology classes, which correspond to obstructions or invariants related to the Poisson structure. The symplectic structure associated with Poisson structures provides a geometric foundation for the hamiltonian formalism. However, the latter does not automatically lead to a Poisson structure. Hence the term "quasi-Poisson structures". According to [2], the terminology "quasi" is motivated by the relation to quasi-Poisson Lie groups [1], which are the classical limits of quasi-Hopf algebras. Historically, quasi-Poisson structures appeared as a finite-dimensional alternative to infinite-dimensional constructions of Poisson structures on moduli spaces, proposed particularly in [5, 4, 7, 8]. Quasi-Poisson structures then appear as a more natural technique for constructing Poisson structures. They have indeed good reduction properties, allowing to obtain a Poisson structure when we proceed to the quotient. For Poisson structure $P$, operator $d_P := [P, \cdot]$ is differential (i.e. $d_P^2 = 0$), called the Lichnerowicz-Poisson differential (or LP-differential) and the corresponding cohomology is the Lichnerowicz-Poisson cohomology (or LP-cohomology) which plays an important role in obstruction to geometric quantization [12]. The LP-cohomology complex was defined in [6]. Example of Poisson cohomology of Poisson algebras are given in [10] and [11]. A usefull reference

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for Poisson cohomology of Poisson algebras is [9]. However, for quasi-Poisson structure $Q$, differential does not exists. The approach for quasi-Poisson cohomology was first developped in [1, 2] as part of a G-manifold where G is a Lie Group and $\mathfrak{g}$ its corresponding Lie algebra which is equipped with an invariant inner product. From the Poisson bracket $\{\cdot,\cdot\}$ of $\mathfrak{g}$ and the invariant inner product, they constructed an invariant element $\phi \in \wedge^3\mathfrak{g}$, called the Cartan 3-tensor of the Lie algebra with an invariant product. The quasi-manifolds that were studied by these precursor authors are G-manifolds $M$ together with an invariant bivector field $P$, such that the Schouten-Nijenhuis bracket $[P,P]$ equals the trivector field $\phi_M$ generated by $\phi$. In this paper, we consider quasi-Poisson structure $Q$ on affine space $\mathbb{F}^3$ with its function algebra $\mathcal{A} = \mathbb{F}[x_1, x_2, x_3]$. It defines a skew-symmetric biderivation $\{\cdot,\cdot\}$ on $\mathcal{A}$ and a bivector field $Q$, such that the Schouten bracket $[Q,Q]$ equals the trivector field $\phi_{\mathbb{F}^3}$, derived from its Jacobiator $\mathcal{J}$. Notably, this structure does not satisfy the Jacobi identity, as $\phi_{\mathbb{F}^3}(dx_1, dx_2, dx_3) = 2\mathcal{J}(x_1, x_2, x_3)$ where $\mathbb{F}$ is a field of characteristic zero. The article also discusses the Lichnerowicz-Poisson coboundary operators $\delta^0_Q : \mathfrak{X}^0(\mathcal{A}) \longrightarrow \mathfrak{X}^1(\mathcal{A})$, $\delta^1_Q : \mathfrak{X}^1(\mathcal{A}) \longrightarrow \mathfrak{X}^2(\mathcal{A})$ and $\delta^2_Q : \mathfrak{X}^2(\mathcal{A}) \longrightarrow \mathfrak{X}^3(\mathcal{A})$, which map alternating $k-$derivations in $\mathfrak{X}^k(\mathcal{A})$. Importantly, the compositions $\delta^1_Q \circ \delta^0_Q$ and $\delta^2_Q \circ \delta^1_Q$ does not vanishes, presenting significant challenges for cohomology computations. The question that arises is : does there exist a subalgebra of $\mathcal{A}$ in which these obstructions vanish? In other words, can a truly stable subcomplex be constructed on subalgebra of $\mathcal{A}$ ? Another question is whether the twisted term $\phi_{\mathbb{F}^3}$ (generated by Jacobiator $\mathcal{J}$) is coboundary ? The analysis of all these problems has, in turn, enabled us to obtain a certain number of main results. Starting from initial algebra $\mathcal{A}$, we establish the existence of a class of Poisson structures $P$ satisfying

$$\delta^2_P(\cdot) := [Q,Q] = \phi_{\mathbb{F}^3}. \tag{1.1}$$

We also propose some general properties of the quasi-Poisson structure associated with the quasi-Poisson algebra. These properties will have helped to clarify the conditions for obtaining a true Poisson structure in linear combinaison cases of quasi-Poisson structures on $\mathbb{F}^3$. Condition (1.1) enables us to affirm that under certain constraints, the Jacobi anomaly can be the coboundary of Poisson cohomology. Finally, we construct a Poisson subalgebra, specifically a Poisson Ideal in which we have subcomplex and we demonstrate its stability. For illustrations of main results, we give an example of quasi-Poisson structure denoted $Q_0 = -x_2^n \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{1}{n} x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} - \frac{1}{n} x_1 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}$, where $n \geq 1$ be an integer.

## 2. Preliminaries

### 2.1. **Conventions and notations.**

In the sequel we consider once and for all the following conventions :

• $\mathbb{F}$ is an arbitrary field of characteristic zero. Vector spaces, the (quasi-) Poisson bracket and (quasi-) Poisson algebras are defined over $\mathbb{F}$ unless otherwise stated. The coordinates on $\mathbb{F}^3$ are $(x_1, x_2, x_3)$ and $\partial_i$ is the partial derivative $\frac{\partial}{\partial_i}$.

• $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket, $Q$ the bivector wich does not form a Lie algebra denoted by $Q = a\partial_1 \wedge \partial_2 + b\partial_2 \wedge \partial_3 + c\partial_3 \wedge \partial_1$ and $\mathcal{J}$ is its Jacobiator.
$\overrightarrow{Q} = (b, c, a)$, $a, b, c$ are generators brackets and $A_Q = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$ denoted quasi-Poisson matrix of $Q$.

Notations $\cdot$ and $\times$ denote respectively the inner and the cross products in $\mathcal{A}^3$, while $\overrightarrow{\nabla}$, $\overrightarrow{\nabla}\times$, $Div$ denote respectively the gradient, the curl and the divergence operators, $\langle\cdot|\cdot\rangle$ is the standard inner product.

### 2.2. On commutative polynomial Poisson algebras and related structures.

2.2.1. *Poisson algebras.*

**Definition 2.1.** [9] A $\mathbb{F}$-Poisson algebra is defined as an $\mathbb{F}$-vector space $\mathbb{A}$ that is equipped with two distinct multiplication operations $(F, G) \mapsto F \cdot G$ and $(F, G) \mapsto \{F, G\}$, such that $(\mathbb{A}, \cdot)$ forms a commutative and associative algebra over $\mathbb{F}$, with multiplicative identity denoted by 1, $(\mathbb{A}, \{\cdot,\cdot\})$ constitutes a Lie algebra over $\mathbb{F}$ and the two multiplications are compatible in the sense that for any elements $F$, $G$ and $H$ in $\mathbb{A}$, $\{F \cdot G, H\} = F \cdot \{G, H\} + G \cdot \{F, H\}$.

A Poisson structure $\{\cdot,\cdot\}$ on $\mathcal{A}$ is entirely determined by the values of $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_1\}$.

*Remark* 2.2. Let $a, b, c \in \mathcal{A}$. $\mathfrak{X}^1(\mathcal{A}) = \{a\partial_1 + b\partial_2 + c\partial_3\}$, $\mathfrak{X}^2(\mathcal{A}) = \{a\partial_1 \wedge \partial_2 + b\partial_2 \wedge \partial_3 + c\partial_3 \wedge \partial_1\}$, $\mathfrak{X}^3(\mathcal{A}) = \{a\partial_1 \wedge \partial_2 \wedge \partial_3\}$ and $\mathfrak{X}^k(\mathcal{A}) = \{0\}$, for all $k \geq 4$.

Note that the biderivation property of the Poisson bracket leads to a fundamental operation which allows one to associate to elements of $\mathcal{A}$, derivations of $\mathcal{A}$. This operation, which we define now, corresponds in the Hamiltonian formulation of classical mechanics to deriving the equations of a given Hamiltonian.

**Definition 2.3.** Consider generators bracket. For any elements $f$ and $H$ in $\mathbb{A}$,

(1) $Ham(\mathcal{A}) = \{a_1\partial_1 + a_2\partial_2 + a_3\partial_3\}$ with $a_1 = \{x_1, x_2\}\partial_2 H + \{x_1, x_3\}\partial_3 H$, $a_2 = -\{x_1, x_2\}\partial_1 H - \{x_2, x_3\}\partial_3 H$, $a_3 = \{x_3, x_1\}\partial_1 H + \{x_2, x_3\}\partial_2 H$.
(2) $Cas(\mathcal{A}) = \{H \in \mathcal{A} \mid b_1\partial_1 H + b_2\partial_2 H + b_3\partial_3 H = 0\}$ with $b_1 = -\{x_1, x_2\}\partial_2 f + \{x_3, x_1\}\partial_3 f$, $b_2 = \{x_1, x_2\}\partial_1 f + \{x_2, x_3\}\partial_3 f$, $b_3 = -\{x_3, x_1\}\partial_1 f - \{x_2, x_3\}\partial_2 f$.

**Proposition 2.4.** [9] *Let $(\mathbb{A}, \cdot, \{,\})$ be a general Poisson algebra.*

(1) *$Cas(\mathbb{A})$ is a subalgebra of $(\mathbb{A}, \cdot)$, that includes the image of $\mathbb{F}$ in $\mathbb{A}$, through $a \longmapsto a \cdot 1$ and, $Ham(\mathcal{A})$ is generally not an $\mathcal{A}$-module.*
(2) *$Ham(\mathcal{A})$ forms a Lie subalgebra of $\mathfrak{X}^1(\mathcal{A})$ and the following sequence of Lie algebras is exact : $0 \xrightarrow{0} Cas(\mathcal{A}) \longrightarrow \mathcal{A} \xrightarrow{-\mathcal{X}} Ham(\mathcal{A}) \longrightarrow 0$.*

**Proposition 2.5.** [9]

(1) The following bracket is Poisson structure on $\mathcal{A}$.
$\{f, g\} = \frac{\partial\varphi}{\partial x_1}(\frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial x_3}) + \frac{\partial\varphi}{\partial x_2}(\frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_1} - \frac{\partial g}{\partial x_3}\frac{\partial f}{\partial x_1}) + \frac{\partial\varphi}{\partial x_3}(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2})$
with $\varphi, f, g \in \mathcal{A}$ where $\{x_1, x_2\} = \frac{\partial\varphi}{\partial x_3}$, $\{x_2, x_3\} = \frac{\partial\varphi}{\partial x_1}$ and $\{x_3, x_1\} = \frac{\partial\varphi}{\partial x_2}$.

(2) There exists a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{A}$ if and only if, $\{x_1,x_2\}(\frac{\partial\{x_3,x_1\}}{\partial x_1}-\frac{\partial\{x_2,x_3\}}{\partial x_2})+\{x_2,x_3\}(\frac{\partial\{x_1,x_2\}}{\partial x_2}-\frac{\partial\{x_3,x_1\}}{\partial x_3})+\{x_3,x_1\}(\frac{\partial\{x_2,x_3\}}{\partial x_3}-\frac{\partial\{x_1,x_2\}}{\partial x_1})=0.$

2.2.2. *Poisson subalgebra and Poisson Ideals.*

**Definition 2.6.** [9] Let $(\mathcal{A}_1,\cdot_1,\{,\}_1)$ and $(\mathcal{A}_2,\cdot_1,\{,\}_2)$ be two Poisson algebras over $\mathbb{F}$. A linear map $\phi:\mathcal{A}_1\longrightarrow\mathcal{A}_2$ is called a morphism of Poisson algebra if for all $F,G\in\mathcal{A}_1$, $\phi(f\cdot_1 g)=\phi(f)\cdot_2\phi(g)$ and $\phi(\{f,g\}_1)=\{\phi(f),\phi(g)\}_2$.

If $\phi:\mathcal{A}_1\longrightarrow\mathcal{A}_2$ is a morphism of Poisson algebras and $\phi$ is bijective, then $\phi^{-1}:\mathcal{A}_2\longrightarrow\mathcal{A}_1$ is also a morphism of Poisson algebras.

**Definition 2.7.** [9] Let $\mathbb{A}=(\mathbb{A},\cdot,\{,\})$ be an arbitrary Poisson algebra. A vector subspace $B\subset\mathbb{A}$ is called a Poisson subalgebra if

$$B\cdot B\subset B\quad and\quad \{B,B\}\subset B. \tag{2.1}$$

A vector subspace $\mathfrak{I}\subset\mathbb{A}$ is called a Poisson ideal if : $\mathfrak{I}\cdot\mathbb{A}\subset\mathfrak{I}$ and $\{\mathfrak{I},\mathbb{A}\}\subset\mathfrak{I}$.

Note that $B$ becomes itself a Poisson algebra, when equipped with the restrictions of both multiplications to $B$ and the inclusion map $\iota:B\longrightarrow\mathbb{A}$ is as a morphism of Poisson algebras. The quotient $\mathbb{A}_{/\mathfrak{I}}$ inherits a Poisson structure from $\mathbb{A}$, ensuring that the canonical projection $p:\mathbb{A}\longrightarrow\mathbb{A}_{/\mathfrak{I}}$ serves as a morphism of Poisson algebras.

2.2.3. *LP-Poisson cohomology complex.*

**Definition 2.8.** [9] Consider an arbitrary Poisson algebra $(\mathbb{A},\cdot,\pi=\{,\})$ and let $k$ be an integer. For all $f_0,\dots,f_k\in\mathcal{A}$ and $Q\in\mathfrak{X}^k(\mathcal{A})$, the LP-cohomology coboundary $\delta^*_\pi:\mathfrak{X}^*(\mathbb{A})\longrightarrow\mathfrak{X}^{*+1}(\mathbb{A})$ is given by :

$$\delta^k_\pi(Q)(f_0,\dots,f_k):=\sum_{i=0}^{k}(-1)^i\{f_i,Q(f_0,\dots,\widehat{f_i},\dots,f_k)\}+\sum_{0\le i<j\le k}(-1)^{i+j}\Psi$$

with $\delta^*_\pi\circ\delta^*_\pi=0$ where $\Psi=Q(\{f_i,f_j\},f_0,\dots,\widehat{f_i},\dots,\widehat{f_j},\dots,f_k))$ and $(\mathfrak{X}^*,\delta^*_\pi)$ is the Poisson complex of $(\mathcal{A},\cdot,\{,\})$.

2.2.4. *Schouten-Nijenhuis bracket and Quasi-Poisson structures.*

**Theorem 2.9.** [13, 3] *Consider the polynomial algebra $\mathcal{A}$. There exists on $\mathfrak{X}(\mathcal{A})$ a bracket $[\cdot,\cdot]$ called the Schouten-Nijenhuis bracket and verifying :*

(a) *If $A\in\mathfrak{X}^a(\mathcal{A})$ and $B\in\mathfrak{X}^b(\mathcal{A})$ then, $[A,B]\in\mathfrak{X}^{a+b-1}(\mathcal{A})$.*
(b) *If $A\in\mathfrak{X}^a(\mathcal{A})$ and $B\in\mathfrak{X}^b(\mathcal{A})$ then, $[A,B]=-(-1)^{(a-1)(b-1)}[B,A]$.*
(c) *If $A\in\mathfrak{X}^a(\mathcal{A})$, $B\in\mathfrak{X}^b(\mathcal{A})$ and $C\in\mathfrak{X}^c(\mathcal{A})$ then,*

$$[A,B\wedge C]=[A,B]\wedge C+(-1)^{(a-1)b}B\wedge[A,C],$$

$$[A\wedge B,C]=A\wedge[B,C]+(-1)^{(c-1)b}[A,C]\wedge B.$$

(d) *If $A\in\mathfrak{X}^a(\mathcal{A})$, $B\in\mathfrak{X}^b(\mathcal{A})$ and $C\in\mathfrak{X}^c(\mathcal{A})$ then,*

$$(-1)^{(a-1)(c-1)}[A,[B,C]]+(-1)^{(b-1)(a-1)}[B,[C,A]]+(-1)^{(c-1)(b-1)}[C,[A,B]].$$

Note that if $X \in \mathfrak{X}^1(\mathcal{A})$, $A \in \mathfrak{X}^a(\mathcal{A})$ and $f \in C^\infty(\mathcal{A})$, the Schouten-Nijenhuis bracket of $X$ and $A$ coincides with the Lie bracket and $[X, f] = X(f)$.

**Definition 2.10.** [9] Let $f \in \mathcal{A}$, $P \in \mathfrak{X}^p(\mathcal{A})$ and $P \in \mathfrak{X}^p(\mathcal{A})$, $[P, f] = i_X P$ where $i_f P(f_2, \dots, f_p) := P(f, f_2, \dots, f_p)$ and $i_f(P \wedge Q) = i_f P \wedge Q + (-1)^p P \wedge i_f Q$ with $f_2, \dots, f_p \in \mathcal{A}$ and, every derivation $X \in \mathfrak{X}^1(\mathcal{A})$ defines a graded $\mathbb{F}$-linear map of degree 0, $\mathcal{L}_X : \mathfrak{X}^\bullet(\mathcal{A}) \longrightarrow \mathfrak{X}^\bullet(\mathcal{A})$ which is called the *Lie derivative* with respect to $X$ defined by $\mathcal{L}_X P(f_1, \dots, f_p) := X(P(f_1, \dots, f_p)) - \sum_{i=1}^{p} P(f_1, \dots, X(f_i), \dots, f_p)$, for $P \in \mathfrak{X}^p(\mathcal{A})$, $f_1, \dots, f_p \in \mathcal{A}$.

Note that, for $X \in \mathfrak{X}^1(\mathcal{A})$, $i_f X = X(f)$ and $i_f g := 0$ with $g$ in $\mathfrak{X}^0(\mathcal{A})$.
It follows from the definition that for all $P, Q \in \mathfrak{X}^\bullet(\mathcal{A})$,

$$\mathcal{L}_X(P \wedge Q) = (\mathcal{L}_X P) \wedge Q + P \wedge \mathcal{L}_X Q$$

and, for any bivector field $Q \in \mathfrak{X}^2(\mathcal{A})$ and for every vectors field $X, Y \in \mathfrak{X}^1(\mathcal{A})$,

$$\mathcal{L}_Q(X \wedge Y) = \mathcal{L}_X Q \wedge Y - X \wedge \mathcal{L}_Y Q. \tag{2.2}$$

The coboundary operator to Schouten-Nijenhuis bracket is related as

$$\delta_\pi^{\bullet+1}(\cdot) = -[\cdot, \pi].$$

Given a skew-symmetric biderivation $\{\cdot, \cdot\}$ on $\mathcal{A}$, the *jacobiator* $\mathcal{J}$ is the tri-linear map $\mathcal{A}^3 \longrightarrow \mathcal{A}$ given by

$$\mathcal{J}(x_1, x_2, x_3) = \{x_1, \{x_2, x_3\}\} + \{x_2, \{x_3, x_1\}\} + \{x_3, \{x_1, x_2\}\}.$$

**Definition 2.11.** A *quasi-Poisson structure* on $\mathbb{F}^3$ equipped with $\mathcal{A}$, is a skew-symmetric biderivation $\{\cdot, \cdot\}_Q$ on $\mathcal{A}$, equipped with a bivector $Q$ such that the Schouten bracket $[Q, Q]$ equals the trivector field $\phi_{\mathbb{F}^3}$ generated by its Jacobiator $\mathcal{J}$, which is not satisfy the Jacobi identity; instead

$$\phi_{\mathbb{F}^3}(dx_1, dx_2, dx_3) = 2\mathcal{J}(x_1, x_2, x_3).$$

## 3. Quasi-Poisson Matrix and construction of a class of Poisson structures

Consider initial algebra $\mathcal{A}$, we want to establish the existence of a class of Poisson structures $P$ satisfying condition (1.1). Before going to the essential point, we need to clarify the following facts :

### 3.1. Some properties of Quasi-Poisson structures on $\mathbb{F}^3$.

**Proposition 3.1.** *Consider bivector field $Q$ decomposes as*

$$Q = a\partial_1 \wedge \partial_2 + b\partial_2 \wedge \partial_3 + c\partial_3 \wedge \partial_1. \tag{3.1}$$

*$Q$ is Poisson structure if and only if $\langle curl \overrightarrow{Q} \mid \overrightarrow{Q} \rangle = 0$ and $\phi_{\mathbb{F}^3} = 2\langle curl \overrightarrow{Q} \mid \overrightarrow{Q} \rangle$.*

*Proof.* Let $Q = a\partial_1 \wedge \partial_2 + b\partial_2 \wedge \partial_3 + c\partial_3 \wedge \partial_1$. It satisfies the Jacobi identity if and only if $\mathcal{J}(x_1, x_2, x_3) = 0$, which is equivalent to

$$a(\partial_1 c - \partial_2 b) + b(\partial_2 a - \partial_3 c) + c(\partial_3 b - \partial_1 a) = 0. \tag{3.2}$$

Using vector $\overrightarrow{Q}$ given by $(b, c, a)$, the standard inner product and the curl operator, condition (3.2) can be rewritten as $\langle \overrightarrow{Q} \mid curl\ \overrightarrow{Q} \rangle = 0$. From Definition 2.11, we have $\phi_{\mathbb{F}^3} = 2\langle \overrightarrow{Q} \mid curl\ \overrightarrow{Q} \rangle$. □

**Proposition 3.2.** *For every function $\rho \in \mathcal{A}$ ($\rho \neq 0$), $fQ$ is also quasi-Poisson structure on $\mathbb{F}^3$. However, if $\rho\overrightarrow{Q}$ is the gradient of a polynomial function $\varphi$, i.e. such that $\frac{\partial\varphi}{\partial z} = \rho a$, $\frac{\partial\varphi}{\partial x} = \rho b$ and $\frac{\partial\varphi}{\partial y} = \rho c$. Then, $\rho Q$ is Poisson structure.*

*Proof.* Let $\rho$ be a polynomial function. A simple computation shows that

$$[\rho Q, \rho Q] = \rho^2 [Q, Q].$$

Since $Q$ is a quasi-Poisson structure, $[Q, Q] \neq 0$, so that $\rho Q$ is Poisson structure if and only if $\rho = 0$. This shows that for $\rho \neq 0$, $\rho Q$ is quasi-Poisson structure.

It follows that when $\rho\overrightarrow{Q}$ is the gradient of a polynomial function $\varphi$, i.e. : $\frac{\partial\varphi}{\partial z} = \rho a$, $\frac{\partial\varphi}{\partial x} = \rho b$, $\frac{\partial\varphi}{\partial y} = \rho c$. Then, $\rho Q$ becomes a Poisson structure and it admits $\varphi$ as Casimir function. □

Note that

$$[Q, Q] = 2 \langle\ curl\ \overrightarrow{Q} \mid \overrightarrow{Q}\ \rangle\ \partial_1 \wedge \partial_2 \wedge \partial_3.$$

**Proposition 3.3.** *Consider the following quasi-Poisson structures :*

$$\begin{aligned} Q_1 &= a_1\partial_1 \wedge \partial_2 + b_1\partial_2 \wedge \partial_3 + c_1\partial_3 \wedge \partial_1, \\ Q_2 &= a_2\partial_1 \wedge \partial_2 + b_2\partial_2 \wedge \partial_3 + c_2\partial_3 \wedge \partial_1. \end{aligned}$$

*$Q_1 + Q_2$ is quasi-Poisson on $\mathbb{F}^3$ if and only if*

$$\phi^1_{\mathbb{F}^3} + \phi^2_{\mathbb{F}^3} + \langle \overrightarrow{Q_1} \mid curl\ \overrightarrow{Q_2} \rangle + \langle \overrightarrow{Q_2} \mid curl\ \overrightarrow{Q_1} \rangle = 0.$$

*Proof.* This result follows from the following computation

$$\frac{1}{2}[Q_1 + Q_2, Q_1 + Q_2] = \left( \frac{1}{2}[Q_1, Q_1] + \frac{1}{2}[Q_2, Q_2] + \Delta \right) \partial_1 \wedge \partial_2 \wedge \partial_3$$

with $\Delta = b_1(\partial_2 a_2 - \partial_3 c_2) + c_1(\partial_3 b_2 - \partial_1 a_2) + a_1(\partial_1 c_2 - \partial_2 b_2) + b_2(\partial_2 a_1 - \partial_3 c_1) + c_2(\partial_3 b_1 - \partial_1 a_1) + a_2(\partial_1 c_1 - \partial_2 b_1)$.

□

Let's give an example of quasi-Poisson structure.

**Example 3.4.** Bivector field $Q_0$ defined as follows is quasi-Poisson structure.

$$Q_0 = -x_2^n \partial_1 \wedge \partial_2 - \frac{1}{n} x_2 \partial_2 \wedge \partial_3 - \frac{1}{n} x_1 \partial_3 \wedge \partial_1$$

where $n \geq 1$ be an integer, $\langle curl\ \overrightarrow{Q_0} \mid \overrightarrow{Q_0} \rangle = x_2^n$ and $[Q, Q] = 2x_2^n \partial_1 \wedge \partial_2 \wedge \partial_3$.

3.2. **Quasi-Poisson Matrix on $\mathbb{F}^3$.**
In this section, we present how the quasi-Poisson structure $Q$, can be encoded in a skew-symmetric matrix. For $f, g \in \mathcal{A}$,

$$Q(dF, dG) = a\frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_2} + b\frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_3} + c\frac{\partial F}{\partial x_3}\frac{\partial G}{\partial x_1}.$$

If we review the structure functions (generators) $a, b, c$ as the elements of a (skew-symmetric) $3 \times 3$ matrix $A_Q$, then the quasi-Poisson structure be expressed in a compact form by $Q(dF, dG) = (\overrightarrow{\nabla} F)A_Q(\overrightarrow{\nabla} G)^\top$ where $\overrightarrow{\nabla}$ is the gradient operator. The matrix $A_Q$ is called the *quasi-Poisson matrix* of $Q$.

**Definition 3.5.** We call *quasi-Poisson matrix* of $Q$, any skew-symmetric matrix denoted $A_Q$ which amounts to the condition

$$Q(dF, dG) = (\overrightarrow{\nabla} F)A_Q(\overrightarrow{\nabla} G)^\top.$$

Explicitly, we have : $A_Q = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$ when $Q = a\partial_1\wedge\partial_2 + b\partial_2\wedge\partial_3 + c\partial_3\wedge\partial_1$.

The following lemma will be used in the following result.

**Lemma 3.6.** *Let* $X = \sum_{i=1}^{3} X_i\partial_i$ *and* $Y = \sum_{j=1}^{3} Y_j\partial_j$ *be two vectors field on* $\mathbb{F}^3$. *Then,*

$$(curl\ \overrightarrow{Q} - A_Q\overrightarrow{\nabla}) = \Big(\partial_2 a + a\partial_2 - \partial_3 c - c\partial_3, -\partial_1 a - a\partial_1 + \partial_3 b + b\partial_3, -\partial_2 b - b\partial_2 + \partial_1 c + c\partial_1\Big),$$

$$[X\wedge Y, Q] = \langle curl\ \overrightarrow{Q} - A_Q\overrightarrow{\nabla} \mid \overrightarrow{X\wedge Y}\rangle\partial_1\wedge\partial_2\wedge\partial_3$$

*where* $X\wedge Y$ *is the vector* $(X_2Y_3 - X_3Y_2,\ X_3Y_1 - X_1Y_3,\ X_1Y_2 - X_2Y_1)$ *and* $A_Q$ *the quasi-Poisson matrix of* $A$.

*Proof.* By definition,

$$curl\ \overrightarrow{Q} - A_Q\overrightarrow{\nabla} = (\partial_2 a - \partial_3 c + a\partial_2 - c\partial_3, \partial_3 b - \partial_1 a - a\partial_1 + b\partial_3, \partial_1 c - \partial_2 b + c\partial_1 - b\partial_2).$$

Since $[X\wedge Y, Q] = \mathcal{L}_Q(X\wedge Y)$ and according to (2.2), we have

$$[X\wedge Y, Q] = \mathcal{L}_X Q\wedge Y - X\wedge\mathcal{L}_Y Q.$$

By computation, we obtain

$\mathcal{L}_X Q = ((\langle\overrightarrow{X} \mid \overrightarrow{\nabla} a\rangle - a\partial_1 X_1 + b\partial_3 X_1 - a\partial_2 X_2 + c\partial_3 X_2)\partial_1\wedge\partial_2 + (\langle\overrightarrow{X} \mid \overrightarrow{\nabla} b\rangle + c\partial_1 X_2 - b\partial_2 X_2 + a\partial_1 X_3 - b\partial_3 X_3)\partial_2\wedge\partial_3 + (\langle\overrightarrow{X} \mid \overrightarrow{\nabla} c\rangle - c\partial_1 X_1 + b\partial_2 X_1 + a\partial_2 X_3 - c\partial_3 X_3)\partial_3\wedge\partial_1$,

$\mathcal{L}_Y Q = ((\langle\overrightarrow{Y} \mid \overrightarrow{\nabla} a\rangle - a\partial_1 Y_1 + b\partial_3 Y_1 - a\partial_2 Y_2 + c\partial_3 Y_2)\partial_1\wedge\partial_2 + (\langle\overrightarrow{Y} \mid \overrightarrow{\nabla} b\rangle + c\partial_1 Y_2 - b\partial_2 Y_2 + a\partial_1 Y_3 - b\partial_3 Y_3\Big)\partial_2\wedge\partial_3 + \Big(\langle\overrightarrow{Y} \mid \overrightarrow{\nabla} c\rangle - c\partial_1 Y_1 + b\partial_2 Y_1 + a\partial_2 Y_3 - c\partial_3 Y_3\Big)\partial_3\wedge\partial_1$,

and

$$\mathcal{L}_X Q\wedge Y - X\wedge\mathcal{L}_Y Q = \left\langle \begin{pmatrix} \partial_2 a - \partial_3 c \\ \partial_3 b - \partial_1 a \\ \partial_1 c - \partial_2 b \end{pmatrix} + \begin{pmatrix} 0 & -a & c \\ a & 0 & -b \\ -c & b & 0 \end{pmatrix}\overrightarrow{\nabla} \,\middle|\, \begin{pmatrix} X_2Y_3 - X_3Y_2 \\ X_3Y_1 - X_1Y_3 \\ X_1Y_2 - X_2Y_1 \end{pmatrix} \right\rangle.$$

It follows that $[X\wedge Y, Q] = \langle curl\ \overrightarrow{Q} - A_Q\overrightarrow{\nabla} \mid \overrightarrow{X\wedge Y}\rangle\partial_1\wedge\partial_2\wedge\partial_3$. □

### 3.3. **Construction of a class of Poisson structures.**

**Theorem 3.7.** *Consider quasi-Poisson $Q$ on $\mathbb{F}^3$ equipped with $\mathcal{A}$. Let $B$ be an arbitrary bivector defined as $B = a_B\partial_1 \wedge \partial_2 + b_B\partial_2 \wedge \partial_3 + c_B\partial_3 \wedge \partial_1$ and let $P$ be non-zero bivector defined by $P = a_P\partial_1 \wedge \partial_2 + b_P\partial_2 \wedge \partial_3 + c_P\partial_3 \wedge \partial_1$ verifying $[P, B] = \phi_{\mathbb{F}^3}\partial_1\wedge\partial_2\wedge\partial_3$ where $\phi_{\mathbb{F}^3}$ equals $[Q, Q](dx_1, dx_2, dx_3)$. Then, $P$ is Poisson structure if and only if*

$$\langle \overrightarrow{\nabla} \mid \overrightarrow{B} \wedge \overrightarrow{P}\rangle + \phi_{\mathbb{F}^3} = 0 \tag{3.3}$$

*and*

$$\langle curl\ \overrightarrow{P} \mid \overrightarrow{P}\rangle = 0, \tag{3.4}$$

*which are equivalent to :*

$$\partial_1(-a_Bc_P + c_Ba_P) + \partial_2(-b_Ba_P + a_Bb_P) + \partial_3(-c_Bb_P + b_Bc_P) = \phi_{\mathbb{F}^3} \tag{3.5}$$

*and*

$$a_P\partial_1c_P + b_P\partial_2a_P + c_P\partial_3b_P = c_P\partial_1a_P + a_P\partial_2b_P + b_P\partial_3c_P \tag{3.6}$$

*respectively.*

*Proof.* Since $Q$ is quasi-Poisson, $[Q, Q]$ equals $\phi_{\mathbb{F}^3}\partial_1 \wedge \partial_2 \wedge \partial_3$ which does not vanish. According to Lemma 3.6, $[P, B] = \langle curl\overrightarrow{B} - A_B\overrightarrow{\nabla} \mid \overrightarrow{P}\rangle\partial_1 \wedge \partial_2 \wedge \partial_3$ where $curl\ \overrightarrow{B} = (\partial_2a_B - \partial_3c_B, \partial_3b_B - \partial_1a_B, \partial_1c_B - \partial_2b_B)$, $\overrightarrow{P} = (b_P, c_P, a_P)$ and $A_B = \begin{pmatrix} 0 & a_B & -c_B \\ -a_B & 0 & b_B \\ c_B & -b_B & 0 \end{pmatrix}$. Then, $[P, B](dx_1, dx_2, dx_3)$ equals $\langle curl\overrightarrow{B} \mid \overrightarrow{P}\rangle + \langle curl - A_B\overrightarrow{\nabla} \mid \overrightarrow{P}\rangle$. Let us calculate $\langle curl\overrightarrow{B} \mid \overrightarrow{P}\rangle$ and $\langle curl - A_B\overrightarrow{\nabla} \mid \overrightarrow{P}\rangle$.

By simple computation, $\langle curl\overrightarrow{B} \mid \overrightarrow{P}\rangle$ equals

$$(\partial_2a_B - \partial_3c_B)b_P + (\partial_3b_B - \partial_1a_B)c_P + (\partial_1c_B - \partial_2b_B)a_P$$

and

$$\langle curl - A_B\overrightarrow{\nabla} \mid \overrightarrow{P}\rangle = (-a_B\partial_2 + c_B\partial_3)b_P + (a_B\partial_1 - b_B\partial_3)c_P + (-c_B\partial_1 + b_B\partial_2)a_P.$$

It follows that,

$$[P, B](dx_1, dx_2, dx_3) = \partial_1(a_Bc_P - c_Ba_P) + \partial_2(b_Ba_P - a_Bb_P) + \partial_3(c_Bb_P - b_Bc_P)$$

which is equivalent to : $-\langle\overrightarrow{\nabla}|\overrightarrow{B} \wedge \overrightarrow{P}\rangle$ with $\overrightarrow{\nabla} = (\partial_1, \partial_2, \partial_3)$. Then, the differential equation (3.3), which corresponds to (3.5). Furthermore, we know that the condition $P$ is Poisson structure is equivalent to $[P, P] = 0$, which implies (3.4) and corresponds to (3.6). This completes the proof of this theorem. □

**Corollary 3.8.** *Let $Q$ be quasi-Poisson where $[Q, Q]$ equals $\phi_{\mathbb{F}^3}\partial_1\wedge\partial_2\wedge\partial_3$ and let $B$ be an arbitrary non-zero bivector defined as $B = a_B\partial_1\wedge\partial_2+b_B\partial_2\wedge\partial_3+c_B\partial_3\wedge\partial_1$. Let $P$ be Poisson structure defined by*

$$P = \frac{\partial\varphi}{\partial x_3}\partial_1 \wedge \partial_2 + \frac{\partial\varphi}{\partial x_1}\partial_2 \wedge \partial_3 + \frac{\partial\varphi}{\partial x_2}\partial_3 \wedge \partial_1$$

*where $\varphi \in \mathcal{A}$. Then, $P$ satisfies the condition $[P, B] = [Q, Q]$ if and only if*

$$\langle curl\overrightarrow{B} \mid \overrightarrow{\nabla}\varphi\rangle = \phi_{\mathbb{F}^3}. \tag{3.7}$$

*In particular,*

$$\textit{For curl } \overrightarrow{B} := e_x = (1,0,0),\ \varphi \textit{ equals } \int \phi_{\mathbb{F}^3} dx_1. \tag{3.8}$$

$$\textit{For curl } \overrightarrow{B} := e_y = (0,1,0),\ \varphi \textit{ equals } \int \phi_{\mathbb{F}^3} dx_2. \tag{3.9}$$

$$\textit{For curl } \overrightarrow{B} := e_z = (0,1,0),\ \varphi \textit{ equals } \int \phi_{\mathbb{F}^3} dx_3. \tag{3.10}$$

*Proof.* Let $Q$ be quasi-Poisson where $[Q,Q]$ equals $\phi_{\mathbb{F}^3}\partial_1 \wedge \partial_2 \wedge \partial_3$ and let $B = a_B\partial_1 \wedge \partial_2 + b_B\partial_2 \wedge \partial_3 + c_B\partial_3 \wedge \partial_1$ be an arbitrary non-zero bivector. Consider Poisson structure $P = \frac{\partial\varphi}{\partial x_3}\partial_1 \wedge \partial_2 + \frac{\partial\varphi}{\partial x_1}\partial_2 \wedge \partial_3 + \frac{\partial\varphi}{\partial x_2}\partial_3 \wedge \partial_1$ with $\varphi \in \mathcal{A}$.
Let us show that $[P,B] = [Q,Q]$ if and only if $\langle \overrightarrow{B} | curl\ \overrightarrow{\nabla}\varphi \rangle = \phi_{\mathbb{F}^3}(dx_1, dx_2, dx_3)$.
Since $P$ is Poisson structure by assumption, it follows from Theorem 3.7 that,

$$\partial_1(-a_B\frac{\partial\varphi}{\partial x_2}+c_B\frac{\partial\varphi}{\partial x_3})+\partial_2(-b_B\frac{\partial\varphi}{\partial x_3}+a_B\frac{\partial\varphi}{\partial x_1})+\partial_3(-c_B\frac{\partial\varphi}{\partial x_1}+b_B\frac{\partial\varphi}{\partial x_2}) = \phi_{\mathbb{F}^3} \tag{3.11}$$

which is equivalent to

$$\langle (\partial_1, \partial_2, \partial_3) | (b_B, c_B, a_B) \wedge (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) \rangle + \phi_{\mathbb{F}^3} = 0$$

where $\overrightarrow{\nabla} = (\partial_1, \partial_2, \partial_3)$, $\overrightarrow{B} = (b_B, c_B, a_B)$ and $\overrightarrow{P} = (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) = \overrightarrow{\nabla}\varphi$.
It follows from (3.11) that,

$$-b_B\left(\partial_2(\frac{\partial\varphi}{\partial x_3}) - \partial_3(\frac{\partial\varphi}{\partial x_2})\right) - c_B\left(\partial_3(\frac{\partial\varphi}{\partial x_1}) - \partial_1(\frac{\partial\varphi}{\partial x_3})\right) - a_B\left(\partial_1(\frac{\partial\varphi}{\partial x_2}) - \partial_2(\frac{\partial\varphi}{\partial x_1})\right)$$
$$+\frac{\partial\varphi}{\partial x_1}\left(\partial_2 a_B - \partial_3 c_B\right) + \frac{\partial\varphi}{\partial x_2}\left(\partial_3 b_B - \partial_1 a_B\right) + \frac{\partial\varphi}{\partial x_3}\left(\partial_1 c_B - \partial_2 b_B\right) = \phi_{\mathbb{F}^3}.$$

which corresponds to

$$\langle -\overrightarrow{B} | curl\ \overrightarrow{\nabla}\varphi \rangle + \langle curl\ \overrightarrow{B} | \overrightarrow{\nabla}\varphi \rangle = \phi_{\mathbb{F}^3}. \tag{3.12}$$

By simple computation, we have $curl(\overrightarrow{\nabla}\varphi) = 0$. It implies that $\langle -\overrightarrow{B} | curl\ \overrightarrow{\nabla}\varphi \rangle$ equals zero. So, that (3.12) becomes $\langle curl\ \overrightarrow{B} | \overrightarrow{\nabla}\varphi \rangle = \phi_{\mathbb{F}^3}$. This completes the proof of (3.10). Let's now demonstrate relations (3.8), (3.9) and (3.10). To do this, let us consider the three cases that need to be distinguished.

- For $curl(\overrightarrow{B}) := e_x = (1,0,0)$, $\langle curl\overrightarrow{B} | \overrightarrow{\nabla}\varphi \rangle = \langle (1,0,0) | (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) \rangle = \frac{\partial\varphi}{\partial x_1}$.
Applying (3.10), it follows that $\varphi = \int \phi_{\mathbb{F}^3} dx_1$.

- For $curl(\overrightarrow{B}) := e_y = (0,1,0)$, $\langle curl\overrightarrow{B} | \overrightarrow{\nabla}\varphi \rangle = \langle (0,1,0) | (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) \rangle = \frac{\partial\varphi}{\partial x_2}$.
Applying (3.10), it follows that $\varphi = \int \phi_{\mathbb{F}^3} dx_2$.

- For $curl(\overrightarrow{B}) := e_z = (0,0,1)$, $\langle curl\overrightarrow{B} | \overrightarrow{\nabla}\varphi \rangle = \langle (0,0,1) | (\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}) \rangle = \frac{\partial\varphi}{\partial x_3}$.
Applying (3.10), it follows that $\varphi = \int \phi_{\mathbb{F}^3} dx_3$.

This concludes the proof of the corollary. □

**Example 3.9.** Consider quasi-Poisson structure $Q_0$ defined in Example 3.4. Let's determine a Poisson bivector $P$ verifying $[Q_0,Q_0]=[P,B]$ where $B$ is an arbitrary bivector field described in Corollary 3.8. Since $[Q,Q]=\phi_{\mathbb{F}^3}\partial_1\wedge\partial_2\wedge\partial_3$, $\phi_{\mathbb{F}^3}(dx_1,dx_2,dx_3)=2\mathcal{J}=2x_2^n$ and according to (3.8), (3.9) and (3.10) respectively,

(i) $P=\frac{\partial k_{23}}{\partial x_3}\partial_1\wedge\partial_2+(2x_1x_2^n+k_{23})\partial_2\wedge\partial_3+(2nx_1x_2^{n-1}+\frac{\partial k_{23}}{\partial x_2})\partial_3\wedge\partial_1$
for $curl\ \overrightarrow{B}:=e_x=(1,0,0)$ with $k_{23}\in\mathbb{F}[x_2,x_3]$.

(ii) $P=\frac{\partial k_{13}}{\partial x_3}\partial_1\wedge\partial_2+\frac{\partial k_{13}}{\partial x_1}\partial_2\wedge\partial_3+(\frac{2}{n+1}x_2^{n+1}+k_{13})\partial_3\wedge\partial_1$
for $curl\ \overrightarrow{B}:=e_y=(0,1,0)$, with $k_{13}\in\mathbb{F}[x_1,x_3]$.

(iii) $P=(2x_2^nx_3+k_{12})\partial_1\wedge\partial_2+\frac{\partial k_{12}}{\partial x_1}\partial_2\wedge\partial_3+(2x_2^{n-1}x_3+\frac{\partial k_{12}}{\partial x_2})\partial_3\wedge\partial_1$
for $curl\ \overrightarrow{B}:=e_z=(0,0,1)$, with $k_{12}\in\mathbb{F}[x_1,x_2]$.

## 4. Quasi-Poisson structures on $\mathcal{A}$ and Poisson subalgebras associated

### 4.1. LP-cohomology complex and isomorphism of complexes.

In the following, we construct a complex $(\mathfrak{X}^*,d_\pi^*)$ isomorphic to $(\mathfrak{X}^*,\delta_\pi^*)$ in the case where $\mathbb{A}$ is $\mathcal{A}$. Suppose that $(\mathbb{A},\cdot,\{,\})$ be an Poisson algebra. Its LP-cohomology complex is defined as follows

$$(\mathfrak{X}^*,\delta_\pi^*)=\left(0\xrightarrow{0}\mathfrak{X}^0(\mathcal{A})\xrightarrow{\delta_\pi^0}\mathfrak{X}^1(\mathcal{A})\xrightarrow{\delta_\pi^1}\mathfrak{X}^2(\mathcal{A})\xrightarrow{\delta_\pi^2}\mathfrak{X}^3(\mathcal{A})\longrightarrow 0\right)\tag{4.1}$$

computes it Poisson cohomology. Each of the Poisson coboundary operators associated to the Poisson algebra $(\mathcal{A},\cdot,\{,\})$ denoted $\delta_\pi^k$ can now be written in compact form :

$$\begin{aligned}\delta_\pi^0(F)&=\overrightarrow{\nabla}F\times\overrightarrow{\nabla}\varphi,\quad\forall F\in\mathcal{A};\\ \delta_\pi^1(F')&=-\overrightarrow{\nabla}(\langle\overrightarrow{F}\mid\overrightarrow{\nabla}\varphi\rangle)+Div(\overrightarrow{F})\overrightarrow{\nabla}\varphi,\quad\forall F'\in\mathcal{A}^3;\\ \delta_\pi^2(\overrightarrow{F})&=-Div(\overrightarrow{F}\times\overrightarrow{\nabla}\varphi),\quad\forall\overrightarrow{F}\in\mathcal{A}^3\end{aligned}$$

with $\overrightarrow{\nabla}\varphi=(\{x_2,x_3\},\{x_3,x_1\},\{x_1,x_2\})$. Then, $(\mathfrak{X}^*,\delta_\pi^*)$ is LP-cohomology complex. Indeed, for all $F\in\mathcal{A}$ and $\overrightarrow{F}=(F_1,F_2,F_3)\in\mathcal{A}^3$, we have

$$\delta_\pi^1\circ\delta_\pi^0(F)=-\langle curl\ \overrightarrow{\nabla}\varphi\mid\overrightarrow{\nabla}F\rangle\overrightarrow{\nabla}\varphi,\tag{4.2}$$

and

$$\delta_\pi^2\circ\delta_\pi^1(\overrightarrow{F})=\langle curl\ \overrightarrow{\nabla}\varphi\mid\overrightarrow{\nabla}\varphi\rangle Div(\overrightarrow{F}).\tag{4.3}$$

Since $\pi=\{,\}$ is Poisson structure, it follows that $\delta_\pi^1\circ\delta^0$ and $\delta_\pi^2\circ\delta^1$ defined in (4.2) and (4.3) equals zero.

**Proposition 4.1.** *The LP-Cohomology Poisson complex of Poisson algebra $(\mathcal{A},\cdot,\{,\})$ is isomorphic to*

$$(\mathfrak{X}^*,d_\pi^*)=\left(0\xrightarrow{0}\mathcal{A}\xrightarrow{d_\pi^0}\mathcal{A}\times\mathcal{A}\times\mathcal{A}\xrightarrow{d_\pi^1}\mathcal{A}\times\mathcal{A}\times\mathcal{A}\xrightarrow{d_\pi^2}\mathcal{A}\longrightarrow 0\right)\tag{4.4}$$

*with*

$$d_\pi^* = \delta_\pi^*.$$

*Proof.* For the sake of simplicity, we consider the following isomorphisms :

$$f_0 : \mathfrak{X}^0(\mathcal{A}) \longrightarrow \mathcal{A}, \quad a \longmapsto a, \tag{4.5}$$

$$f_1 : \mathfrak{X}^1(\mathcal{A}) \longrightarrow \mathcal{A}\times\mathcal{A}\times\mathcal{A}, \quad a_1\partial_x + a_2\partial_y + a_3\partial_z \longmapsto (a_1, a_2, a_3), \tag{4.6}$$

$$f_2 : \mathfrak{X}^2(\mathcal{A}) \longrightarrow \mathcal{A}\times\mathcal{A}\times\mathcal{A}, \quad a\partial_1\wedge\partial_2 + b\partial_2\wedge\partial_3 + c\partial_3\wedge\partial_1 \longmapsto (a, b, c) \tag{4.7}$$

and

$$f_3 : \mathfrak{X}^2(\mathcal{A}) \longrightarrow \mathcal{A}, \quad a\partial_x\wedge\partial_y\wedge\partial_z \longmapsto a. \tag{4.8}$$

We have $f_1\circ\delta_\pi^0 = d_\pi^0\circ f_0$, $f_2\circ\delta_\pi^1 = d_\pi^1\circ f_1$ and $f_3\circ\delta_\pi^2 = d_\pi^2\circ f_2$. It follows that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{X}^0(\mathcal{A}) \arrow[r, "\delta_\pi^0"] \arrow[d, "f_0"] & \mathfrak{X}^1(\mathcal{A}) \arrow[r, "\delta_\pi^1"] \arrow[d, "f_1"] & \mathfrak{X}^2(\mathcal{A}) \arrow[r, "\delta_\pi^2"] \arrow[d, "f_2"] & \mathfrak{X}^3(\mathcal{A}) \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r, "d_\pi^0"] & \mathcal{A}\times\mathcal{A}\times\mathcal{A} \arrow[r, "d_\pi^1"] & \mathcal{A}\times\mathcal{A}\times\mathcal{A} \arrow[r, "d_\pi^2"] & \mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

where $f_0, f_1, f_2$ and $f_3$ are defined in (4.5), (4.6), (4.7) and (4.8). The above commutator diagram involves that the Poisson complex defined by (4.1) is isomorphic to the Poisson complex $(\mathfrak{X}_0^*, d_{\pi_0}^*)$ defined by (4.4). □

### 4.2. **Construction of Poisson subalgebras associated with $Q$.**

Our problem in this section is to construct Poisson subalgebras associated to any quasi-Poisson structures $Q = \{,\}_Q$. According to Definition 2.7, the problem can be reduced to constructing a vector subspace $\mathcal{B}\subset\mathcal{A}$ satisfying (2.1) (i.e. $\mathcal{B}\cdot\mathcal{B}\subset\mathcal{B}$ and $\{\mathcal{B},\mathcal{B}\}\subset\mathcal{B}$). In the following proposition, we are demonstrating that $Cas(\mathcal{A})$ makes a Poisson subalgebra of every Poisson algebra $(\mathcal{A},\cdot,\{,\}_P)$.

**Proposition 4.2.** *$Cas(\mathcal{A})$ is Poisson subalgebra of any Poisson algebra $(\mathcal{A},\cdot,\{,\}_P)$.*

*Proof.* Indeed, for $H_1, H_2$ in $Cas(\mathcal{A})$, $a, b$ in $\mathbb{F}$ and for every polynomial function $F$ of $\mathcal{A}$, $\{F, aH_1 + bH_2\}_P = a\{F, H_1\}_P + b\{F, H_2\}_P$ which equals zero since $H_1, H_2 \in Cas(\mathcal{A})$ since, $\{\cdot,\cdot\}_P$ is a biderivation, $H_1\cdot H_2 \in Cas(\mathcal{A})$. Additionally, $\{F, \{H_1, H_2\}_P\}_P$ equals zero, which means that $\{Cas(\mathcal{A}), Cas(\mathcal{A})\}_P \subseteq Cas(\mathcal{A})$. $Cas(\mathcal{A})$ is therefore a Poisson algebra of $(\mathcal{A},\cdot,\{,\}_P)$. □

*Remark* 4.3. $Cas(\mathcal{A})$ do not leads automatically to Poisson subalgebra for quasi-Poisson algebra $(\mathcal{A},\cdot, Q = \{,\}_Q)$.

**Example 4.4.** Consider the quasi-Poisson structure $Q_0$.

Its jacobiator $\mathcal{J}$ which does not satisfy the Jacobi identity is defined as $\mathcal{J}(x_1, x_2, x_3) = x_2^n$. The question arises is to know if $\mathcal{J} \in Cas(\mathcal{A})$. For all $F$ in $\mathcal{A}$, we have :

$$\{F, \mathcal{J}\}_Q = \mathcal{J}(-\frac{\partial\mathcal{J}}{\partial x_2}\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_3}).$$

Since $Q_0$ is a quasi-Poisson structure, its Jacobiator $\mathcal{J}$ is not zero, which implies that $\mathcal{J}$ is not a Casimir.

Let $\mathcal{B}$ the set of functions $F \in \mathcal{A}$ which are solutions of the following differential equation

$$\left\langle A_Q(\overrightarrow{\nabla}\mathcal{J})^\top \mid \overrightarrow{\nabla}F\right\rangle = 0$$

where $A_Q$ is the quasi-Poisson matrix of quasi-Poisson structure $Q = \{,\}_Q$. Then, $\mathcal{B}$ is non-empty subspace of $\mathcal{A}$.

**Proposition 4.5.** *For any quasi-Poisson structure $Q = \{,\}_Q$ on $\mathcal{A}$, $Cas(\mathcal{B})$ is Poisson subalgebra of $(\mathcal{A}, \cdot, \{,\}_Q)$ with*

$$\mathcal{B} = \left\{F \in \mathcal{A} : \left\langle A_Q(\overrightarrow{\nabla}\mathcal{J})^\top \mid \overrightarrow{\nabla}F\right\rangle = 0\right\}, \tag{4.9}$$

*where $A_Q$ is the quasi-Poisson matrix of $Q$, $\mathcal{J}$ its Jacobiator and $\overrightarrow{\nabla}$ the gradient operator.*

*Proof.* It is sufficient to prove that $\mathcal{J} \in Cas(\mathcal{B})$ which means that for every polynomial function $\varphi \in \mathcal{B}$, $\{\varphi, \mathcal{J}\}$ is zero. Since $\{\varphi, \mathcal{J}\}$ equals $Q(d\varphi, d\mathcal{J})$, the bracket $\{\varphi, \mathcal{J}\}$ becomes $(\overrightarrow{\nabla}\varphi)A_Q(\overrightarrow{\nabla}\mathcal{J})^\top$ which is equivalent to

$$\{\varphi, \mathcal{J}\} = \langle A_Q(\overrightarrow{\nabla}\mathcal{J})^\top \mid \overrightarrow{\nabla}F\rangle.$$

Since $\varphi \in \mathcal{B}$, it follows that $\{\varphi, \mathcal{J}\} = 0$, so $\mathcal{J}$ is a Casimir of $\mathcal{B}$. $\square$

Let $\mathfrak{I}$ be an ideal of polynomial functions vanishing on $\mathbb{F}^3$. For an element $F$ in $\mathcal{A}$, we denote the coset $F + \mathfrak{I}$ as $\overline{\mathcal{A}} = \mathcal{A}/_{\mathfrak{I}}$, with $\overline{a}, \overline{b}, \overline{c}$ representing the generators of $\overline{\mathcal{A}}$. Consequently, $\mathfrak{I}$ includes all relations among these generators. Let $Q$ be the bivector field on $\mathcal{A}^3$ defined in (3.1), such that $Q(dF, dG) = \{F, G\}_Q$, where $Q(dF, dG) = \{F, G\}_Q$ denotes the quasi-Poisson bracket. The quotient $\overline{\mathcal{A}}$ inherits the structure of an associative algebra from $\mathcal{A}$. Given that $Q = \{\cdot,\cdot\}_Q$ represents quasi-Poisson structure on $\mathcal{A}$, a pertinent question arises : what conditions must be ideal $\mathfrak{I}$ satisfy for $\overline{\mathcal{A}}$ to possess a Poisson structure $P = \{\cdot,\cdot\}_P$, that is derived from $Q$ ? We would like to define Poisson structure $\{\cdot,\cdot\}_P$ of $\overline{\mathcal{A}}$ by setting

$$\overline{\{F, G\}_Q} = \{\overline{F}, \overline{G}\}_P$$

for all $F, G \in \mathcal{A}$ and $\overline{F}, \overline{G} \in \overline{\mathcal{A}}$. In general, this problem is not well-defined.

**Proposition 4.6.** *Let $\mathfrak{I}$ be an ideal of $\mathcal{A}$. Let $Q = a\partial_1\wedge\partial_2 + b\partial_2\wedge\partial_3 + c\partial_3\wedge\partial_1$ be a quasi-Poisson bivector field on $\mathbb{F}^3$ $(a, b, c \in \mathcal{A})$. There exists a unique well-defined Poisson bivector field $P$ on $\overline{\mathcal{A}} = \mathcal{A}/_{\mathfrak{I}}$ defined by*

$$P(d\,\overline{F}, d\,\overline{G}) = \overline{Q(dF, dG)} \quad for\ \overline{F}, \overline{G} \in \overline{\mathcal{A}} = \mathcal{A}/_{\mathfrak{I}} \tag{4.10}$$

*if and only if*

$$\phi_{\mathbb{F}^3} \in \mathfrak{I} \tag{4.11}$$

*where $\phi_{\mathbb{F}^3}$ is the trivector field defined in Definition 2.11.*

*Proof.* Let $\{\cdot,\cdot\}_Q$ on $\mathcal{A}$ and $\{\cdot,\cdot\}_P$ on $\overline{\mathcal{A}}$ be two skew-symmetries biderivations verifying $\{F, G\}_Q = Q(dF, dG)$, $\{F, G\}_P = P(dF, dG)$ and satisfying condition (4.10).

It is clear that $P$ is well-defined. Also $P$ is a skew-symmetric biderivation since $Q$ is one. Since $P$ is well-defined,

$$\{\{\overline{F},\overline{G}\}_P,\overline{H}\}_P=\{\{F,G\}_Q,H\}_Q$$

for all $F,G,H\in\mathcal{A}$, so that $\{\cdot,\cdot\}_P$ is a Poisson bracket on $\overline{\mathcal{A}}$ if and only if

$$\{\{F,G\}_Q,H\}_Q+\{\{G,H\}_Q,F\}_Q+\{\{H,F\}_Q,G\}_Q\in\mathfrak{I},\quad for\ all\ F,G,H\in\mathcal{A}$$

which is equivalent to (4.11) :

$$\phi_{\mathbb{F}^3}(dx_1,dx_2,dx_3)=2\mathcal{J}(x_1,x_2,x_3)\in\mathfrak{I}.$$

Conversely, given a Poisson structure $\{\cdot,\cdot\}_P$ on $\overline{\mathcal{A}}$, the Poisson bracket $\{\overline{F},\cdot\}_P$ of any two elements $\overline{F}$ and $\overline{G}$ of $\mathcal{A}$ can be computed from the skew-symmetric biderivation $\{\cdot,\cdot\}_Q$ of $\mathcal{A}$, namely

$$\{\overline{F},\overline{G}\}_Q=\overline{a}\overline{\frac{\partial F}{\partial x_1}}\,\overline{\frac{\partial G}{\partial x_2}}+\overline{b}\overline{\frac{\partial F}{\partial x_2}}\,\overline{\frac{\partial G}{\partial x_3}}+\overline{c}\overline{\frac{\partial F}{\partial x_3}}\,\overline{\frac{\partial G}{\partial x_1}},$$

Finally, let us consider the canonical projection $p:\mathcal{A}\longrightarrow\overline{\mathcal{A}},\ F\longmapsto p(F)=\overline{F}$. Since this projection is surjective, if such a Poisson structure $P$ exists, then it is unique. $\square$

**Proposition 4.7.** *The complex described in Proposition 4.1 becomes a Poisson complex for $Q$ if and only if $\langle curl\ \overrightarrow{Q}\mid\overrightarrow{\nabla}F\rangle\overrightarrow{Q}=0$ and $\langle curl\ \overrightarrow{Q}\mid\overrightarrow{Q}\rangle Div(\overrightarrow{F})=0$ for all $F\in\mathcal{A}$, $\overrightarrow{F}\in\mathcal{A}^3$.*

*Proof.* Let $(\mathfrak{X}^*,d_Q^*)$ be the complex defined by

$$(\mathfrak{X}^*,d_Q^*)=\left(0\xrightarrow{0}\mathfrak{X}^0(\mathcal{A})\xrightarrow{d_Q^0}\mathfrak{X}^1(\mathcal{A})\xrightarrow{d_Q^1}\mathfrak{X}^2(\mathcal{A})\xrightarrow{d_Q^2}\mathcal{A}\longrightarrow 0\right)$$

where

$$\begin{aligned}
d_Q^0(F) &= \overrightarrow{\nabla}F\times\overrightarrow{Q},\quad \forall F\in\mathcal{A}\simeq\mathfrak{X}^0(\mathcal{A}),\\
d_Q^1(\overrightarrow{F}) &= -\overrightarrow{\nabla}(\langle\overrightarrow{F}\mid\overrightarrow{Q}\rangle)+Div(\overrightarrow{F})\overrightarrow{Q},\quad \forall\overrightarrow{F}\in\mathcal{A}^3\simeq\mathfrak{X}^1(\mathcal{A}),\\
d_Q^2(\overrightarrow{G}) &= -Div(\overrightarrow{F}\times\overrightarrow{Q}),\quad \forall\overrightarrow{G}\in\mathcal{A}^3\simeq\mathfrak{X}^2(\mathcal{A})
\end{aligned}$$

where $\overrightarrow{Q}=(b,c,a)$, $Div(\overrightarrow{F})=\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}+\frac{\partial F_3}{\partial x_3}$ and $\times$ the cross product.

By computation, we obtain $d_Q^1\circ d_Q^0(F)=-\langle curl\ \overrightarrow{Q}\mid\overrightarrow{\nabla}F\rangle\overrightarrow{Q}$ and

$$d_Q^2\circ d_Q^1(\overrightarrow{F})=\langle curl\ \overrightarrow{Q}\mid\overrightarrow{Q}\rangle Div(\overrightarrow{F}).$$

Since $Q$ is quasi-Poisson structure, $d_Q^1\circ d_Q^0(F)$ and $d_Q^2\circ d_Q^1(\overrightarrow{F})$ does not vanishes. $\square$

**Lemma 4.8.** *Let $\mathcal{B}'$ be an ideal $\{F\in\mathcal{A}\mid\langle curl\ \overrightarrow{Q}\mid\overrightarrow{\nabla}F\rangle\}$ and consider*

$$I=\mathcal{B}\bigcap\mathcal{B}'\tag{4.12}$$

*where $\mathcal{B}$ is described in Proposition 4.5 by (4.9). Then, $I$ is a non-trivial ideal of $\mathcal{A}$.*

*Proof.* Since $\mathcal{B}$ and $\mathcal{B}'$ are ideals of $\mathcal{A}$, their intersection is obviously an ideal of $\mathcal{A}$ containing at least the trivial element. It remains to prove that $I$ contains at least one non-trivial element. Their intersection contains the zero polynomial as well as all constant polynomials. Explicitly,

$$\mathcal{B}' = \{F \in \mathcal{A} : (-a\partial_2\mathcal{J}+c\partial_3\mathcal{J})\partial_1 F+(a\partial_1\mathcal{J}-b\partial_3\mathcal{J})\partial_2 F+(-c\partial_1\mathcal{J}+b\partial_2\mathcal{J})\partial_3 F = 0\},$$

$$\textit{and } \mathcal{B} = \{F \in \mathcal{A} : (\partial_2 a - \partial_3 c)\partial_1 F + (\partial_3 b - \partial_1 a)\partial_2 F + (\partial_1 c - \partial_2 b)\partial_3 F = 0\}.$$

Using the curl operator, the problem can be summarized as finding a non-trivial vector $\overrightarrow{\nabla} F$ that is simultaneously orthogonal to non-zero vectors $A_Q(\overrightarrow{\nabla}\mathcal{J})^\top$ and $Curl\ \overrightarrow{Q}$. ☐

**Example 4.9.** Consider quasi-Poisson structure $Q_0$. By computation, we have

$$d^1_{Q_0} \circ d^0_{Q_0}(F) = x_2^n(-\partial_1 F, -x_1 x_2^{-1}\partial_1 F, -n x_2^{n-1}\partial_1 F),\ \ d^2_{Q_0} \circ d^1_{Q_0}(\overrightarrow{F}) = x_2^n Div(\overrightarrow{F}),$$

$\mathcal{B}_{Q_0} = \{F \in \mathcal{A} : -x_2^{n-1})\partial_1 F + \partial_3 F) = 0\}$ *and* $\mathcal{B'}_{Q_0} = \{F \in \mathcal{A} : -n x_2^{n-1}\partial_1 F = 0\}$. It follows that its jacobiator $x_2^n \in I = \mathcal{B}_{Q_0} \bigcap \mathcal{B'}_{Q_0}$.

In the following theorem, we present a LP-cohomology subcomplex induced by $(\mathfrak{X}^*, \delta^*_Q)$.

**Theorem 4.10.** *Consider the following graded map*

$$(\mathfrak{X}^*_I, \delta^*_Q) := \left(0 \xrightarrow{0} \mathfrak{X}^0(I) \xrightarrow{\delta^0_Q} \mathfrak{X}^1(I) \xrightarrow{\delta^1_Q} \mathfrak{X}^2(I) \xrightarrow{\delta^2_Q} \mathfrak{X}^3(I) \longrightarrow 0\right)$$

*with*

$$\begin{array}{rcl} \delta^0_Q(F) & = & \overrightarrow{\nabla} F \times \overrightarrow{Q}, \quad \forall F \in I \simeq \mathfrak{X}^0(I); \\ \delta^1_Q(\overrightarrow{F}) & = & -\overrightarrow{\nabla}(\langle \overrightarrow{F} \mid \overrightarrow{Q} \rangle) + Div(\overrightarrow{F})\overrightarrow{Q}, \quad \forall \overrightarrow{F} \in I^3 \simeq \mathfrak{X}^1(I); \\ \delta^2_Q(\overrightarrow{G}) & = & -Div(\overrightarrow{F} \times \overrightarrow{Q}), \quad \forall \overrightarrow{G} \in I^3 \simeq \mathfrak{X}^2(I). \end{array}$$

*Then, $(\mathfrak{X}^*_I, \delta^*_Q)$ is a stable LP-cohomology subcomplex which computes Poisson cohomology of the quasi-Poisson algebra $(\mathcal{A}, Q = \{,\}_Q)$ when $\mathcal{A}$ is reduced to Ideal $I$ defined on (4.12).*

*Proof.* It follows from Lemma 4.8 that is equivalent to $\delta^1_Q \circ \delta^0_Q(F)$ and $\delta^2_Q \circ \delta^1_Q(F)$ simultaneously vanish on $I$. The stability of the complex needs to be justified. To do this, we must show that the values $\delta^0_Q(F)$, $\delta^1_Q(\overrightarrow{F})$ and $\delta^2_Q(\overrightarrow{G})$ are absolutely contained in $I$, $I^3$ and $I^3$ respectively. Specifically, $(\mathfrak{X}^*_I, d^*_Q)$ is equivalent to

$$(0 \xrightarrow{0} I \xrightarrow{d^0_Q} I \times I \times I \xrightarrow{d^1_Q} I \times I \times I \xrightarrow{d^2_Q} I \longrightarrow 0)$$

with $d^0_Q(F) = (d^0_x, d^0_y, d^0_z)$, $d^1_Q(F_1, F_2, F_3) = (d^1_x, d^1_y, d^1_z)$ and $d^2_Q(G_1, G_2, G_3) = d^2$ where $d^0_x = a\frac{\partial F}{\partial x_2} - c\frac{\partial F}{\partial x_3}, d^0_y = b\frac{\partial F}{\partial x_3} - a\frac{\partial F}{\partial x_1},\ d^0_z = c\frac{\partial F}{\partial x_1} - b\frac{\partial F}{\partial x_2}$,

$d^1_x = -\frac{\partial b}{\partial x_1}F_1 + (-\frac{\partial c}{\partial x_1} - c\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2})F_2 + (-\frac{\partial a}{\partial x_1} - a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_3})F_3$,

$d^1_y = (-\frac{\partial b}{\partial x_2} - b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_1})F_1 - \frac{\partial c}{\partial x_2}F_2 + (-\frac{\partial a}{\partial x_2} - a\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3})F_3$,

$d_z^1 = (-\frac{\partial b}{\partial x_3} - b\frac{\partial}{\partial x_3} + a\frac{\partial}{\partial x_1})F_1 + (-\frac{\partial c}{\partial x_3} + a\frac{\partial}{\partial x_2} - c\frac{\partial}{\partial x_3})F_2 - \frac{\partial a}{\partial x_3}F_3$,
and $d^2 = (a\frac{\partial}{\partial x_2} - c\frac{\partial}{\partial x_3})G_1 + (-a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_3})G_2 + (c\frac{\partial}{\partial x_1} - b\frac{\partial}{\partial x_2})G_3$.
Explicitly, $I$ is the set of polynomials $F$ satisfying equations

$$(\partial_2 a - \partial_3 c)\partial_1 F + (\partial_3 b - \partial_1 a)\partial_2 F + (\partial_1 c - \partial_2 b)\partial_3 F = 0 \tag{4.13}$$

and

$$(-a\partial_2\mathcal{J} + c\partial_3\mathcal{J})\partial_1 F + (a\partial_1\mathcal{J} - b\partial_3\mathcal{J})\partial_2 F + (-c\partial_1\mathcal{J} + b\partial_2\mathcal{J})\partial_3 F = 0. \tag{4.14}$$

The objective is to demonstrate that each of the polynomials $d_x^0$, $d_y^0$, $d_z^0$, $d_x^1$, $d_y^1$, $d_z^1$ and $d^2$, is an element of $I$. By computation, we have :
$\partial_1\mathcal{J} = \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1} - \frac{\partial c}{\partial x_3}\frac{\partial b}{\partial x_1} + \frac{\partial b}{\partial x_3}\frac{\partial c}{\partial x_1} - \frac{\partial a}{\partial x_1}\frac{\partial c}{\partial x_1} + \frac{\partial c}{\partial x_1}\frac{\partial a}{\partial x_1} - \frac{\partial b}{\partial x_2}\frac{\partial a}{\partial x_1} + \frac{\partial^2 a}{\partial x_1\partial x_2}b - \frac{\partial^2 c}{\partial x_1\partial x_3}b + \frac{\partial^2 b}{\partial x_1\partial x_3}c - \frac{\partial^2 a}{\partial x_1^2}c + \frac{\partial^2 c}{\partial x_1^2}a - \frac{\partial^2 b}{\partial x_1\partial x_2}a$, $\partial_2\mathcal{J} = \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_2} - \frac{\partial c}{\partial x_3}\frac{\partial b}{\partial x_2} + \frac{\partial b}{\partial x_3}\frac{\partial c}{\partial x_2} - \frac{\partial a}{\partial x_1}\frac{\partial c}{\partial x_2} + \frac{\partial c}{\partial x_1}\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_2}\frac{\partial a}{\partial x_2} + \frac{\partial^2 a}{\partial x_2^2}b - \frac{\partial^2 c}{\partial x_2\partial x_3}b + \frac{\partial^2 b}{\partial x_2\partial x_3}c - \frac{\partial^2 a}{\partial x_1\partial x_2}c + \frac{\partial^2 c}{\partial x_1\partial x_2}a - \frac{\partial^2 b}{\partial x_2^2}a$, $\partial_3\mathcal{J} = \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_3} - \frac{\partial c}{\partial x_3}\frac{\partial b}{\partial x_3} + \frac{\partial b}{\partial x_3}\frac{\partial c}{\partial x_3} - \frac{\partial a}{\partial x_1}\frac{\partial c}{\partial x_3} + \frac{\partial c}{\partial x_1}\frac{\partial a}{\partial x_3} - \frac{\partial b}{\partial x_2}\frac{\partial a}{\partial x_3} + \frac{\partial^2 a}{\partial x_2\partial x_3}b - \frac{\partial^2 c}{\partial x_3^2}b + \frac{\partial^2 b}{\partial x_3^2}c - \frac{\partial^2 a}{\partial x_1\partial x_3}c + \frac{\partial^2 c}{\partial x_1\partial x_3}a - \frac{\partial^2 b}{\partial x_2\partial x_3}a$,
$\partial_1 d_x^0 = \frac{\partial a}{\partial x_1}\frac{\partial F}{\partial x_2} - \frac{\partial c}{\partial x_1}\frac{\partial F}{\partial x_3} + a\frac{\partial^2 F}{\partial x_1\partial x_2} - c\frac{\partial^2 F}{\partial x_1\partial x_3}$, $\partial_2 d_x^0 = \frac{\partial a}{\partial x_2}\frac{\partial F}{\partial x_2} - \frac{\partial c}{\partial x_2}\frac{\partial F}{\partial x_3} + a\frac{\partial^2 F}{\partial x_2^2} - c\frac{\partial^2 F}{\partial x_2\partial x_3}$ and $\partial_3 d_x^0 = \frac{\partial a}{\partial x_3}\frac{\partial F}{\partial x_2} - \frac{\partial c}{\partial x_3}\frac{\partial F}{\partial x_3} + a\frac{\partial^2 F}{\partial x_2\partial x_3} - c\frac{\partial^2 F}{\partial x_3^2}$.

By substituting the partial derivatives with their values, we demonstrate that $d_x^0$ satisfies conditions (4.13) and (4.14). Consequently, $d_x^0 \in I$. Similarly, we establish the membership of $d_y^0$, $d_z^0$, $d_x^1$, $d_y^1$, $d_z^1$ and $d^2$ in $I$. □

To illustrate Theorem 4.10, let's consider an example of application.

**Example 4.11.** Consider quasi-Poisson structure $Q_0$ with $a = -x_2^2$, $b = -\frac{1}{n}x_2$, $c = -\frac{1}{n}x_1$ and $\mathcal{J} = x_2^n$. The LP-cohomology subcomplex described in Theorem 4.10 is equivalent to

$$(0 \xrightarrow{0} I \xrightarrow{d^0_{Q_0}} I\times I\times I \xrightarrow{d^1_{Q_0}} I\times I\times I \xrightarrow{d^2_{Q_0}} I \longrightarrow 0)$$

with $d^0_{Q_0}(F) = (d_x^0, d_y^0, d_z^0)$, $d^1_Q(F_1,F_2,F_3) = (d_x^1, d_y^1, d_z^1)$ and $d^2_Q(G_1,G_2,G_3) = d^2$ where
$d_x^0 = -x_2^n\frac{\partial F}{\partial x_2} + \frac{1}{n}x_1\frac{\partial F}{\partial x_3}, d_y^0 = -\frac{1}{n}x_2\frac{\partial F}{\partial x_3} + x_2^n\frac{\partial F}{\partial x_1}$, $d_z^0 = -\frac{1}{n}x_1\frac{\partial F}{\partial x_1} + \frac{1}{n}x_2\frac{\partial F}{\partial x_2}$,
$d_x^1 = (\frac{1}{n} + \frac{1}{n}x_1\frac{\partial}{\partial x_1} - \frac{1}{n}x_2\frac{\partial}{\partial x_2})F_2 + (x_2^n\frac{\partial}{\partial x_1} - \frac{1}{n}x_2\frac{\partial}{\partial x_3})F_3$,
$d_y^1 = (\frac{1}{n} + \frac{1}{n}x_2\frac{\partial}{\partial x_2} - \frac{1}{n}x_1\frac{\partial}{\partial x_1})F_1 + (nx_2^{n-1} + x_2^n\frac{\partial}{\partial x_2} - \frac{1}{n}x_1\frac{\partial}{\partial x_3})F_3$,
$d_z^1 = (\frac{1}{n}x_2\frac{\partial}{\partial x_3} - x_2^n\frac{\partial}{\partial x_1})F_1 + (-x_2^n\frac{\partial}{\partial x_2} + \frac{1}{n}x_1\frac{\partial}{\partial x_3})F_2$
and $d^2 = (-x_2^n\frac{\partial}{\partial x_2} + \frac{1}{n}x_1\frac{\partial}{\partial x_3})G_1 + (x_2^n\frac{\partial}{\partial x_1} - \frac{1}{n}x_2\frac{\partial}{\partial x_3})G_2 + (-\frac{1}{n}x_1\frac{\partial}{\partial x_1} + \frac{1}{n}x_2\frac{\partial}{\partial x_2})G_3$.
Explicitly, $I$ is the set of polynomials $F$ satisfying equation

$$\mathcal{J}(nx_2^{-1}\partial_1 F) = 0 \tag{4.15}$$

and

$$\mathcal{J}(nx_2^{n-1}\partial_1 F - \partial_3 F) = 0. \tag{4.16}$$

Since $\mathcal{J}$ vanishes on Ideal $I$ , automatically every polynomial $F$ of $I$ satisfies the required conditions (4.15) and (4.16). It is sufficient to conclude that each of the polynomials $d_x^0$, $d_y^0$, $d_z^0$, $d_x^1$, $d_y^1$, $d_z^1$ and $d^2$ were in $I$. This consolidates the result obtained in Theorem 4.10.

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