Gulf Journal of Mathematics
Vol 1, Issue 2 (2013) 193-209

# ON A NEW FRACTIONAL COMPLEX SYSTEM

RABHA W. IBRAHIM[1] *

Abstract. In this work, we suggest a new fractional complex system (three and five equations) of type Newton-Leipnik systems. The fractional calculus are taken in sense of the Caputo derivatives. The stability and stabilization of such a system are studied. The technique based on stability theory of fractional-order systems. Numerical solutions are imposed. Moreover, conditions of unique solution are illustrated.

## 1. Introduction and preliminaries

Fractional-order chaotic systems have been considered excessively in current years in many fields such as chaotic phenomena, chaotic control, chaotic synchronization, and other related studies. It has been provided that many fractional-order dynamical systems, as some well-known integer-order systems, can be presented complex bifurcation and chaotic phenomena. For example, Lorenz system, Chen system, Liu system, Newton-Leipnik system and the unified system also exhibit chaotic behavior. Recently, researchers imposed several types of chaotic nonlinear systems with complex variables. These systems which involving complex variables are employed to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators. As special models, the chaotic complex Newton-Leipnik systems which are used in physics. The complex model is, in fact, the outcome of the study of Baroclinic models (such as Newton-Leipnik Model). This model corresponds to the equilibrium state of the atmosphere in which surfaces of constant density are not parallel to the surface of constant gravitational potential [1, 2, 3, 4, 5, 6, 7].

The five dimensional complex model is often used to explain and simulate the physics of tuned lasers. These studies have their origin in the shaping expression by Haken [8] of the isomorphism between the three equations of the real Lorenz model and the three real equations for a single mode laser operating with its resonant cavity tuned to resonance with the material transition.

*Date*: Received: Jun 19, 2013; Accepted: Aug 28, 2013.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 02.30.Fn; 02.30.Ks.

*Key words and phrases.* Fractional calculus, fractional differential equations, fractional complex system, unit disk, analytic solution.

In this paper, we introduce a new fractional complex system (three and five equations) of type Newton-Leipnik systems. The fractional calculus are taken in sense of the Caputo derivatives. The stability and stabilization of such a system are studied. The technique based on stability theory of fractional-order systems. Numerical solutions are imposed. Furthermore, conditions of unique solution are calculated.

The idea of the fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Abel in 1823 investigated the generalized tautochrone problem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Since that time the fractional calculus has haggard the attention of many researchers in all area of sciences. Recently, the theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities and convolution structures for various subclasses of analytic functions and the works in the research monographs [9, 10].

**Definition 1.1.** The fractional (arbitrary) order integral of the function $f$ of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product, $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0,\ t \leq 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

**Definition 1.2.** The fractional (arbitrary) order derivative of the function $f$ of order $0 \leq \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

**Definition 1.3.** The Caputo fractional derivative of order $\mu > 0$ is defined, for a smooth function $f(t)$ by

$${}^c D^\mu f(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu]+1$, (the notation $[\mu]$ stands for the largest integer not greater than $\mu$ ).

Note that there is a relationship between Riemann-Liouville differential operator and the Caputo operator

$$D_a^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{f(a)}{(t-a)^\mu} + {}^c D_a^\mu f(t);$$

and they are equivalent in a physical problem (i.e., a problem which specifies the initial conditions).

## 2. Propose system

The NewtonLeipnik system is described by the following non-linear differential equations

$$\begin{aligned} x^{\cdot} &= -ax + y + 10yz \\ y^{\cdot} &= -x - 0.4y + 5xz \\ z^{\cdot} &= bz - 5xy \end{aligned} \tag{2.1}$$

and for the fractional order (see [1])

$$\begin{aligned} {}^{c}D^{\mu_1}x &= -ax + y + 10yz \\ {}^{c}D^{\mu_2}y &= -x - 0.4y + 5xz \\ {}^{c}D^{\mu_3}z &= bz - 5xy, \end{aligned} \tag{2.2}$$

where $0 < \mu_i \leq 1$, $i = 1, 2, 3$.

Here we turn our attention toward the system of the form

$$\begin{aligned} x^{\cdot} &= \sigma(-x - y) + yz \\ y^{\cdot} &= x - ry + 5xz \\ z^{\cdot} &= bz - 5xy \end{aligned} \tag{2.3}$$

and for fractional order

$$\begin{aligned} {}^{c}D^{\mu_1}x &= \sigma(-x - y) + yz \\ {}^{c}D^{\mu_2}y &= x - ry + 5xz \\ {}^{c}D^{\mu_3}z &= bz - 5xy, \end{aligned} \tag{2.4}$$

### 2.1. **Complex system.**

In complex variables, system (2.3) can be named as

$$\begin{aligned} \xi_1^{\cdot} &= \sigma(-\xi_1 - \xi_2) + \xi_2\xi_3 \\ \xi_2^{\cdot} &= \xi_1 - r\xi_2 + 5\xi_1\xi_3 \\ \xi_3^{\cdot} &= b\xi_3 - \frac{5}{2}(\xi_1^*\xi_2 + \xi_1\xi_2^*), \end{aligned} \tag{2.5}$$

where $\xi_1$ and $\xi_2$ are complex variables, i.e. $\xi_1 = u_1 + iu_2$, $\xi_1^* = u_1 - iu_2$ and $\xi_2 = u_3 + iu_4$, $\xi_2^* = u_3 - iu_4$ but $\xi_3 := u_5$ is real variable and $\sigma$, $r$, $b$ are real parameters. It assumed that $u_k, k = 1, ..., 5$ is a function in $t$. Also for fractional order, we obtain the system

$$\begin{aligned} {}^{c}D^{\mu_1}\xi_1 &= \sigma(-\xi_1 - \xi_2) + \xi_2\xi_3 \\ {}^{c}D^{\mu_2}\xi_2 &= \xi_1 - r\xi_2 + 5\xi_1\xi_3 \\ {}^{c}D^{\mu_3}\xi_3 &= b\xi_3 - \frac{5}{2}(\xi_1^*\xi_2 + \xi_1\xi_2^*), \end{aligned} \tag{2.6}$$

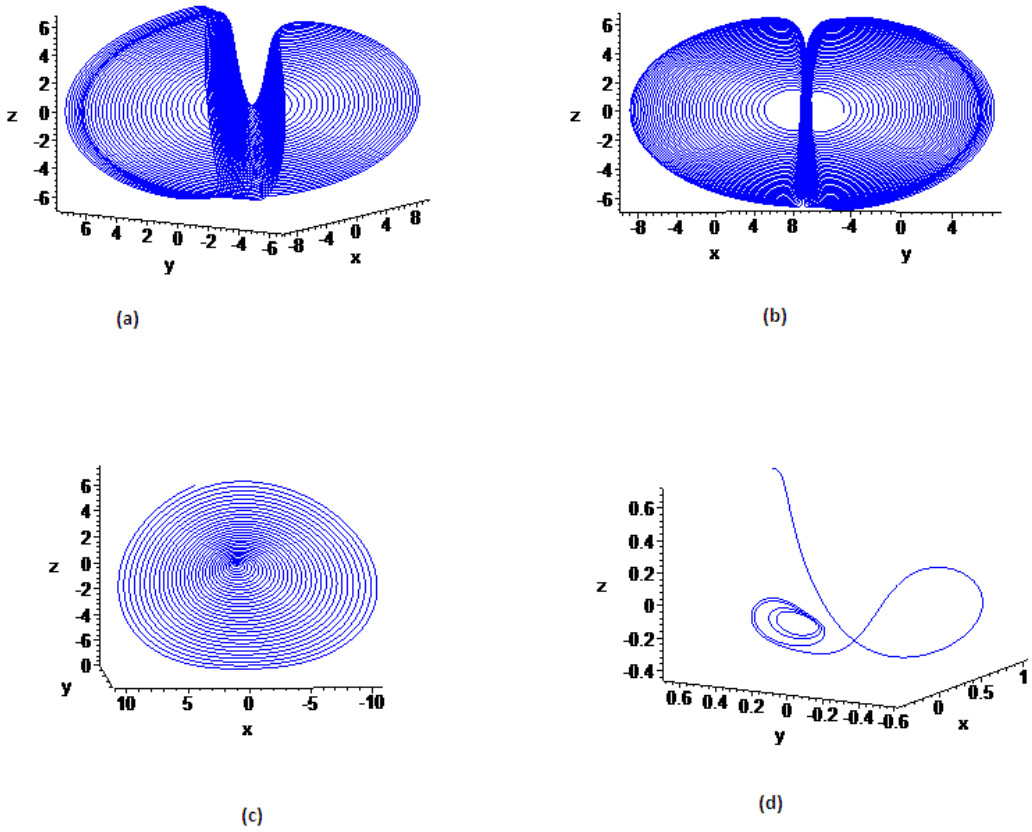



FIGURE 1. Newton-Leipnik fractional system: $(\mu_1, \mu_2, \mu_3) = (0.996, 0.992, 0.995)$

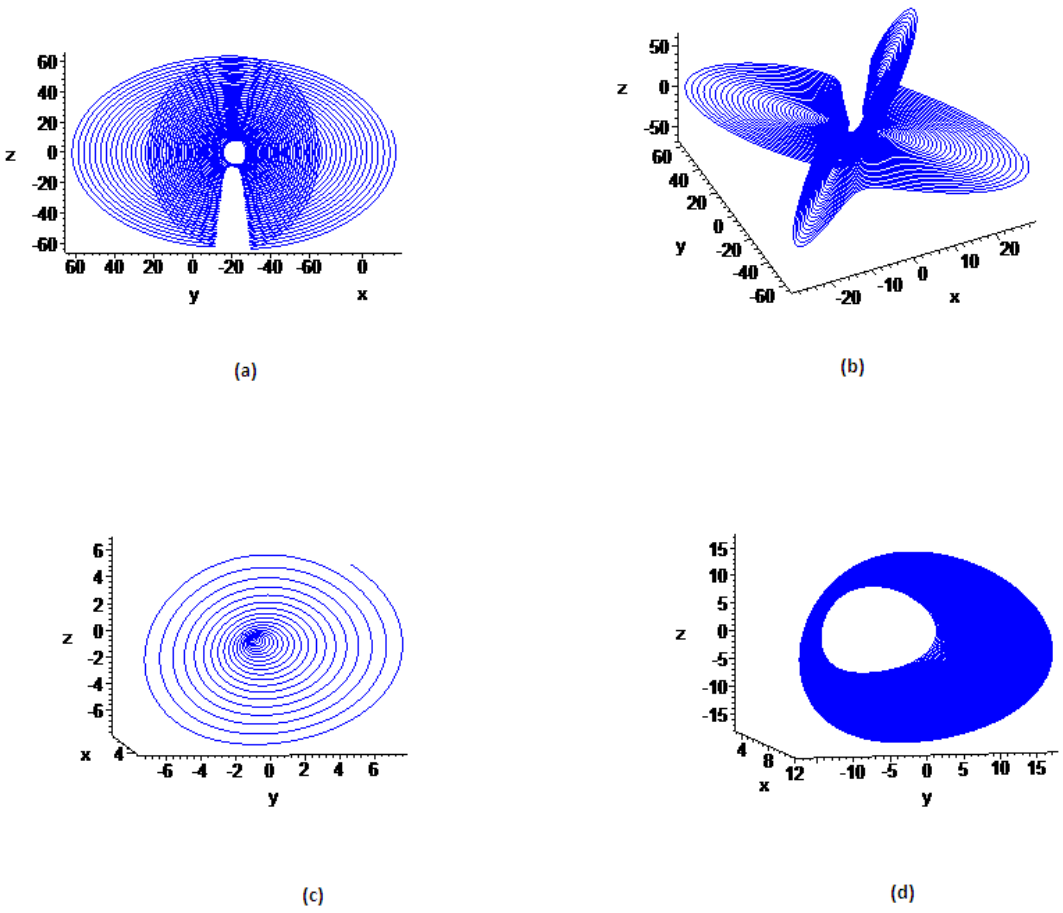



FIGURE 2. System (2.4): $(\mu_1, \mu_2, \mu_3) = (0.998, 0.997, 0.995), (x(0), y(0), z(0)) = (4, 5, 6)$

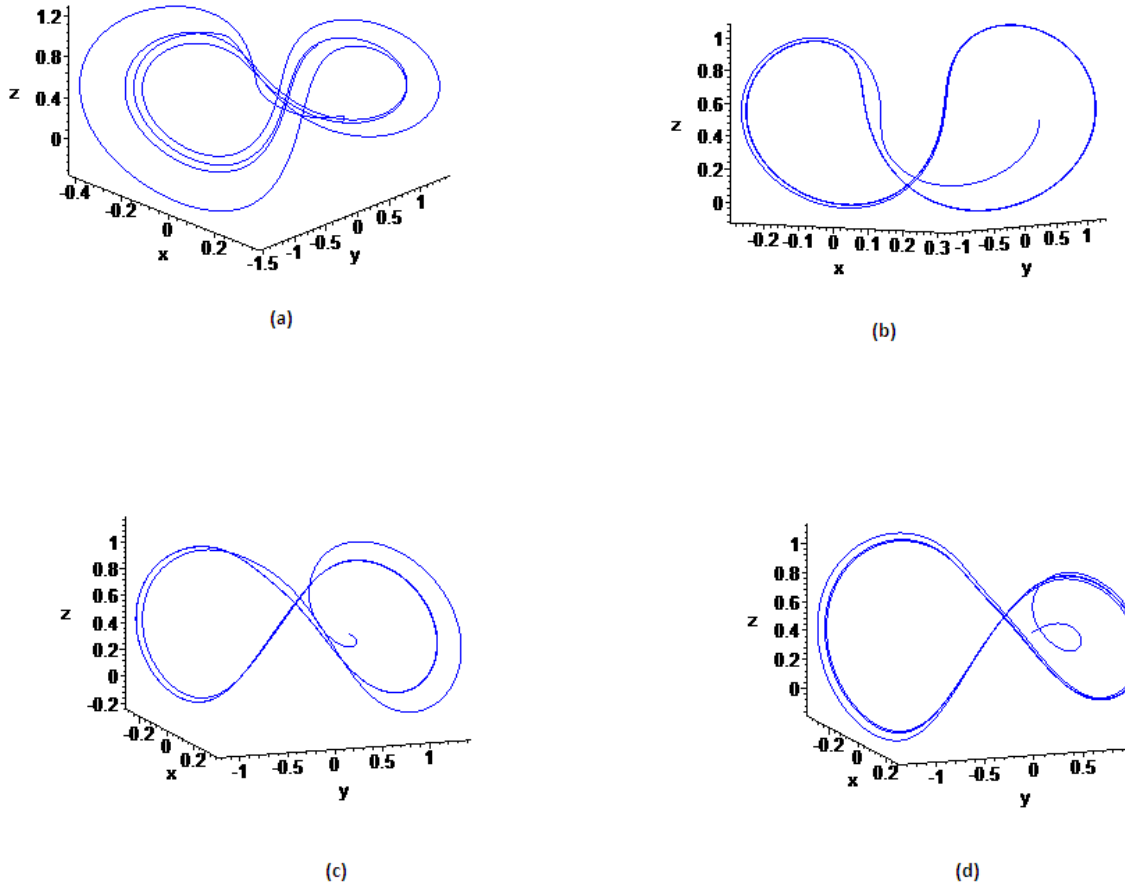



FIGURE 3. System (2.4) with different initial values

Now for real variables, system (2.6) reduces to the system of five equations

$$\begin{aligned} {}^cD^{\mu_1}u_1 &= -\sigma u_1 - \sigma u_3 + u_1u_5 \\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 - \sigma u_4 + u_2u_5 \\ {}^cD^{\mu_2}u_3 &= u_1 - ru_3 + 5u_1u_5 \\ {}^cD^{\mu_2}u_4 &= u_2 - ru_4 + 5u_2u_5 \\ {}^cD^{\mu_3}u_5 &= bu_5 + 5(u_1u_3 + u_2u_4). \end{aligned} \tag{2.7}$$

## 3. Stability of $p_0$

In this section, we shall discuss the stability of the system (2.7). For this purpose we need the following preliminaries which can be found in [11]:

**Definition 3.1.** The zero solution of the equation ${}^cD^{\mu}u = f(t, u(t))$, $\mu \in (0, 1]$ is said to be stable if, for any initial values $u_0$, there exists $\epsilon > 0$ such that $\|u(t)\| \leq \epsilon$, $\forall t > t_0$. The zero is said to be asymptotically stable if it is stable and $\|u(t)\| \to 0$, $t \to \infty$.

**Lemma 3.2.** *Assume the system of the form*

$${}^cD^{\mu}u = Au(t) + h(u(t)),\ \mu \in (0, 1], \tag{3.1}$$

*where* $u(t) = (u_1, ..., u_n(t))^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. *If* $|arg(spec(A))| > \mu\pi/2$, $\mu\|A\| > 1$, *spec*$(A)$ *denotes the eigenvalues of* $A$ *and* $\|.\|$ *denotes the* $l_2$ *norm; and*

$$\lim_{u\to 0}(\|h(u(t))\|/\|u(t)\|) = 0,$$

*then the system* (3.1) *is asymptotically stable.*

**Lemma 3.3.** *Assume the controller system with the linear feedback control input*

$${}^cD^{\mu}u = (A + BK)u(t) + h(u(t)),\ \mu \in (0, 1], \tag{3.2}$$

*where* $K \in \mathbb{R}^{1\times n}$ *is a feedback,* $B \in \mathbb{R}^{n\times 1}$, $u(t) = (u_1, ..., u_n(t))^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. *If* $|arg(spec(A))| > \mu\pi/2$, $\mu\|A + BK\| > 1$, *and* $\lim_{u\to 0}(\|h(u(t))\|/\|u(t)\|) = 0$, *then the system* (3.2) *is asymptotically stable.*

**Theorem 3.4.** *If* $b < 0$, $\sigma > r > 0$ *then the system 2.7 is asymptotically stable at the equilibrium point* $p_0 = (0, 0, 0, 0, 0)$.

**Proof.** System 2.7 can be written in the form (2.3), where

$$A = \begin{pmatrix} -\sigma & 0 & -\sigma & 0 & 0 \\ 0 & -\sigma & 0 & -\sigma & 0 \\ 1 & 0 & -r & 0 & 0 \\ 0 & 1 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \quad \text{and} \quad h(u) = \begin{pmatrix} u_1u_5 \\ u_2u_5 \\ 5u_1u_5 \\ 5u_2u_5 \\ 5(u_1u_3 + u_2u_4) \end{pmatrix}.$$

For $u_1 = u_3$ and $u_2 = u_4$, we have

$$\begin{aligned}\lim_{u\to 0}\frac{\|h(u(t))\|}{\|u(t)\|} &= \lim_{u\to 0}\frac{\sqrt{26(u_1u_5)^2+26(u_2u_5)^2+25(u_1u_3+u_2u_4)^2}}{\sqrt{(u_1)^2+(u_2)^2+(u_3)^2+(u_4)^2+(u_5)^2}}\\ &\leq \sqrt{26}\lim_{u\to 0}\frac{\sqrt{u_5^2(u_1^2+u_2^2)+(u_1^2+u_2^2)^2}}{\sqrt{(u_1^2+u_2^2)}}\\ &= \sqrt{26}\lim_{u\to 0}\sqrt{u_1^2+u_2^2+u_5^2}\\ &= 0.\end{aligned}$$

Moreover the characteristic equation of the system is

$$(b-\lambda)\big((\sigma+\lambda)(r+\lambda)+\sigma\big)^2 = 0$$

The eigenvalues are

$$\lambda_1 = b, \quad \lambda_{2,3} = \frac{-(\sigma+r)\pm\sqrt{(\sigma+r)^2-4\sigma(r+1)}}{2},$$

where $\lambda_{2,3}$ are of multiplicity 2. Since $-(\sigma+r)<0$, therefore

$$|arg(\lambda_i)| > \frac{\pi}{2} > \frac{\pi}{2}\mu, \quad \mu = max(\mu_i,\, i=1,2,3) \tag{3.3}$$

and for suitable choice of the parameters $b, \sigma, \ell$ and $\mu_i$ we have $\|A\|\underline{\mu} > 1$, $\underline{\mu} := min(\mu_i)$. According to Lemma 3.2, it implies that the equilibrium point $p_0$ of system (2.7) is asymptotically stable. This completes the proof.

The system (2.7) can be assumed in the form (3.2), we derive the following result:

**Theorem 3.5.** *If $b<0$, $\sigma > r > 0$ then the controlled system (2.7) is asymptotically stable at the equilibrium point $p_0 = (0,0,0,0,0)$.*

**Proof.** Applying control input $v(t) = BKu(t)$ on (2.7), where $B = (\sigma,0,0,0,0)^T$ and $K = (0,0,1,0,0)$ are taken in which $|arg(spec(A+BK))| > \frac{\mu\pi}{2}$, where the characteristic equation of the system is

$$(b-\lambda)(\sigma+\lambda)(\lambda+r)[\lambda^2+(r+\sigma)\lambda+\sigma(r+1)] = 0.$$

Obviously, the eigenvalues of the system are

$$\lambda_1 = b, \quad \lambda_2 = -\sigma, \quad \lambda_3 = -r,$$

and

$$\lambda_{4,5} = \frac{-(\sigma+r)\pm\sqrt{(\sigma+r)^2-4\sigma(r+1)}}{2}.$$

Furthermore, $\mu\|A+BK\| > 1$; hence in view of Lemma 3.3, controlled system (2.7) is asymptotically stable at $p_0$.

## 4. Stabilizing $p_0$

In this section, we design a controller for fractional-order chaotic system (2.7) via fractional-order derivative. For this peropus we need the following results which can be found in [12]:

**Lemma 4.1.** *The fixed points of the following nonlinear commensurate fractional-order autonomous system:*

$$ {}^{c}D^{\mu}u = f(u),\ \mu \in (0,1), \tag{4.1}$$

*is asymptotically stable if all eigenvalues* $(\lambda)$ *of the Jacobian matrix evaluated at the fixed points satisfy* $|arg\lambda| > 0.5\pi\mu$, *where* $0 < \mu < 1, u \in R^n, f : R^n \to R^n$ *are continuous nonlinear vector functions, and the fixed points of this nonlinear commensurate fractional-order system are calculated by solving equation* $f(u) = 0$.

We impose the following results:

**Theorem 4.2.** *Assume the controlled fractional-order chaotic system*

$$\begin{aligned}
{}^{c}D^{\mu_1}u_1 &= -\sigma u_1 - \sigma u_3 + u_1u_5\\
{}^{c}D^{\mu_1}u_2 &= -\sigma u_2 - \sigma u_4 + u_2u_5\\
{}^{c}D^{\mu_2}u_3 &= u_1 - ru_3 + 5u_1u_5 + f_1(u_1)\\
{}^{c}D^{\mu_2}u_4 &= u_2 - ru_4 + 5u_2u_5\\
{}^{c}D^{\mu_3}u_5 &= bu_5 + 5(u_1u_3 + u_2u_4),
\end{aligned} \tag{4.2}$$

*where* $f_1(u_1) = -k_{11}\,{}^{c}D^{\mu_1}u_1 - k_{12}u_1$ *is the fractional-order controller, and* $k_{1i}$, $i = 1, 2$ *is the feedback coefficient. If* $b < 0, \sigma > 0$,

$$k_{11}\sigma < r, \quad \text{and} \quad k_{12} = 1 + k_{11}\sigma,$$

*then system* (4.2) *will asymptotically converge to the unstable equilibrium point* $p_0$.

**Proof.** The Jacobi matrix of the controlled fractional-order chaotic system (4.2) at $p_0$ is

$$J = \begin{pmatrix} -\sigma & 0 & -\sigma & 0 & 0\\ 0 & -\sigma & 0 & -\sigma & 0\\ 1 + k_{11}\sigma - k_{12} & 0 & k_{11}\sigma - r & 0 & 0\\ 0 & 1 & 0 & -r & 0\\ 0 & 0 & 0 & 0 & b \end{pmatrix}.$$

Since $k_{12} = 1 + k_{11}\sigma$, we have

$$J = \begin{pmatrix} -\sigma & 0 & -\sigma & 0 & 0\\ 0 & -\sigma & 0 & -\sigma & 0\\ 0 & 0 & k_{11}\sigma - r & 0 & 0\\ 0 & 1 & 0 & -r & 0\\ 0 & 0 & 0 & 0 & b \end{pmatrix}.$$

Thus the characteristic equation takes the form

$$(b-\lambda)(\sigma+\lambda)(k_{11}\sigma-r-\lambda)[\lambda^2+(r+\sigma)\lambda+\sigma(r+1)]=0$$

which leads to the eigenvalues

$$\lambda_1=b,\quad \lambda_2=-\sigma,\quad \lambda_3=k_{11}\sigma-r,\quad \lambda_{4,5}=\frac{-(\sigma+r)\pm\sqrt{(\sigma+r)^2-4\sigma(r+1)}}{2}$$

Since $b<0,\sigma>0$, and $k_{11}\sigma<r$ therefore,

$$|arg\lambda_i|>0.5\pi\mu,\quad i=1,...,5,$$

where $\mu:=\max\mu_j$, $j=1,2,3$. According to Lemma 4.1, it yields that the equilibrium point $p_0$ of system (2.7) is asymptotically stable, that is, the unstable equilibrium point $p_0$ in fractional order Lorenz system (2.7) can be stabilized via fractional-order derivative. The proof is completed.

In the same manner of Theorem 4.1, we have the asymptotically converge to the unstable equilibrium point $p_0$ for the system

$$\begin{aligned}
{}^cD^{\mu_1}u_1&=-\sigma u_1-\sigma u_3+u_1u_5\\
{}^cD^{\mu_1}u_2&=-\sigma u_2-\sigma u_4+u_2u_5\\
{}^cD^{\mu_2}u_3&=u_1-ru_3+5u_1u_5\\
{}^cD^{\mu_2}u_4&=u_2-ru_4+5u_2u_5+f_2(u_2)\\
{}^cD^{\mu_3}u_5&=bu_5+5(u_1u_3+u_2u_4),
\end{aligned}\tag{4.3}$$

where $f_2(u_2)=-k_{21}\,{}^cD^{\mu_1}u_2-k_{22}u_2$ is the fractional-order controller, and $k_{2i}$, $i=1,2$ is the feedback coefficient.

**Theorem 4.3.** *Assume the controlled fractional-order chaotic system*

$$\begin{aligned}
{}^cD^{\mu_1}u_1&=-\sigma u_1-\sigma u_3+u_1u_5+f_3(u_3)\\
{}^cD^{\mu_1}u_2&=-\sigma u_2-\sigma u_4+u_2u_5\\
{}^cD^{\mu_2}u_3&=u_1-ru_3+5u_1u_5\\
{}^cD^{\mu_2}u_4&=u_2-ru_4+5u_2u_5\\
{}^cD^{\mu_3}u_5&=bu_5+5(u_1u_3+u_2u_4),
\end{aligned}\tag{4.4}$$

*where $f_3(u_3)=-k_{31}\,{}^cD^{\mu_2}u_3-k_{32}u_3$ is the fractional-order controller, and $k_{2i}$, $i=1,2$ is the feedback coefficient. If $b<0$, $\sigma>0$, $r>0 k_{32}=-\sigma+rk_{31}$ and $k_{31}>0$ then system (4.4) will asymptotically converge to the unstable equilibrium point $p_0$.*

**Proof.** The Jacobi matrix of the controlled fractional-order chaotic system (4.4) at $p_0$ is

$$J=\begin{pmatrix}
-\sigma-k_{31} & 0 & -\sigma+rk_{31}-k_{32} & 0 & 0\\
0 & -\sigma & 0 & -\sigma & 0\\
1 & 0 & -r & 0 & 0\\
0 & 1 & 0 & -r & 0\\
0 & 0 & 0 & 0 & b
\end{pmatrix}.$$

Because $k_{32} = -\sigma + rk_{31}$, yields

$$J = \begin{pmatrix} -\sigma - k_{31} & 0 & 0 & 0 & 0 \\ 0 & -\sigma & 0 & -\sigma & 0 \\ 1 & 0 & -r & 0 & 0 \\ 0 & 1 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}.$$

the characteristic equation is

$$(b-\lambda)(\sigma + k_{31} + \lambda)(r+\lambda)[\lambda^2 + (r+\sigma)\lambda + \sigma(r+1)] = 0,$$

which implies the eigenvalues

$$\lambda_1 = b, \quad \lambda_2 = -r, \quad \lambda_3 = -\sigma - k_{31}, \quad \lambda_{4,5} = \frac{-(\sigma+r) \pm \sqrt{(\sigma+r)^2 - 4\sigma(r+1)}}{2}.$$

By the assumptions of the theorem and in virtue of Lemma 4.1, we obtain that system (4.4) will asymptotically converge to the unstable equilibrium point $p_0$.

Similarly for the system

$$\begin{aligned} {}^cD^{\mu_1}u_1 &= -\sigma u_1 - \sigma u_3 + u_1 u_5 \\ {}^cD^{\mu_1}u_2 &= -\sigma u_2 - \sigma u_4 + u_2 u_5 + f_4(u_4) \\ {}^cD^{\mu_2}u_3 &= u_1 - r u_3 + 5 u_1 u_5 \\ {}^cD^{\mu_2}u_4 &= u_2 - r u_4 + 5 u_2 u_5 \\ {}^cD^{\mu_3}u_5 &= b u_5 + 5(u_1 u_3 + u_2 u_4), \end{aligned} \tag{4.5}$$

where $f_4(u_4) = -k_{41}\,{}^cD^{\mu_2}u_4 - k_{42}u_4$ is the fractional-order controller, and $k_{2i}$, $i = 1, 2$ is the feedback coefficient.

## 5. Numerical solution

In this section, we discuss the numerical solution of fractional differential equations. All the numerical simulation of fractional-order system in this paper is based on [13]. Suppose the system of the form

$${}^cD^{\mu} = f\big(t, u(t), u(t-\tau)\big), \quad t \in [0, T],\ 0 < \mu \le 1 \tag{5.1}$$

with the initial condition

$$u(t) = \psi(t), \quad t \in [-\tau, 0].$$

Set

$$h = T/N,\ t_n = nh,\ n = -k, -k+1, ..., 0, 1, ..., N,$$

where $k$, and $N$ are integers such that $h = \tau/k$. Let

$$u_h(t_j) = \psi(t_j),\ j = -k, -k+1, ..., 0$$

and

$$u_h(t_j - \tau) = u_h(jh - kh) = u_h(t_{j-k}), \quad j = 0, 1, ..., N.$$

Our aim is to compute $u_h(t_{n+1})$

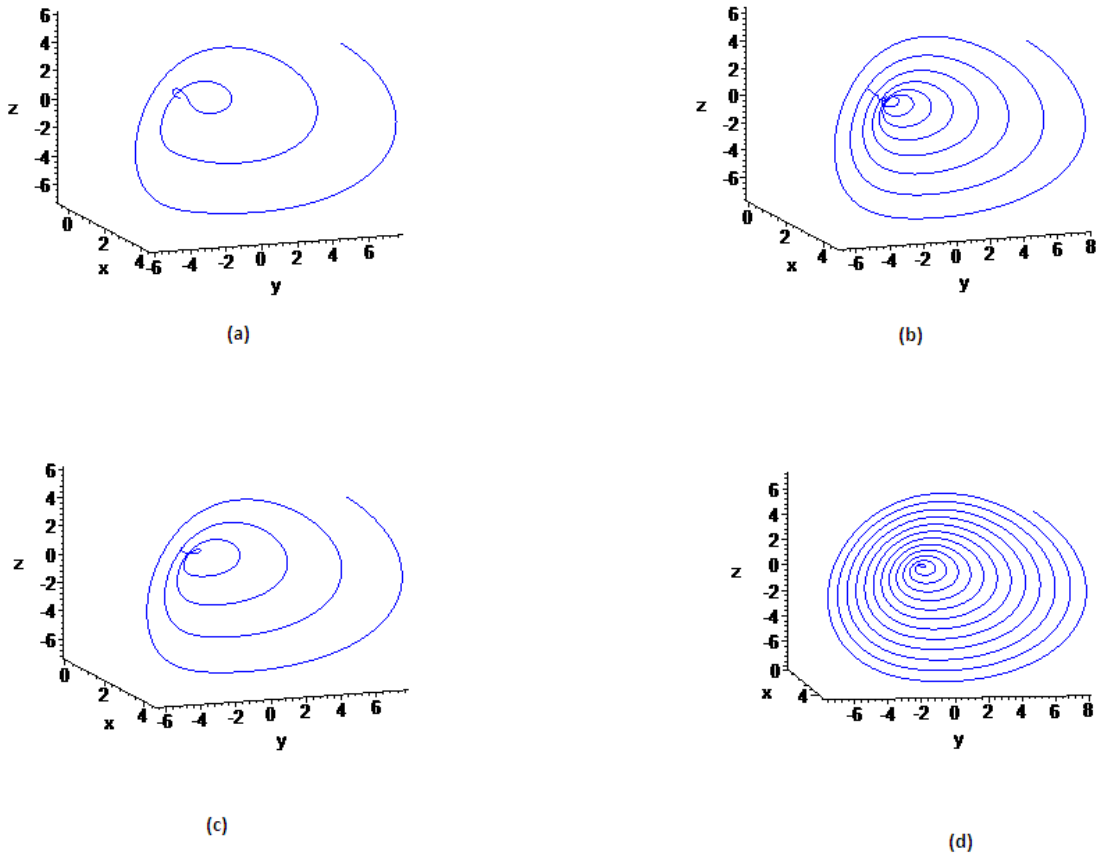



FIGURE 4. Asymptotically stable of system (2.4)

$$u(t_{n+1}) = \psi(0) + \frac{1}{\Gamma(\mu)} \int_0^{t_{n+1}} (t_{n+1} - \zeta)^{\mu-1} f(\zeta, u(\zeta), u(\zeta - \tau)) d\zeta. \tag{5.2}$$

By applying the product trapezoidal quadrature formula we have

$$\begin{aligned} u(t_{n+1}) &= \psi(0) + \frac{h^\mu}{\Gamma(\mu+2)} f(t_{n+1}, u_h(t_{n+1}), u_h(t_{n+1} - \tau)) \\ &\quad + \frac{h^\mu}{\Gamma(\mu+2)} \sum_{j=0}^{n} \alpha_{j,n+1} f(t_j, u_h({}_t j), u_h(t_j - \tau)) \\ &= \psi(0) + \frac{h^\mu}{\Gamma(\mu+2)} f(t_{n+1}, u_h(t_{n+1}), u_h(t_{n+1-k})) \\ &\quad + \frac{h^\mu}{\Gamma(\mu+2)} \sum_{j=0}^{n} \alpha_{j,n+1} f(t_j, u_h({}_t j), u_h(t_{j-k})) \end{aligned} \tag{5.3}$$

where

$$\alpha_{j,n+1} = \begin{cases} n^{\mu+1} - (n-\mu)(n+1)^\mu, & j = 0 \\ (n-j+2)^{\mu+1} + (n-j)^{\mu+1} - 2(n-j+1)^{\mu+1}, & 1 \le j \le n \\ 1, & j = n+1. \end{cases}$$

Now replace the term $u_h(t_{n+1})$ by the approximate predictor term $u_h^p(t_{n+1})$ such that

$$\begin{aligned} u_h^p(t_{n+1}) &= \psi(0) + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \beta_{j,n+1} f(t_j, u_h({}_t j), u_h(t_j - \tau)) \\ &= \psi(0) + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \beta_{j,n+1} f(t_j, u_h({}_t j), u_h(t_{j-k})) \end{aligned} \tag{5.4}$$

where

$$\beta_{j,n+1} = \frac{h^\mu}{\mu}[(n+1-j)^\mu - (n-j)^\mu].$$

With the initial condition $\big(u_1(0),...,u_5(0)\big)$, the system (2.7) can be described as

$$\begin{aligned}
u_1(n+1) &= u_1(0) + \frac{h^{\mu_1}}{\Gamma(\mu_1+2)}[-\sigma\big(u_1^p(n+1) + u_3^p(n+1) + u_1^p(n+1)u_5^p(n+1)\big) \\
&\quad + \sum_{j=0}^{n} \alpha_{1,j,n+1} \times \sigma\big(-u_1(j) - u_3(j) + u_1(j)u_5(j)\big)], \\
u_2(n+1) &= u_2(0) + \frac{h^{\mu_1}}{\Gamma(\mu_1+2)}[\sigma\big(-u_2^p(n+1) - u_4^p(n+1) + u_2^p(n+1)u_5^p(n+1)\big) \\
&\quad + \sum_{j=0}^{n} \alpha_{2,j,n+1} \times \sigma\big(-u_2(j) - u_4(j) + u_2(j)u_5(j)\big)], \\
u_3(n+1) &= u_3(0) + \frac{h^{\mu_2}}{\Gamma(\mu_2+2)}[\big(u_1^p(n+1) - ru_3^p(n+1) + 5u_1^p(n+1)u_5^p(n+1)\big) \\
&\quad + \sum_{j=0}^{n} \alpha_{3,j,n+1}\big(u_1(j) - ru_3^p(n+1) + 5u_1(j)u_5(j)\big)] \\
u_4(n+1) &= u_4(0) + \frac{h^{\mu_2}}{\Gamma(\mu_2+2)}[\big(u_2^p(n+1) - ru_4^p(n+1) + 5u_2^p(n+1)u_5^p(n+1)\big) \\
&\quad + \sum_{j=0}^{n} \alpha_{4,j,n+1}\big(u_2(j) - ru_4(j) + 5u_2(j)u_5(j)\big)] \\
u_5(n+1) &= u_5(0) + \frac{h^{\mu_3}}{\Gamma(\mu_3+2)}[bu_5^p(n+1) + 5\big(u_1^p(n+1)u_3^p(n+1) + u_2^p(n+1)u_4^p(n+1)\big) \\
&\quad + \sum_{j=0}^{n} \alpha_{5,j,n+1}\big(bu_5(j) + 5(u_1(j)u_3(j) + u_2(j)u_4(j))\big)],
\end{aligned}$$

where

$$\begin{aligned}
u_1^p(n+1) &= u_1(0) + \sum_{j=0}^{n} \beta_{1,j,n+1} \times \big(-\sigma u_1(j) - \sigma u_3(j) + u_1(j)u_5(j)\big) \\
u_2^p(n+1) &= u_2(0) + \sum_{j=0}^{n} \beta_{2,j,n+1} \times \big(-\sigma u_2(j) - \sigma u_4(j) + u_2(j)u_5(j)\big) \\
u_3^p(n+1) &= u_3(0) + \sum_{j=0}^{n} \beta_{3,j,n+1} \times \big(u_1(j) - ru_3(j) + 5u_1(j)u_5(j)\big) \\
u_4^p(n+1) &= u_4(0) + \sum_{j=0}^{n} \beta_{4,j,n+1} \times \big(u_2(j) - ru_4(j) + 5u_2(j)u_5(j)\big) \\
u_5^p(n+1) &= u_5(0) + \sum_{j=0}^{n} \beta_{5,j,n+1} \times \big(bu_5(j) + 5(u_1(j)u_3(j) + u_2(j)u_4(j))\big).
\end{aligned}$$

The error of this approximation can be computed as follows:

$$|u_i(t_n) - u_i(n)| = o(h^{\mu}), \quad p = min(2, 1 + \max \mu_{1,2,3}).$$

## 6. Existence and uniqueness

In this section we shall discus the existence and uniqueness of system (2.7). Let $\xi_j$, $j = 1, 2, 3$ be in the unit disk $U := \{z \in \mathbb{C} : |z| \leq 1\}$ and $t \in J = [0, T]$. Moreover, we assume that $u_1, ..., u_5$ are continuous in $J$; we denote this set by $C(J)$. Let $\mathcal{S} = \{(u_1, ..., u_5)^{\mathcal{T}} : u_1, ..., u_5 \in C(J)\}$ with the norm

$$\|\xi_1\|_{\mathcal{S}} = \|u_1\| + \|u_2\| = \sup_{t\in J} |u_1(t)| + \sup_{t\in J} |u_2(t)|,$$

$$\|\xi_2\|_{\mathcal{S}} = \|u_3\| + \|u_4\| = \sup_{t\in J} |u_3(t)| + \sup_{t\in J} |u_4(t)|$$

and

$$\|\xi_3\|_{\mathcal{S}} = \|u_5\| = \sup_{t\in J} |u_5(t)|.$$

**Theorem 6.1.** *Let $|\sigma| \geq 1$, $|r| \geq 1$ in system (2.7) withe the initial condition $\big(u_1(0), ..., u_5(0)\big)^{\mathcal{T}}$. If*

$$\frac{|\sigma|T^{\mu_1}}{\Gamma(\mu_1 + 1)} + \frac{(1 + |r|)T^{\mu_2}}{\Gamma(\mu_2 + 1)} + \frac{(1 + |b|)T^{\mu_3}}{\Gamma(\mu_3 + 1)} < \frac{1}{10}, \quad T \in J$$

$$(T \in J, 0 < \mu_j \leq 1,\ j = 1, 2, 3)$$

*then (2.7) has a unique solution in $\mathcal{S}$.*

**Proof.** The system (2.7) can be reduced to the matrix form

$$\begin{aligned}
\Big({}^{c}D^{\mu_1}u_1, {}^{c}D^{\mu_1}u_2\Big)^{\mathcal{T}} &= \Big(-\sigma u_1 - \sigma u_3 + u_1u_5,\ -\sigma u_2 - \sigma u_4 + u_2u_5\Big)^{\mathcal{T}} \\
\Big({}^{c}D^{\mu_2}u_3, {}^{c}D^{\mu_2}u_4\Big)^{\mathcal{T}} &= \Big(u_1 - ru_3 + 5u_1u_5,\ u_2 - ru_4 + 5u_2u_5\Big)^{\mathcal{T}} \\
{}^{c}D^{\mu_3}u_5 &= bu_5 + 5(u_1u_3 + u_2u_4).
\end{aligned} \tag{6.1}$$

Operating (6.1) by $I^{\mu_1}, I^{\mu_2}$ and $I^{\mu_3}$ respectively yields

$$\begin{aligned}
\Big(u_1, u_2\Big)^{\mathcal{T}} &= \Big(u_1(0) - \sigma I^{\mu_1}u_1 - \sigma I^{\mu_1}u_3 + I^{\mu_1}u_1u_5,\ u_2(0) - \sigma I^{\mu_1}u_2 - \sigma I^{\mu_1}u_4 + I^{\mu_1}u_2u_5\Big)^{\mathcal{T}} \\
\Big(u_3,\ u_4\Big)^{\mathcal{T}} &= \Big(u_3(0) + I^{\mu_2}u_1 - rI^{\mu_2}u_3 + 5I^{\mu_2}u_1u_5, u_4(0) + I^{\mu_2}u_2 - rI^{\mu_2}u_4 + 5I^{\mu_2}u_2u_5\Big)^{\mathcal{T}} \\
u_5 &= u_5(0) + bI^{\mu_3}u_5 + 5I^{\mu_3}(u_1u_3 + u_2u_4).
\end{aligned} \tag{6.2}$$

Define the operator $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ by

$$\begin{aligned}\mathcal{P}(u_1,...,u_5)^{\mathcal{T}} &= (u_1(0),...,u_5(0))^{\mathcal{T}} + \Big( -\sigma I^{\mu_1}u_1 - \sigma I^{\mu_1}u_3 \\ &\quad + I^{\mu_1}u_1u_5,\ -\sigma I^{\mu_1}u_2 - \sigma I^{\mu_1}u_4 + I^{\mu_1}u_2u_4, \\ &\quad I^{\mu_2}u_1 - rI^{\mu_2}u_3 + 5I^{\mu_2}u_1u_5, I^{\mu_2}u_2 - rI^{\mu_2}u_4 + 5I^{\mu_2}u_2u_5, \\ &\quad bI^{\mu_3}u_5 + 5I^{\mu_3}(u_1u_3 + u_2u_4)\Big)^{\mathcal{T}}\end{aligned} \tag{6.3}$$

Since $|u_i| < 1$, thus a computation implies

$$\begin{aligned}&\|\mathcal{P}(u_1,...,u_5)^{\mathcal{T}} - \mathcal{P}(v_1,...,v_5)^{\mathcal{T}}\|_{\mathcal{S}} \\ &= \Big\| \Big( -\sigma I^{\mu_1}(u_1 - v_1) - \sigma I^{\mu_1}(u_3 - v_3) + I^{\mu_1}(u_1u_5 - v_1v_5), \\ &- \sigma I^{\mu_1}(u_2 - v_2) - \sigma I^{\mu_1}(u_4 - v_4) + I^{\mu_1}(u_2u_4 - v_2v_4), \\ &I^{\mu_2}(u_1 - v_1) - rI^{\mu_2}(u_3 - v_3) + 5I^{\mu_2}(u_1u_5 - v_1v_5), \\ &I^{\mu_2}(u_2 - v_2) - rI^{\mu_2}(u_4 - v_4) + 5I^{\mu_2}(u_2u_5 - v_2v_5), \\ &bI^{\mu_3}(u_5 - v_5) + 5I^{\mu_3}[(u_1u_3 + u_2u_4) - (v_1v_3 + v_2v_4)]\Big)^{\mathcal{T}} \Big\|_{\mathcal{S}} \\ &\leq 10[\frac{|\sigma|T^{\mu_1}}{\Gamma(\mu_1+1)} + \frac{(1+|r|)T^{\mu_2}}{\Gamma(\mu_2+1)} + \frac{(1+|b|)T^{\mu_3}}{\Gamma(\mu_3+1)}]\times \\ &\qquad \|(u_1,...,u_5)^{\mathcal{T}} - (v_1,...,v_5)^{\mathcal{T}}\|_{\mathcal{S}}\end{aligned}$$

for $|v_i| < 1$. Hence by the contraction fixed point theorem, the problem (2.7) has a unique solution in $\mathcal{S}$.

## 7. Conclusion and discussion

We considered the complex fractional system of five equations. The fractional system is taken for different values of $\mu_i$, $i = 1, 2, 3$ where $\mu_1 = 0.996$, $\mu_2 = 0.997$, $\mu_3 = 0.998$. We modified numerical method for solving this system based on work due to Matignon. This system was taken the Newton-Leipnik type systems ( Fig. 1). Fig. 2 shows the propose system with different values of its parameters where (a-d) has the value $(\sigma, r, b)$=(0.5,0.5,0.5), (0.5,0.5,0.5), (1,0.5, 0.5) respectively with the initial valued (4,5,6) while Fig.3 shows the system for different initial values. The asymptotic stability is showed in Fig.4 with the initial valued (4,5,6) for some $b < 0$ and $\sigma > r > 0$. The stability and stabilization of such a system are studied at the equilibrium point $p_0 = (0, 0, 0, 0, 0)$. The technique based on stability theory of fractional-order systems. We consider a controller for fractional-order chaotic system (2.7) via fractional-order derivative based on the method due to Diethelm and Ford. The existence and uniqueness are established in Section 6, by imposing the sufficient condition on the coefficients of the system. The complex variables are taken in the unit disk. From above, we conclude that the fractional-order systems of complex variable and we pointed that they are more suitable than integer-order ones in biological, economic, and social systems. These complex systems, of five equations, are applied

to characterize the physics of a laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators.

## References


1. Long-Jye Sheu et all., *Chaos in the Newton-Leipnik system with fractional order,* Chaos, Solitons and Fractals **36** (2008), 98-103.
2. S. Panchey, N. K. Vitanov, *On asymptotic properties of some complex Lorenz-like systems,* Journal of the Calcutta Mathematical Society, **1**(2005), 121-130.
3. Lu Jun-Jie, L. Chong-Xin, *Realization of fractional-order Liu chaotic system by circuit,* Chinese Phys. **16** (2007), 1586.
4. X. Wang, M. Wang, *Dynamic analysis of the fractional-order Liu system and its synchronization,* Chaos **17** (2007), 1-6.
5. X. Gao, *Chaotic Dynamics of Fractional-Order Liu System,* Applied Mechanics and Materials, **1327** (2011), 55-57.
6. L.Chen, Y. Chai, R. Wu, J. Yang, *Stability and stabilization of a class of nonlinear fractional order system with Caputo derivative,* IEEE Transaction on Circuits and Systems (2012) 1-5.
7. A. M. A. El-Sayed, E. Ahmed, and H. A. A. El-Saka, *Dynamic properties of the fractional-order logistic equation of complex variables,* Abstract and Applied Analysis Volume 2012, Article ID 251715, 12 pages doi:10.1155/2012/251715.
8. H. Haken, *Analogy between higher instabilities in fluids and lasers,* Phys. Lett. A **53** (1975), 77-88.
9. I. Podlubny, Fractional Differential Equations, Acad. Press, London, 1999.
10. A.A. Kilbas, H. M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V, Amsterdam, 2006.
11. L.Chen, Y. Chai, R. Wu, J. Yang, *Stability and stabilization of a class of nonlinear fractional order system with Caputo derivative,* IEEE Transaction on Circuits and Systems (2012), 1-5.
12. K. Diethelm and N. J. Ford, *Analysis of fractional differential equations,* Journal of Mathematical Analysis and Applications, **265** (2002), 229-248.
13. D. Matignon, *Stability result on fractional differential equations with applications to control processing,* in Proceedings of the (IMACS-SMC 96), **2**(1996), 963-968.


[1] Institute of Mathematical Sciences, University Malaya, 50603, Malaysia.
*E-mail address*: rabhaibrahim@yahoo.com