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HYDROSTATIC APPROXIMATION OF THE 3D COMPRESSIBLE PRIMITIVE EQUATIONS

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Abstract. In this paper, we obtain the 3D compressible primitive equations approximation with gravity by taking the small aspect ratio limit to the Navier-Stokes equations. We use the versatile relative entropy inequality to prove rigorously the limit from the compressible Navier-Stokes equations with a pressure law of the form $p(\rho) = \rho^2$ to the compressible primitive equations.

1. Introduction

The motion and state of the atmosphere, which is a specific compressible fluid, are modeled using the general hydrodynamic and thermodynamic equations that incorporate Coriolis force and gravity. However, these equations are highly complex and computationally demanding. To address this, the authors in [3] take advantage of the significant difference in vertical and horizontal scales in the atmosphere. They derive the compressible primitive equations (CPE) from the compressible Navier-Stokes equations, exploiting the small aspect ratio between these two directions, as in planetary-scale geophysical models. In the CPE, the hydrostatic balance equation replaces the vertical momentum component, also known as the quasi-static equilibrium equation. This hydrostatic approximation proves to be accurate enough for practical applications and has become a fundamental equation in atmospheric science. It is reliable and useful because it captures the dominance of gravity and pressure in the vertical direction, which is challenging to observe in reality. Further insights from a meteorological standpoint can be found in [18].

We note that atmospheric flow in meteorology, water flow in oceanography, and limnology are all described by the Navier-Stokes equations. Large-scale geophysical motions are constrained by an aspect ratio ε where the depth is much smaller than the horizontal width, defined by

$$\varepsilon = \frac{\text{characteristic depth}}{\text{characteristic width}}.$$

Since ε is very small in most geophysical domains, asymptotic models have been used see, e.g., [2, 12]. One such model is the primitive equations model; see, e.g.,

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[4, 5], wherein the unknown flow variables are velocity, pressure, temperature, and salinity (in the case of an ocean). Besides, most geophysical fluids are stratified (i.e., density is a known function of the temperature (and salinity, if any)) and have a free surface.
We consider this compressible anisotropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho U) = 0, \\ \partial_t(\rho U) + \ \mathrm{div}(\rho U \otimes U) + \nabla p(\rho) - \mu_1 \Delta_x U - \mu_2 \partial_{zz} U = \rho f, \\ p(\rho) = \rho^2 \end{cases} \tag{1.1}$$

in the thin domain $(0, T) \times \Omega_\varepsilon$. Here $t > 0$, $x = (x_1, x_2)$ and z are time, horizontal and vertical variables respectively, $\Omega_\varepsilon = \{(x, z); x \in \mathbb{T}^2, 0 < z < \varepsilon\}$ with $\mathbb{T}^2$ a bi-dimensional torus. The unknown functions ρ, $U = (u, v)$ and p represent the density, velocities, and pressure of the medium, respectively. $\mathrm{div} U = \mathrm{div}_x u + \partial_z v$ (with $\mathrm{div}_x = \partial_{x_1} + \partial_{x_2}$) is divergence operator and $\nabla = (\nabla_x, \partial_z)$ the gradient. (μ_1, μ_2) are viscosities in the horizontal and vertical directions respectively, which are supposed to be constant in this work. The term f is the gravity strength given as follows:

$$f = \tfrac{1}{\varepsilon} g \kappa,$$

where g is the gravitational constant and $\kappa = (0, 0, 1)^T$ (X^T stands for the transpose of tensor X).
Pedlosky [17] highlighted that in the ocean and atmosphere, which are thin layers compared to the radius of the sphere, the pressure variance between two points on a vertical line is solely determined by the fluid weight between those points.
We invite readers to look at the paper [9] and the references mentioned for more literature on CNS, PE, and CPE.
We suppose $\mu_1 = 1$ and $\mu_2 = \varepsilon^2$. According to Azérad and Guillén [1], it is crucial to take into account the scaling of anisotropic viscosities mentioned above, as it plays a fundamental role in deriving the Primitive Equations (PE). Based on this assumption, the system is reconfigured as follows

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) + \partial_z(\rho v) = 0, \\ \partial_t(\rho u) + \ \mathrm{div}_x(\rho u \otimes u) + \partial_z(\rho u v) + \nabla_x p(\rho) = \Delta_x u + \varepsilon^2 \partial_{zz} u, \\ \partial_t(\rho v) + \ \mathrm{div}_x(\rho u v) + \partial_z(\rho v^2) + \partial_z p(\rho) - \Delta_x v - \varepsilon^2 \partial_{zz} v = \tfrac{1}{\varepsilon} g \rho, \\ p(\rho) = \rho^2. \end{cases} \tag{1.2}$$

We perform a vertical scaling to make the domain-independent of ε, that is, $t = \tilde{t}$, $x = \tilde{x}$ and $z = \varepsilon \tilde{z}$. Omitting the symbol "*tilde*", the new fixed domain is written $\Omega = \{(x, z); x \in \mathbb{T}^2, 0 < z < 1\}$.
As in [11], the corresponding kinematic scaling is,

$$U_\varepsilon = (u_\varepsilon, v_\varepsilon),\ u_\varepsilon(t, x, z) = u(t, x, \varepsilon z),\ v_\varepsilon(t, x, z) = \tfrac{1}{\varepsilon} v(t, x, \varepsilon z),\ p_\varepsilon(t, x, z) = p(t, x, \varepsilon z),$$

for any $(x, z) \in \Omega := \mathbb{T}^2 \times (0, 1)$. Then, the system (1.2) becomes the following compressible scaled Navier-Stokes equations (CNS):

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon) + \partial_z(\rho_\varepsilon v_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \partial_z(\rho_\varepsilon u_\varepsilon v_\varepsilon) + \nabla_x p(\rho_\varepsilon) = \Delta_x u_\varepsilon + \partial_{zz} u_\varepsilon, \\ \varepsilon^2 \Big(\partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon v_\varepsilon^2) - \Delta_x v_\varepsilon - \partial_{zz} v_\varepsilon \Big) + \partial_z p(\rho_\varepsilon) = g\rho_\varepsilon, \\ p(\rho_\varepsilon) = \rho_\varepsilon^2. \end{cases} \tag{1.3}$$

We make the following boundary conditions:

$$\begin{cases} u_\varepsilon \text{ and } \rho_\varepsilon \text{ are periodic in the directions } x_1,\, x_2 \text{ respectively}, \\ v_\varepsilon|_{z=0} = v_\varepsilon|_{z=1} = 0, \\ \partial_z u_\varepsilon|_{z=0} = \partial_z u_\varepsilon|_{z=1} = 0. \end{cases} \tag{1.4}$$

The initial conditions are

$$\rho_\varepsilon(0,x,z)\, U_\varepsilon(0,x,z) = m_0(x,z), \qquad \rho_\varepsilon(0,x,z) = \rho_0(x,z). \tag{1.5}$$

In this work, our goal is to prove that as $\varepsilon \to 0$, the system (1.3) converges in a certain sense to the following compressible primitive equations (CPE):

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) + \partial_z(\rho v) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \partial_z(\rho u v) + \nabla_x p(\rho) = \Delta_x u + \partial_{zz} u, \\ \partial_z p(\rho) = g\rho, \\ p(\rho) = \rho^2. \end{cases} \tag{1.6}$$

This is given in Theorem 3.4 which announces the main result of this study, the convergence of a weak solution of the CNS (1.3) to the unique strong solution of the CPE (1.6).
According to [13], this limit problem (1.6), by the change of variable $\rho(t,x,z) = \xi(t,x) + \varphi(z)$ with $\varphi(z) = \frac{1}{2} gz$ and $\xi(t,x)$ an unknown function, is equivalent to

$$\begin{cases} \partial_t \xi + u \cdot \nabla_x \xi + (\xi + \varphi) \operatorname{div}_x(u) + \partial_z(\xi v + \varphi v) = 0, \\ (\xi + \varphi)\left(\partial_t u + u \cdot \nabla_x u + v \partial_z u\right) + 2(\xi + \varphi)\nabla_x \xi = \Delta_x u + \partial_{zz} u, \\ \partial_z \xi = 0. \end{cases} \tag{1.7}$$

Thus, they obtain

$$\partial_z(\rho v) = -\widetilde{u} \cdot \nabla_x \xi - \xi \operatorname{div}_x \widetilde{u} - \tfrac{g}{2} \widetilde{z \operatorname{div}_x u}, \tag{1.8}$$

where $\widetilde{f} = f - \bar{f}$ with $\bar{f} = \int_0^1 f dz$.
Therefore, the vertical velocity v is determined, thanks to the boundary condition (1.4), by the relation

$$\rho v(t,x,z) = -\int_0^z \Big(\operatorname{div}_x(\xi \widetilde{u}) + \tfrac{1}{2} g \widetilde{z \operatorname{div}_x u} \Big) dz. \tag{1.9}$$

The following sections of this document are structured as follows:

Section 2 : This section provides a review of essential inequalities that will be used throughout the paper.

Section 3 : We introduce the concepts of weak solutions, strong solutions, and relative energy and establish the relative entropy inequality. The main theorem of the paper is also stated in this section.

Section 4 : This section focuses on proving convergence results, and establishing the core findings of the paper.

2. Preliminaries

This section lays the groundwork for subsequent proofs by presenting fundamental inequalities. We begin with the generalized Poincaré inequality, a key tool that will be employed throughout our analysis.

Lemma 2.1. *([6], Lemma 3.2) Let $2 \leq q \leq 6$, and $\rho > 0$ such that $0 < \int_\Omega \rho \, dxdz = m \leq \infty$ and $\int_\Omega \rho^\delta \, dxdz \leq \theta$, with $\delta > 1$, then*

$$\|h\|_{L^q(\Omega)} \leq C\|\nabla h\|_{L^2(\Omega)} + \|\sqrt{\rho} h\|_{L^2(\Omega)},$$

where $C = C(m, \theta)$.

The following is the famous Gagliardo-Nirenberg inequality (for the proof, [16]).

Lemma 2.2. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a function defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $1 \leq q, r \leq \infty$, and m a natural number. Suppose that a real number θ and a natural number j are such that*

$$\frac{1}{p} = \frac{j}{n} + (\frac{1}{r} - \frac{m}{n})\theta + \frac{1-\theta}{q} \text{ and } \frac{j}{m} \leq \theta \leq 1,$$

then there exists constant C independent of u such that

$$\|D^j u\|_p \leq C\|D^m u\|^\theta_{L^r(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}.$$

3. Main result

Before showing our main result, we give the definition of a weak solution for CNS and a strong solution for CPE.

3.1. Weak solutions of CNS.

Here, we give firstly the definition of weak solutions to (1.3) and second, give the energy inequality.

Definition 3.1. We say that $(\rho_\varepsilon, u_\varepsilon, v_\varepsilon)$ is a finite energy weak solution to the system (1.3), with boundary conditions (1.4) and initial conditions (1.5) if

- we have the following regularities

$$\begin{cases} \rho_\varepsilon^2 \in L^\infty\big(0,T; L^1(\Omega)\big), & \rho_\varepsilon \in C\big(0,T; L^1(\Omega)\big), \\ \sqrt{\rho_\varepsilon}\, v_\varepsilon \in L^\infty\big(0,T; L^2(\Omega)\big), & \sqrt{\rho_\varepsilon}\, u_\varepsilon \in L^\infty\big(0,T; L^2(\Omega)^2\big), \\ v_\varepsilon \in L^2\big(0,T; H^1(\Omega)\big), & u_\varepsilon \in L^2\big(0,T; H^1(\Omega)^2\big), \end{cases} \tag{3.1}$$

- the continuity equation

$$\left[\int_\Omega \rho_\varepsilon\, \varphi\, dxdz\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left(\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon u_\varepsilon \cdot \nabla_x \varphi + \rho_\varepsilon v_\varepsilon \partial_z \varphi\right) dxdzdt, \tag{3.2}$$

holds for all $\varphi \in C_c^\infty([0,T)\times\Omega)$;

- the momentum equations

$$\begin{aligned}&\left[\int_\Omega \rho_\varepsilon u_\varepsilon\, \phi_H\, dxdz\right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \rho_\varepsilon u_\varepsilon \partial_t \phi_H\, dxdzdt - \int_0^\tau \int_\Omega \rho_\varepsilon u_\varepsilon v_\varepsilon \partial_z \phi_H\, dxdzdt\\ &+ \int_0^\tau \int_\Omega (\nabla_x u_\varepsilon - \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) : \nabla_x \phi_H\, dxdzdt + \int_0^\tau \int_\Omega \partial_z u_\varepsilon \partial_z \phi_H\, dxdzdt\\ &- \int_0^\tau \int_\Omega p(\rho_\varepsilon)\ \mathrm{div}_x \phi_H\, dxdzdt = 0\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\varepsilon^2 \left[\int_\Omega \rho_\varepsilon v_\varepsilon\, \phi_3\, dxdz\right]_{t=0}^{t=\tau} - \varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon v_\varepsilon \partial_t \phi_3\, dxdzdt - \varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3\, dxdzdt\\ &+\varepsilon^2 \int_0^\tau \int_\Omega \nabla_x v_\varepsilon : \nabla_x \phi_3\, dxdzdt + \varepsilon^2 \int_0^\tau \int_\Omega \left(\partial_z v_\varepsilon - \rho_\varepsilon v_\varepsilon^2\right)\partial_z \phi_3\, dxdzdt\\ &- \int_0^\tau \int_\Omega p(\rho_\varepsilon)\partial_z \phi_3\, dxdzdt = \int_0^\tau \int_\Omega g\rho_\varepsilon \phi_3\, dxdzdt,\end{aligned} \tag{3.4}$$

hold for all $\phi_H \in C_c^\infty([0,T)\times\Omega)^2$, $\phi_3 \in C_c^\infty([0,T)\times\Omega)$ and for a.a $\tau \in (0,T)$.

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}&\left[\int_\Omega \rho_\varepsilon \left(u_\varepsilon\, \phi_H + \varepsilon^2 v_\varepsilon\, \phi_3\right)\, dxdz\right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \rho_\varepsilon \left(u_\varepsilon \partial_t \phi_H + \varepsilon^2 v_\varepsilon \partial_t \phi_3\right)\, dxdzdt\\ &- \int_0^\tau \int_\Omega \rho_\varepsilon \left(u_\varepsilon v_\varepsilon \partial_z \phi_H + \varepsilon^2 v_\varepsilon^2 \partial_z \phi_3\right)\, dxdzdt + \int_0^\tau \int_\Omega \left(\nabla_x u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 \nabla_x v_\varepsilon : \nabla_x \phi_3\right) dxdzdt\\ &- \int_0^\tau \int_\Omega \rho_\varepsilon \left(u_\varepsilon \otimes u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3\, dxdzdt\right) + \int_0^\tau \int_\Omega \left(\partial_z u_\varepsilon \partial_z \phi_H + \varepsilon^2 \partial_z v_\varepsilon \partial_z \phi_3\right) dxdzdt\\ &- \int_0^\tau \int_\Omega p(\rho_\varepsilon)\left(\ \mathrm{div}_x \phi_H + \partial_z \phi_3\right)\, dxdzdt = \int_0^\tau \int_\Omega g\rho_\varepsilon \phi_3\, dxdzdt.\end{aligned} \tag{3.5}$$

- the energy inequality

$$\begin{aligned}&\left[\int_\Omega \left[\tfrac{1}{2}\rho_\varepsilon \left(|u_\varepsilon|^2 + \varepsilon^2 v_\varepsilon^2\right) + P(\rho_\varepsilon)\right] dxdz\right]_0^\tau\\ &\qquad + \int_0^\tau \int_\Omega \left(|\nabla_x u_\varepsilon|^2 + |\partial_z u_\varepsilon|^2\right) dxdzdt + \varepsilon^2 \int_0^\tau \int_\Omega \left(|\nabla_x v_\varepsilon|^2 + |\partial_z v_\varepsilon|^2\right) dxdzdt\\ &\leq \int_0^\tau \int_\Omega g\rho_\varepsilon v_\varepsilon\, dxdzdt\\ &\leq \frac{g}{2}\left(\int_0^\tau \int_\Omega \rho_\varepsilon dxdzdt + \int_0^\tau \int_\Omega \rho_\varepsilon v_\varepsilon^2 dxdzdt\right),\end{aligned} \tag{3.6}$$

holds, where $P(\rho)=\rho\int_1^{\rho}\frac{p(y)}{y^2}dy=\rho^2-\rho$.

We assume that the initial data satisfy

$$\begin{cases} 0<\underline{\rho}\le\rho_0=\xi_0+\frac{1}{2}gz\le M<+\infty, \\ u_0\in H^2(\Omega), \quad \rho_0\in H^2(\Omega), \quad \partial_z u_0|_{z=0,1}=0, \\ \frac{|m_0|^2}{\rho_0}\in L^1(\Omega). \end{cases} \tag{3.7}$$

3.2. Strong solution of CPE.

Xin Liu and Edriss S. Titi demonstrated in [13] that the CPE system (1.6), with boundary conditions (1.4) and initial conditions (3.7), has a unique strong solution in $\Omega\times(0,T)$. This solution depends continuously on the initial data. From their theorem 1, it follows that the couple solution (r,ϕ), (where $\phi=(\phi_H=(\phi_{1H},\phi_{2H}),\phi_3)$) satisfies following regularities

$$\begin{cases} r\in L^\infty\left(0,T;H^2(\Omega)\right),\ \partial_t r\in L^\infty\left(0,T;H^1(\Omega)\right),\ r>0 \text{ for all } (t,x,z), \\ \phi_H\in L^\infty\left(0,T;H^2(\Omega)\right)\cap L^2\left(0,T;H^3(\Omega)\right), \\ \partial_t\phi_H\in L^2\left(0,T;H^1(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Omega)\right). \end{cases}$$

3.3. Relative entropy inequality.

Motivated by [7, 8] and [10, 15], for any finite energy weak solution (U,ρ), where $U=(u,v)$, to the CNS system (1.3), we introduce the relative energy functional

$$\begin{aligned} E\Big([\rho,U]|[r,\phi]\Big) &= \int_\Omega\Big(\frac{1}{2}\rho|u-\phi_H|^2+\frac{\varepsilon^2}{2}\rho|v-\phi_3|^2+P(\rho)-P(r)-P'(r)(\rho-r)\Big)dxdz \\ &= \int_\Omega\Big(\frac{1}{2}\rho|u|^2+\frac{\varepsilon^2}{2}\rho|v|^2+P(\rho)\Big)dxdz-\int_\Omega\Big(\rho u\phi_H+\varepsilon^2\rho v\phi_3\Big)dxdz \\ &\quad+\int_\Omega\rho\Big(\frac{1}{2}|\phi_H|^2+\frac{\varepsilon^2}{2}|\phi_3|^2-P'(r)\Big)dxdz+\int_\Omega p(r)dxdz. \end{aligned} \tag{3.8}$$

$(r,\phi)=(r,\phi_H,\phi_3)$ designates the strong solution of problem (1.6) showed in [13]. r is a strictly positive function define on $\Omega\times(0,T)$.

From [15], we derive the following lemma.

Lemma 3.2. *A general function $G=G(\rho)$ can be decomposed into an essential and residual part, namely,*

$$G=G_{ess}+G_{res},$$

where

$$G_{ess}:=\begin{cases} G & \text{on} \quad \rho\in(\frac{r}{2},2r), \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 < a_1 < a_2 < \infty$. *Then there exists* $C = C(a_1, a_2) > 0$ *such that for all* $\rho \in [0, \infty)$ *and* $r \in [a_1, a_2]$,

$$E\Big([\rho, U]|[r, \phi]\Big) \geq C \int_{\Omega} \left(\rho|u - \phi_H|^2 + \varepsilon^2\rho|v - \phi_3|^2 + |\rho - r|^2_{ess} + 1_{res} + \rho^2_{res}\right) dxdz \tag{3.9}$$

Thus, we deduce from (3.9) that

$$\begin{cases} \int_{\Omega} |\rho - r|^2_{ess} dxdz = \int_{\Omega} \chi_{\{\frac{r}{2} < \rho < 2r\}}(\rho - r)^2 dxdz \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)(t), \\ \int_{\Omega} 1_{res} dxdz \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)(t), \\ \int_{\Omega} \rho^2_{res} dxdz \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)(t), \\ \int_{\Omega} \rho\left(|u - \phi_H|^2 + \varepsilon^2|v - \phi_3|^2\right) dxdz \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)(t). \end{cases} \tag{3.10}$$

Moreover, from [7] subsection 4.1, we have

$$\begin{cases} E\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)(t) \in L^\infty(0, T), \quad \|\rho\|_{L^\alpha(\{\rho \geq 2r\})} \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)^{1/\alpha}, \\ \|\rho^{\alpha/2}\|_{L^2(\{\rho \geq 2r\})} \leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big)^{1/2}, \quad \alpha > 1. \end{cases} \tag{3.11}$$

The symbol C denotes a generic positive constant, which may vary from time to time.
Thanks to the relative entropy, we derive this lemma.

Lemma 3.3. *Let* $0 < a_1 < a_2 < \infty$. *We can find* $c = c(a_1, a_2) > 0$ *such that for all* $\rho \in [0, \infty)$ *and* $r \in [a_1, a_2]$, *we have*

$$P(\rho) - P(r) - (\rho - r)P'(r) > c(\rho - r)^2, \quad \textit{wherever } \rho \in (\tfrac{r}{2}, 2r) \tag{3.12}$$

and

$$P(\rho) - P(r) - (\rho - r)P'(r) > c(\rho^2 + 1), \quad \textit{wherever } \rho \in \mathbb{R}^+ \backslash (\tfrac{r}{2}, 2r). \tag{3.13}$$

Our main result is the following.

Theorem 3.4. *Let* $(\rho_\varepsilon, u_\varepsilon, v_\varepsilon)$ *be a sequence of dissipative weak solutions to the CNS system* (1.3) *from the initial data* (ρ_0, u_0, v_0) *depending on* ε *and satisfies* (3.7). *Suppose that*

$$E\Big([\rho_0, U_0]|[r_0, \phi_0]\Big) \longrightarrow 0,$$

then

$$ess \sup_{t \in (0,T)} E\Big([\rho_\varepsilon, U_\varepsilon]|[r, \phi]\Big) \longrightarrow 0, \tag{3.14}$$

as $\varepsilon \longrightarrow 0$, *and where the couple* (r, ϕ) *satisfies the CPE system* (1.6) *on the time interval* $[0, T]$.

Remark 3.5. To see more clearly the sense of the limit above, we notice that (3.14) implies, for example

$$\begin{aligned}
&\rho_\varepsilon \longrightarrow r \text{ strongly in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&\sqrt{\rho_\varepsilon}U_\varepsilon \longrightarrow \sqrt{r}\phi \text{ strongly in } L^\infty\left(0,T;L^2(\Omega)^3\right),\\
&\rho_\varepsilon U_\varepsilon \longrightarrow r\phi \text{ strongly in } L^\infty\left(0,T;L^{4/3}(\Omega)^3\right).
\end{aligned}$$

To establish the relative entropy inequality, we proceed as:
First, we take $\varphi = \frac{1}{2}|\phi_H|^2$, as a test function in weak formulation of the continuity equation (3.2). We obtain, using identity $\partial_j(\phi_i\phi_i) = 2\phi_i\partial_j\phi_i$,

$$\begin{aligned}
\frac{1}{2}\int_\Omega \rho_\varepsilon(\cdot,\tau)|\phi_H|^2(\cdot,\tau)dxdz &= \frac{1}{2}\int_\Omega \rho_0(\cdot)|\phi_H|^2(\cdot,0)dxdz\\
&+\int_0^\tau\int_\Omega \left(\rho_\varepsilon\phi_H\partial_t\phi_H + \rho_\varepsilon u_\varepsilon\phi_H\cdot\nabla_x\phi_H + \rho_\varepsilon v_\varepsilon\phi_H\partial_z\phi_H\right)dxdzdt. \qquad (3.15)
\end{aligned}$$

Second, we take $\varphi = \frac{\varepsilon^2}{2}|\phi_3|^2$ as test function in weak formulation of the continuity equation (3.2) again to get

$$\begin{aligned}
\frac{\varepsilon^2}{2}\int_\Omega \rho_\varepsilon(\cdot,\tau)|\phi_3|^2(\cdot,\tau)dxdz &= \frac{\varepsilon^2}{2}\int_\Omega \rho_0(\cdot)|\phi_3|^2(\cdot,0)dxdz\\
&+\varepsilon^2\int_0^\tau\int_\Omega \left(\rho_\varepsilon\phi_3\partial_t\phi_3 + \rho_\varepsilon u_\varepsilon\phi_3\cdot\nabla_x\phi_3 + \rho_\varepsilon v_\varepsilon\phi_3\partial_z\phi_3\right)dxdzdt. \qquad (3.16)
\end{aligned}$$

Again, in the weak formulation of the continuity equation, taking $\varphi = P'(r)$ as test function, we have

$$\begin{aligned}
\int_\Omega \rho_\varepsilon(\cdot,\tau)P'(r)(\cdot,\tau)dxdz &= \int_\Omega \rho_0(\cdot)P'(r)(\cdot,0)dxdz\\
&+\int_0^\tau\int_\Omega \left(\rho_\varepsilon\partial_t P'(r) + \rho_\varepsilon u_\varepsilon\nabla_x P'(r) + \rho_\varepsilon v_\varepsilon\partial_z P'(r)\right)dxdzdt. \qquad (3.17)
\end{aligned}$$

Next, we rewrite (3.5) as

$$\begin{aligned}
&\int_\Omega \rho_\varepsilon\left(u_\varepsilon\,\phi_H + \varepsilon^2 v_\varepsilon\,\phi_3\right)(\cdot,\tau)\,dxdz - \int_\Omega \rho_0\left(u_0\,\phi_{H,0} + \varepsilon^2 v_0\,\phi_{3,0}\right)(\cdot)\,dxdz\\
&-\int_0^\tau\int_\Omega \rho_\varepsilon\left(u_\varepsilon\partial_t\phi_H + \varepsilon^2 v_\varepsilon\partial_t\phi_3\right)dxdzdt - \int_0^\tau\int_\Omega \rho_\varepsilon\left(u_\varepsilon v_\varepsilon\partial_z\phi_H + \varepsilon^2 v_\varepsilon^2\partial_z\phi_3\right)dxdzdt\\
&+\int_0^\tau\int_\Omega \left(\nabla_x u_\varepsilon : \nabla_x\phi_H + \varepsilon^2\nabla_x v_\varepsilon : \nabla_x\phi_3\right)dxdzdt\\
&-\int_0^\tau\int_\Omega \left(u_\varepsilon\otimes u_\varepsilon : \nabla_x\phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon\cdot\nabla_x\phi_3\right)dxdzdt\\
&+\int_0^\tau\int_\Omega \rho_\varepsilon\left(\partial_z u_\varepsilon\partial_z\phi_H + \varepsilon^2\partial_z v_\varepsilon\partial_z\phi_3\right)dxdzdt\\
&-\int_0^\tau\int_\Omega p(\rho_\varepsilon)\left(\operatorname{div}_x\phi_H + \partial_z\phi_3\right)dxdzdt = \int_0^\tau\int_\Omega g\rho_\varepsilon\phi_3\,dxdzdt. \qquad (3.18)
\end{aligned}$$

We multiply (3.17) and (3.18) by (-1) and sum up with (3.15), (3.16) and the energy inequality (3.6) to deduce

$$\int_\Omega \Big(\frac{1}{2}\rho_\varepsilon|u_\varepsilon-\phi_H|^2+\frac{\varepsilon^2}{2}\rho_\varepsilon|v_\varepsilon-\phi_3|^2+P(\rho_\varepsilon)-\rho_\varepsilon P'(r)\Big)(\cdot,\tau)dxdz$$
$$-\int_\Omega \Big(\frac{1}{2}\rho_\varepsilon|u_\varepsilon-\phi_H|^2+\frac{\varepsilon^2}{2}\rho_\varepsilon|v_\varepsilon-\phi_3|^2+P(\rho_\varepsilon)-\rho_\varepsilon P'(r)\Big)(\cdot,0)dxdz$$
$$+\int_0^\tau\int_\Omega \left(|\nabla_x u_\varepsilon|^2+|\partial_z u_\varepsilon|^2\right)dxdzdt+\varepsilon^2\int_0^\tau\int_\Omega \left(|\nabla_x v_\varepsilon|^2+|\partial_z v_\varepsilon|^2\right)dxdzdt$$
$$+\int_0^\tau\int_\Omega \rho_\varepsilon\left(u_\varepsilon\partial_t\phi_H+\varepsilon^2 v_\varepsilon\partial_t\phi_3\right)dxdzdt+\int_0^\tau\int_\Omega \rho_\varepsilon\left(u_\varepsilon v_\varepsilon\partial_z\phi_H+\varepsilon^2 v_\varepsilon^2\partial_z\phi_3\right)dxdzdt$$
$$-\int_0^\tau\int_\Omega \left(\nabla_x u_\varepsilon:\nabla_x\phi_H+\varepsilon^2\nabla_x v_\varepsilon\cdot\nabla_x\phi_3\right)dxdzdt-\int_0^\tau\int_\Omega \left(\partial_z u_\varepsilon\partial_z\phi_H+\varepsilon^2\partial_z v_\varepsilon\partial_z\phi_3\right)dxdzdt$$
$$+\int_0^\tau\int_\Omega \rho_\varepsilon\left(u_\varepsilon\otimes u_\varepsilon:\nabla_x\phi_H+\varepsilon^2 u_\varepsilon v_\varepsilon\cdot\nabla_x\phi_3\right)dxdzdt$$
$$\le\int_0^\tau\int_\Omega \left(\rho_\varepsilon\phi_H\partial_t\phi_H+\rho_\varepsilon u_\varepsilon\phi_H\cdot\nabla_x\phi_H+\rho_\varepsilon v_\varepsilon\phi_H\partial_z\phi_H\right)dxdzdt$$
$$+\varepsilon^2\int_0^\tau\int_\Omega \left(\rho_\varepsilon\phi_3\partial_t\phi_3+\rho_\varepsilon u_\varepsilon\phi_3\nabla_x\phi_3+\rho_\varepsilon v_\varepsilon\phi_3\partial_z\phi_3\right)dxdzdt$$
$$-\int_0^\tau\int_\Omega \left(\rho_\varepsilon\partial_t P'(r)+\rho_\varepsilon u_\varepsilon\nabla_x P'(r)+\rho_\varepsilon v_\varepsilon\partial_z P'(r)\right)dxdzdt-\int_0^\tau\int_\Omega g\rho_\varepsilon\phi_3\,dxdzdt$$
$$+\int_0^\tau\int_\Omega g\rho_\varepsilon v_\varepsilon dxdzdt-\int_0^\tau\int_\Omega p(\rho_\varepsilon)\left(\operatorname{div}_x\phi_H+\partial_z\phi_3\right)dxdzdt. \tag{3.19}$$

Observing that the definition of $P(\rho)$ implies

$$p(r)=r^2=rP'(r)-P(r),$$

we immediately achieve

$$\partial_t p(r)=r\partial_t P'(r).$$

Hence, we compute

$$\int_\Omega \left(rP'(r)-P(r)\right)(\cdot,\tau)dxdz-\int_\Omega \left(rP'(r)-P(r)(\cdot,0)\right)dxdz$$
$$=\int_0^\tau\int_\Omega \partial_t p(r)dxdzdt. \tag{3.20}$$

Considering that ϕ satisfies no-slop boundary conditions, we have

$$\int_\Omega \left(r\partial_t P'(r)+r\phi\cdot\nabla P'(r)+p(r)\operatorname{div}\phi\right)dxdz=\int_0^\tau\int_\Omega \partial_t p(r)dxdz. \tag{3.21}$$

From (3.20) and (3.21), we conclude that

$$\int_\Omega \left(rP'(r)-P(r)\right)(\cdot,\tau)dxdz-\int_\Omega \left(rP'(r)-P(r)(\cdot,0)\right)dxdz$$
$$=\int_\Omega \left(r\partial_t P'(r)+r\phi\cdot\nabla P'(r)+p(r)\operatorname{div}\phi\right)dxdz. \tag{3.22}$$

Moreover, we have

$$\int_\Omega \Big(\rho_\varepsilon \phi_H \partial_t \phi_H + \varepsilon^2 \rho_\varepsilon \phi_3 \partial_t \phi_3 - \rho_\varepsilon \left(u_\varepsilon \partial_t \phi_H + \varepsilon^2 v_\varepsilon \partial_t \phi_3\right)\Big) dxdz$$
$$= \int_\Omega \rho_\varepsilon (\phi_H - u_\varepsilon) \partial_t \phi_H \, dxdz + \varepsilon^2 \int_\Omega \rho_\varepsilon (\phi_3 - v_\varepsilon) \partial_t \phi_3 \, dxdz. \quad (3.23)$$

We have by direct computation, $\int_\Omega \mathrm{div}_x \left(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon\right) \phi_H \, dxdz = - \int_\Omega \rho_\varepsilon u_\varepsilon (u_\varepsilon \nabla_x \phi_H) dxdz$. So,

$$\begin{aligned}
&- \int_\Omega \rho_\varepsilon \left(u_\varepsilon \otimes u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3\right) dxdz - \int_0^\tau \int_\Omega \rho_\varepsilon \left(u_\varepsilon v_\varepsilon \partial_z \phi_H + \varepsilon^2 v_\varepsilon^2 \partial_z \phi_3\right) \, dxdzdt \\
&+ \int_\Omega \left(\rho_\varepsilon u_\varepsilon \phi_H \cdot \nabla_x \phi_H + \rho_\varepsilon v_\varepsilon \phi_H \partial_z \phi_H + \varepsilon^2 \rho_\varepsilon u_\varepsilon \phi_3 \nabla_x \phi_3 + \varepsilon^2 \rho_\varepsilon v_\varepsilon \phi_3 \partial_z \phi_3\right) dxdz \\
&= \int_\Omega \rho_\varepsilon \left(u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H\right) \phi_H \, dxdzdt - \int_\Omega \rho_\varepsilon (u_\varepsilon \cdot \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H) u_\varepsilon \, dxdzdt \\
&+ \varepsilon^2 \int_\Omega \rho_\varepsilon \left(u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3\right) dxdz - \varepsilon^2 \int_\Omega \rho_\varepsilon \left(u_\varepsilon \cdot \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3\right) v_\varepsilon dxdz \\
&= \int_\Omega \rho_\varepsilon \left(u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H\right) \left(\phi_H - u_\varepsilon\right) dxdzdt \\
&\quad + \varepsilon^2 \int_\Omega \rho_\varepsilon \left(u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3\right) (\phi_3 - v_\varepsilon) dxdz. \qquad (3.24)
\end{aligned}$$

The sum of (3.24) and (3.23) gives

$$\begin{aligned}
&\int_\Omega \Big(\rho_\varepsilon \phi_H \partial_t \phi_H + \varepsilon^2 \rho_\varepsilon \phi_3 \partial_t \phi_3 - \rho_\varepsilon \left(u_\varepsilon \partial_t \phi_H + \varepsilon^2 v_\varepsilon \partial_t \phi_3\right)\Big) dxdz \\
&\quad - \int_\Omega \rho_\varepsilon \left(u_\varepsilon \otimes u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 u_\varepsilon v_\varepsilon \cdot \nabla_x \phi_3\right) dxdz - \int_0^\tau \int_\Omega \rho_\varepsilon \left(u_\varepsilon v_\varepsilon \partial_z \phi_H + \varepsilon^2 v_\varepsilon^2 \partial_z \phi_3\right) \, dxdzdt \\
&\qquad + \int_\Omega \left(\rho_\varepsilon u_\varepsilon \phi_H \cdot \nabla_x \phi_H + \rho_\varepsilon v_\varepsilon \phi_H \partial_z \phi_H + \varepsilon^2 \rho_\varepsilon u_\varepsilon \phi_3 \nabla_x \phi_3 + \varepsilon^2 \rho_\varepsilon v_\varepsilon \phi_3 \partial_z \phi_3\right) dxdz \\
&\qquad\quad = \int_\Omega \rho_\varepsilon \left(\partial_t \phi_H + u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H\right) \left(\phi_H - u_\varepsilon\right) dxdzdt \\
&\qquad\qquad + \varepsilon^2 \int_\Omega \rho_\varepsilon \left(\partial_t \phi_3 + u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3\right) (\phi_3 - v_\varepsilon) dxdz. \qquad (3.25)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left(|\nabla_x u_\varepsilon|^2 + \varepsilon^2 |\nabla_x v_\varepsilon|^2\right) - \left(\nabla_x u_\varepsilon : \nabla_x \phi_H + \varepsilon^2 \nabla_x v_\varepsilon \cdot \nabla_x \phi_3\right) \\
&\quad + |\partial_z u_\varepsilon|^2 + \varepsilon^2 |\partial_z v_\varepsilon|^2 - \partial_z u_\varepsilon \partial_z \phi_H + \varepsilon^2 \partial_z v_\varepsilon \partial_z \phi_3 \\
&= \nabla u_\varepsilon : \nabla(u_\varepsilon - \phi_H) + \varepsilon^2 \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - \phi_3). \qquad (3.26)
\end{aligned}$$

Putting these remarks (3.22) (3.25),(3.26) , we get the relative entropy inequality defined by

$$\begin{aligned}
&E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)\Big|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big(\nabla u_\varepsilon : \nabla(u_\varepsilon - \phi_H) + \varepsilon^2 |\nabla v_\varepsilon|^2\Big)\, dxdzdt \\
&\leq \int_0^\tau \int_\Omega \Big((r-\rho_\varepsilon)\partial_t P'(r) + (p(r) - p(\rho_\varepsilon))\mathrm{div}\phi + (r\phi - \rho_\varepsilon U_\varepsilon)\cdot \nabla P'(r)\Big) dxdzdt \\
&+ \int_0^\tau \int_\Omega \rho_\varepsilon \left(\partial_t \phi_H + u_\varepsilon \nabla_x \phi_H + v_\varepsilon \partial_z \phi_H\right)\left(\phi_H - u_\varepsilon\right) dxdzdt \\
&+\varepsilon^2 \int_0^\tau \int_\Omega \rho_\varepsilon \left(\partial_t \phi_3 + u_\varepsilon \nabla_x \phi_3 + v_\varepsilon \partial_z \phi_3\right)\left(\phi_3 - v_\varepsilon\right) dxdzdt \\
&- \int_0^\tau \int_\Omega g\rho_\varepsilon(\phi_3 - v_\varepsilon)\, dxdzdt + \varepsilon^2 \int_0^\tau \int_\Omega \nabla v_\varepsilon \cdot \nabla \phi_3 \, dxdzdt. \qquad (3.27)
\end{aligned}$$

4. Convergence

In this section, we will prove our main result. This proof is strongly on the relative energy inequality (3.27). We consider the strong solution (r, ϕ), where $\phi = (\phi_H\,, \phi_3)$, as test function in the relative entropy (3.8). Next, in three steps we prove our main result.

Step 1: Estimate on the relative energy inequality (3.27)
We begin by

$$\begin{aligned}
\int_\Omega \rho_\varepsilon u_\varepsilon(\phi_H - u_\varepsilon)\cdot \nabla_x \phi_H \, dxdz &= \int_\Omega \rho_\varepsilon (u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon)\cdot \nabla_x \phi_H \, dxdz \\
&+ \int_\Omega \rho_\varepsilon \phi_H (\phi_H - u_\varepsilon)\cdot \nabla_x \phi_H \, dxdz. \qquad (4.1)
\end{aligned}$$

As $[r, \phi_H, \phi_3]$ is a strong solution, concerning the first term of (4.1), we have (recall that $\nabla_x \phi_H$ is a bounded function) by using (3.10)

$$\begin{aligned}
\int_\Omega \rho_\varepsilon (u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon)\cdot \nabla_x \phi_H \, dxdz &\leq \int_\Omega |\rho_\varepsilon (u_\varepsilon - \phi_H)(\phi_H - u_\varepsilon)\cdot \nabla_x \phi_H| \, dxdz \\
&\leq C \int_\Omega \rho_\varepsilon \left|\phi_H - u_\varepsilon\right|^2 dxdz \\
&\leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big). \qquad (4.2)
\end{aligned}$$

Again, since $[r, \phi_H, \phi_3]$ is a strong solution of (1.6), we can rearrange the momentum equation $(1.6)_2$ as follows:

$$\begin{aligned}
& r\left[\partial_t \phi_H + \phi_H \cdot \nabla_x \phi_H + \phi_3 \partial_z \phi_H\right] + \phi_H \left[\partial_t r + \mathrm{div}_x(r\phi_H) + \partial_z(r\phi_3)\right] \\
&= \Delta_x \phi_H + \partial_z(\partial_z \phi_H) - \nabla_x p(r),
\end{aligned}$$

which by the means of the continuity equation reduces to

$$\partial_t \phi_H + \phi_H \cdot \nabla_x \phi_H + \phi_3 \partial_z \phi_H = \frac{1}{r}\big(\Delta_x \phi_H + \partial_z(\partial_z \phi_H) - \nabla_x p(r)\big). \qquad (4.3)$$

So we rewrite $\rho_\varepsilon\left(\partial_t\phi_H+u_\varepsilon\nabla_x\phi_H+v_\varepsilon\partial_z\phi_H\right)$ as

$$\begin{aligned}
&\rho_\varepsilon\left(\partial_t\phi_H+u_\varepsilon\nabla_x\phi_H+v_\varepsilon\partial_z\phi_H\right)\\
&= \rho_\varepsilon\Big(\partial_t\phi_H+\phi_H\nabla_x\phi_H+\phi_3\partial_z\phi_H+(u_\varepsilon-\phi_H)\nabla_x\phi_H+(v_\varepsilon-\phi_3)\partial_z\phi_H\Big)\\
&= \frac{\rho_\varepsilon}{r}\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H)-\nabla_x p(r)\Big)+\rho_\varepsilon(u_\varepsilon-\phi_H)\nabla_x\phi_H+\rho_\varepsilon(v_\varepsilon-\phi_3)\partial_z\phi_H.
\end{aligned}$$

Hence, we use (4.2) to get

$$\begin{aligned}
&\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_H+u_\varepsilon\nabla_x\phi_H+v_\varepsilon\partial_z\phi_H\Big)(\phi_H-u_\varepsilon)\,dxdzdt\\
&=\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H)-\nabla_x p(r)\Big)(\phi_H-u_\varepsilon)dxdzdt\\
&\quad+\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H dxdzdt-\int_0^\tau\int_\Omega\rho_\varepsilon(\phi_H-u_\varepsilon)^2dxdzdt\\
&\le\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H)-\nabla_x p(r)\Big)(\phi_H-u_\varepsilon)dxdzdt\\
&+\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H dxdzdt+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big).
\end{aligned}\tag{4.4}$$

Moreover, we have

$$\partial_t\phi_3+u_\varepsilon\nabla_x\phi_3+v_\varepsilon\partial_z\phi_3=\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3+(u_\varepsilon-\phi_H)\nabla_x\phi_3+(v_\varepsilon-\phi_3)\partial_z\phi_3.$$

Thus, using Cauchy's inequality and (3.10), we get

$$\begin{aligned}
&\varepsilon^2\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+u_\varepsilon\nabla_x\phi_3+v_\varepsilon\partial_z\phi_3\Big)(\phi_3-v_\varepsilon)dxdzdt\\
&\le\int_0^\tau\int_\Omega\rho_\varepsilon(\phi_3-v_\varepsilon)^2dxdzdt+\varepsilon^4\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+u_\varepsilon\nabla_x\phi_3+v_\varepsilon\partial_z\phi_3\Big)^2dxdzdt\\
&\le E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+\varepsilon^4\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+u_\varepsilon\nabla_x\phi_3+v_\varepsilon\partial_z\phi_3\Big)^2dxdzdt\\
&\le E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+\varepsilon^4\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdzdt\\
&\quad+\varepsilon^4\int_0^\tau\int_\Omega\rho_\varepsilon\Big((u_\varepsilon-\phi_H)\nabla_x\phi_3+(v_\varepsilon-\phi_3)\partial_z\phi_3\Big)^2dxdzdt.
\end{aligned}\tag{4.5}$$

We estimate $\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz$ as follows:

$$\begin{aligned}
\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz&=\int_\Omega\chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz\\
+\int_\Omega\chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz&+\int_\Omega\chi_{\{\rho_\varepsilon\ge 2r\}}\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}(\rho_\varepsilon-r)\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz\\
&\quad+\int_\Omega \chi_{\{\rho_\varepsilon\leq\frac{r}{2}\}}r\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz+\int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\rho_\varepsilon\Big(\partial_t\phi_3+\phi_H\nabla_x\phi_3+\phi_3\partial_z\phi_3\Big)^2dxdz\\
&\leq C\int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}(\rho_\varepsilon-r)^2dxdz+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+C\int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\rho_\varepsilon^2dxdz+C\\
&\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+C. \qquad (4.6)
\end{aligned}
$$

Using (4.6) in (4.5) holds

$$
\begin{aligned}
&\varepsilon^2\int_0^\tau\int_\Omega\rho_\varepsilon\Big(\partial_t\phi_3+u_\varepsilon\nabla_x\phi_3+v_\varepsilon\partial_z\phi_3\Big)(\phi_3-v_\varepsilon)dxdzdt\\
&\quad\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+o(\varepsilon^2). \qquad (4.7)
\end{aligned}
$$

Moreover, we apply Cauchy-Schwarz's inequality as

$$
\begin{aligned}
\varepsilon^2\int_0^\tau\int_\Omega\nabla v_\varepsilon\cdot\nabla\phi_3\,dxdzdt &\leq \varepsilon^2\left(\int_0^\tau\int_\Omega\frac{|\nabla v_\varepsilon|^2}{2}dxdzdt+\int_0^\tau\int_\Omega\frac{|\nabla\phi_3|^2}{2}dxdzdt\right)\\
&\leq \frac{\varepsilon^2}{2}\int_0^\tau\int_\Omega|\nabla v_\varepsilon|^2dxdzdt+o(\varepsilon^2). \qquad (4.8)
\end{aligned}
$$

Thus, entropy inequality (3.27) becomes

$$
\begin{aligned}
&E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\Big(\nabla u_\varepsilon:\nabla(u_\varepsilon-\phi_H)+\tfrac{\varepsilon^2}{2}|\nabla v_\varepsilon|^2\Big)\,dxdzdt\\
&\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H)\Big)(\phi_H-u_\varepsilon)dxdzdt\\
&-\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}(\phi_H-u_\varepsilon)\nabla_x p(r)dxdzdt+\int_0^\tau\int_\Omega g\rho_\varepsilon(v_\varepsilon-\phi_3)\,dxdzdt\\
&+\int_0^\tau\int_\Omega\Big((r-\rho_\varepsilon)\partial_t P'(r)+(p(r)-p(\rho_\varepsilon))\mathrm{div}\phi+(r\phi-\rho_\varepsilon U_\varepsilon)\cdot\nabla P'(r)\Big)dxdzdt\\
&+\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdzdt. \qquad (4.9)
\end{aligned}
$$

Step 2: Estimate for the non-linear term $\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H dxdzdt$ of (4.9).

Establishing an estimate for the non-linear term $\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H dxdzdt$ is a major difficulty in our analysis. We have

$$
\begin{aligned}
&\int_0^\tau\int_\Omega\rho_\varepsilon(v_\varepsilon-\phi_3)(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdzdt\\
&=\int_0^\tau\int_\Omega\rho_\varepsilon v_\varepsilon(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdzdt-\int_0^\tau\int_\Omega\rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdzdt. \quad (4.10)
\end{aligned}
$$

In a similar way to [7, 9, 10] and using Lemma 2.1., we estimate $\int_\Omega \rho_\varepsilon\phi_3(\phi_H - u_\varepsilon)\partial_z\phi_H\,dxdz$ as follows:

$$
\begin{aligned}
&\int_\Omega \rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdzdt = \int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}\rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz\\
&\quad+\int_\Omega \chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}\rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz+\int_\Omega \chi_{\{\rho_\varepsilon\ge 2r\}}\rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz\\
&\le C\|\chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}(\rho_\varepsilon-r)\|_{L^2(\Omega)}\|\phi_3\partial_z\phi_H\|_{L^3(\Omega)}\|\phi_H-u_\varepsilon\|_{L^6(\Omega)}\\
&\quad+\|\chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}1\|_{L^2(\Omega)}\|r\|_{L^\infty(\Omega)}\|\phi_3\partial_z\phi_H\|_{L^3(\Omega)}\|\phi_H-u_\varepsilon\|_{L^6(\Omega)}\\
&\quad+\int_\Omega \chi_{\{\rho_\varepsilon\ge 2r\}}\rho_\varepsilon\phi_3(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz\\
&\le C\int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}(\rho_\varepsilon-r)^2dxdz + C\int_\Omega \chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}1dxdz\\
&\quad+C\int_\Omega \chi_{\{\rho_\varepsilon\ge 2r\}}\rho_\varepsilon^2dxdz+\lambda\|\phi_H-u_\varepsilon\|^2_{L^6(\Omega)}\\
&\le CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}+\lambda\|\partial_z\phi_H-\partial_z u_\varepsilon\|^2_{L^2(\Omega)}. \quad (4.11)
\end{aligned}
$$

The crucial and difficult challenge of our proof is the analysis of the first term of the right hand of (4.10). Using (1.9), we have

$$
\begin{aligned}
&\int_\Omega \rho_\varepsilon v_\varepsilon(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz=\int_\Omega\left[-\int_0^z\Big(\mathrm{div}_x(\xi\tilde u_\varepsilon)+\tfrac12 gz\widetilde{\mathrm{div}_x u_\varepsilon}\Big)dz\right](\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz\\
&=\int_\Omega\int_0^z\xi_\varepsilon\tilde u_\varepsilon(\phi_H-u_\varepsilon)\partial_z\nabla_x\phi_H\,dzdxdz+\int_\Omega\int_0^z\xi_\varepsilon\tilde u_\varepsilon\,\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dz\,dxdz\\
&\qquad-\int_\Omega\left(\int_0^z\tfrac12 gz\widetilde{\mathrm{div}_x u_\varepsilon}\,dz\right)(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz. \qquad (4.12)
\end{aligned}
$$

In the following, we will estimate the terms on the right hand side of (4.12). Firstly, we deal with $\int_\Omega \xi_\varepsilon\tilde u_\varepsilon(\phi_H-u_\varepsilon)\partial_z\nabla_x\phi_H\,dxdz$ in the following,

$$
\begin{aligned}
&\int_\Omega \xi_\varepsilon\tilde u_\varepsilon(\phi_H-u_\varepsilon)\partial_z\nabla_x\phi_H\,dxdz\\
&\quad=\int_\Omega\xi_\varepsilon\left(\tilde u_\varepsilon-\widetilde{\phi_H}\right)\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz+\int_\Omega\xi_\varepsilon\widetilde{\phi_H}\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&\quad=I_1+I_2. \qquad (4.13)
\end{aligned}
$$

Using Cauchy's inequality, it follows that

$$
\begin{aligned}
I_1&=\int_\Omega\xi_\varepsilon\left(\tilde u_\varepsilon-\widetilde{\phi_H}\right)\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&=\int_\Omega\xi_\varepsilon\left(u_\varepsilon-\phi_H\right)\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz+\int_\Omega\xi_\varepsilon\left(\bar\phi_H-\bar u_\varepsilon\right)\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\partial_z\nabla_x\phi_H\|_{L^\infty(\Omega)}\int_\Omega \xi_\varepsilon|u_\varepsilon-\phi_H|^2dxdz\\
&\quad+\frac{1}{2}\|\partial_z\nabla_x\phi_H\|_{L^\infty(\Omega)}\left(\int_\Omega \xi_\varepsilon|u_\varepsilon-\phi_H|^2dxdz+\int_\Omega \xi_\varepsilon|\bar{u}_\varepsilon-\bar{\phi}_H|^2dxdz\right)\\
&\leq C\int_\Omega \xi_\varepsilon\left|\int_0^1(u_\varepsilon-\phi_H)(s)ds\right|^2dxdz+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\\
&\leq C\int_\Omega \xi_\varepsilon\left(\int_0^1|u_\varepsilon-\phi_H|^2ds\right)dxdz+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\\
&\leq C\int_0^1\int_\Omega \xi_\varepsilon|u_\varepsilon-\phi_H|^2dxdzds+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\\
&\leq C\int_\Omega \xi_\varepsilon|u_\varepsilon-\phi_H|^2dxdz+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\\
&\quad\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big).
\end{aligned}
\tag{4.14}
$$

As in the analysis (4.11), we deal with I_2 as follows:

$$
\begin{aligned}
I_2 \;&=\; \int_\Omega \xi_\varepsilon\widetilde{\phi_H}\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&=\; \int_\Omega \chi_{\{\rho_\varepsilon\leq\frac{r}{2}\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&\quad+\int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&\quad+\int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dxdz\\
&\leq \|\chi_{\{\rho_\varepsilon\leq\frac{r}{2}\}}1\|_{L^2(\Omega)}\|r\|_{L^\infty(\Omega)}\|\widetilde{\phi_H}\partial_z\nabla_x\phi_H\|_{L^3(\Omega)}\|\phi_H-u_\varepsilon\|_{L^6(\Omega)}\\
&\quad+C\|\chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}(\xi_\varepsilon-r)\|_{L^2(\Omega)}\|\widetilde{\phi_H}\partial_z\nabla_x\phi_H\|_{L^3(\Omega)}\|\phi_H-u_\varepsilon\|_{L^6(\Omega)}\\
&\quad+\|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^2_{L^2(\Omega)}\|\widetilde{\phi_H}\partial_z\nabla_x\phi_H\|_{L^3(\Omega)}\|\phi_H-u_\varepsilon\|_{L^6(\Omega)}\\
&\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t)+\lambda\|\nabla\phi_H-\nabla u_\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.15}
$$

Let's consider another non-linear term, $\int_\Omega \xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz$. We have

$$
\begin{aligned}
\int_\Omega \xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz &= \int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz\\
&+\int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz = M_1+M_2.
\end{aligned}
\tag{4.16}
$$

The term M_1 of (4.16) is decomposed as

$$
M_1 \;=\; \int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz
$$

$$
\begin{aligned}
&= \int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\left(\tilde{u}_\varepsilon-\widetilde{\phi_H}\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\qquad +\int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&= M_{11}+M_{12}.
\end{aligned}
\tag{4.17}
$$

M_{11} and M_{12} are treated in a similar way as I_1 and I_2 respectively. Namely,

$$
\begin{aligned}
M_{11} &= \int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\left(\tilde{u}_\varepsilon-\widetilde{\phi_H}\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&= \int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\left(u_\varepsilon-\phi_H\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\qquad +\int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\left(\bar{u}_\varepsilon-\bar{\phi}_H\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\le \frac{1}{2}\|\chi_{\{\rho_\varepsilon<2r\}}\sqrt{\xi_\varepsilon}\|_{L^\infty(\Omega)}\|\partial_z\phi_H\|_{L^\infty(\Omega)}\int_\Omega \chi_{\{\rho_\varepsilon<2r\}}\xi_\varepsilon\left|u_\varepsilon-\phi_H\right|^2dxdz \\
&\quad +\frac{1}{2}\|\chi_{\{\rho_\varepsilon<2r\}}\sqrt{\xi_\varepsilon}\|_{L^\infty(\Omega)}\|\partial_z\phi_H\|_{L^\infty(\Omega)}\int_\Omega \chi_{\{\rho_\varepsilon<2r\}}|\nabla_x(\phi_H-u_\varepsilon)|^2dxdz+CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t) \\
&\le CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t)+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
M_{12} &= \int_\Omega \chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\qquad +\int_\Omega \chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\le \|\chi_{\{\rho_\varepsilon\le\frac{r}{2}\}}1\|_{L^2(\Omega)}\|r\|_{L^\infty(\Omega)}\|\widetilde{\phi_H}\partial_z\phi_H\|_{L^\infty(\Omega)}\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|_{L^2(\Omega)} \\
&\qquad +\|\chi_{\{\frac{r}{2}<\rho_\varepsilon<2r\}}\xi_\varepsilon\|_{L^2(\Omega)}\|\widetilde{\phi_H}\partial_z\phi_H\|_{L^\infty(\Omega)}\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|_{L^2(\Omega)} \\
&\le \lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.19}
$$

Similarly, we decomposition M_2 as

$$
\begin{aligned}
M_2 &= \int_\Omega \chi_{\{\rho_\varepsilon\ge2r\}}\xi_\varepsilon\tilde{u}_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&= \int_\Omega \chi_{\{\rho_\varepsilon\ge2r\}}\xi_\varepsilon\left(\tilde{u}_\varepsilon-\widetilde{\phi_H}\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&\qquad +\int_\Omega \chi_{\{\rho_\varepsilon\ge2r\}}\xi_\varepsilon\widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz \\
&= M_{21}+M_{22}.
\end{aligned}
\tag{4.20}
$$

We have, by using (3.11),

$$\begin{aligned}
M_{22} &= \int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon \widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz\\
&\leq \int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon^{\alpha/2} \widetilde{\phi_H}\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon^{\alpha/2}\|_{L^2(\Omega)}\|\widetilde{\phi_H}\partial_z\phi_H\|_{L^\infty(\Omega)}\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|_{L^2(\Omega)}\\
&\leq C\|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon^{\alpha/2}\|^2_{L^2(\Omega)}+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}\\
&\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t)+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}. \qquad (4.21)
\end{aligned}$$

Due to Hölder inequality and Lemma 2.2, it follows that

$$\begin{aligned}
&M_{21} = \int_\Omega \chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\left(\tilde u_\varepsilon-\widetilde{\phi_H}\right)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)\,dxdz\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|_{L^4(\Omega)}\|\partial_z\phi_H\|_{L^\infty(\Omega)}\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|_{L^2(\Omega)}\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^2_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|^2_{L^4(\Omega)}+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^2_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|_{L^3(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\nabla\tilde u_\varepsilon-\nabla\widetilde{\phi_H})\|_{L^2(\Omega)}\\
&+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^4_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|^2_{L^3(\Omega)}+\lambda\|\nabla_x\tilde u_\varepsilon-\nabla_x\widetilde{\phi_H}\|^2_{L^2(\Omega)}\\
&\quad+\lambda\|\partial_z\tilde u_\varepsilon-\partial_z\widetilde{\phi_H}\|^2_{L^2(\Omega)}+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^4_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|_{L^2(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|_{H^1(\Omega)}\\
&\quad+\lambda\|\nabla_x\tilde u_\varepsilon-\nabla_x\widetilde{\phi_H}\|^2_{L^2(\Omega)}+\lambda\|\partial_z\tilde u_\varepsilon-\partial_z\widetilde{\phi_H}\|^2_{L^2(\Omega)}+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}\\
&\leq \|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^8_{L^4(\Omega)}\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|^2_{L^2(\Omega)}+\lambda\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|^2_{L^2(\Omega)}\\
&+\lambda\|\nabla_x\tilde u_\varepsilon-\nabla_x\widetilde{\phi_H}\|^2_{L^2(\Omega)}+\lambda\|\partial_z\tilde u_\varepsilon-\partial_z\widetilde{\phi_H}\|^2_{L^2(\Omega)}+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2}. \qquad (4.22)
\end{aligned}$$

Lemma 2.2 is used as follows

$$\|f\|^2_{L^4}\leq\|\nabla f\|_{L^2}\|f\|_{L^3}\quad\text{and}\quad \|f\|^2_{L^3}\leq\|f\|_{L^2}\|f\|_{H^1}.$$

We deduce from (3.11),

$$\|\chi_{\{\rho_\varepsilon\geq 2r\}}\xi_\varepsilon\|^8_{L^4(\Omega)}=\left(\int_{\rho_\varepsilon\geq 2r}\xi_\varepsilon^4\,dxdz\right)^{8/4}\leq CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)^2. \qquad (4.23)$$

Recalling (4.14) we have

$$\begin{aligned}
\|\chi_{\{\rho_\varepsilon\geq 2r\}}(\tilde u_\varepsilon-\widetilde{\phi_H})\|^2_{L^2(\Omega)} &= \int_{\rho_\varepsilon\geq 2r}|\tilde u_\varepsilon-\widetilde{\phi_H}|^2dxdz\\
&= \int_{\rho_\varepsilon\geq 2r}\frac{1}{\rho_\varepsilon}\rho_\varepsilon|\tilde u_\varepsilon-\widetilde{\phi_H}|^2dxdz\\
&\leq \frac{1}{\|r\|_{L^\infty(\Omega)}}E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big). \qquad (4.24)
\end{aligned}$$

As in (4.14), we have

$$\begin{cases}\|\nabla_x\tilde{u}_\varepsilon-\nabla_x\widetilde{\phi_H}\|^2_{L^2(\Omega)}\le\|\nabla_x u_\varepsilon-\nabla_x\phi_H\|^2_{L^2(\Omega)},\\ \|\partial_z\tilde{u}_\varepsilon-\partial_z\widetilde{\phi_H}\|^2_{L^2(\Omega)}\le\|\partial_z u_\varepsilon-\partial_z\phi_H\|^2_{L^2(\Omega)}.\end{cases}\tag{4.25}$$

Combining the estimates (4.23), (4.24) and (4.25), we conclude that

$$\begin{aligned}\int_0^\tau M_{21}dt &\le C\int_0^\tau h(t)E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)dt\\ &+\lambda\int_0^\tau\|\nabla_x u_\varepsilon-\nabla_x\phi_H\|^2_{L^2(\Omega)}dt+\lambda\int_0^\tau\|\partial_z u_\varepsilon-\partial_z\phi_H\|^2_{L^2(\Omega)}dt,\end{aligned}\tag{4.26}$$

where $h(t)\in L^1(0,T)$.
Using similar approach, the estimation of remaining quantity in (4.12),
$-\int_\Omega\left(\int_0^z\frac{1}{2}g\widetilde{z\mathrm{div}_x u_\varepsilon}\,dz\right)(\phi_H-u_\varepsilon)\partial_z\phi_H\,dxdz$, can be determined as follows.
Indeed, we have

$$\begin{aligned}&-\int_\Omega\left(\int_0^z\tfrac{1}{2}g\widetilde{z\mathrm{div}_x u_\varepsilon}\,dz\right)(\phi_H-u_\varepsilon)\partial_z\phi_H\,dz\,dxdz\\
&=-\int_\Omega\int_0^z\tfrac{1}{2}gz\mathrm{div}_x u_\varepsilon(\phi_H-u_\varepsilon)\partial_z\phi_H\,dz\,dxdz+\int_\Omega\int_0^1\tfrac{1}{2}g\overline{z\mathrm{div}_x u_\varepsilon}(\phi_H-u_\varepsilon)\partial_z\phi_H\,dz\,dxdz\\
&=\int_\Omega\int_0^z\tfrac{1}{2}gzu_\varepsilon(\phi_H-u_\varepsilon)\partial_z\nabla_x\phi_H\,dz\,dxdz+\int_\Omega\int_0^z\tfrac{1}{2}gzu_\varepsilon\partial_z\phi_H\nabla_x(\phi_H-u_\varepsilon)\,dz\,dxdz\\
&\qquad+\int_\Omega\int_0^1\tfrac{1}{2}g\overline{z\mathrm{div}_x u_\varepsilon}(\phi_H-u_\varepsilon)\partial_z\phi_H\,dz\,dxdz\\
&=\int_\Omega\int_0^z\tfrac{1}{2}gz\left(u_\varepsilon-\phi_H\right)\left(\phi_H-u_\varepsilon\right)\partial_z\nabla_x\phi_H\,dz\,dxdz+\int_\Omega\int_0^z\tfrac{1}{2}gz\phi_H\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)\,dz\,dxdz\\
&\quad+\int_\Omega\int_0^z\tfrac{1}{2}gzu_\varepsilon\partial_z\phi_H\nabla_x(\phi_H-u_\varepsilon)\,dz\,dxdz+\int_\Omega\int_0^1\tfrac{1}{2}g\overline{z\mathrm{div}_x u_\varepsilon}(\phi_H-u_\varepsilon)\partial_z\phi_H\,dz\,dxdz\\
&\le\|\partial_z\nabla_x\phi_H\|_{L^\infty(\Omega)}\int_\Omega\varphi(z)|\phi_H-u_\varepsilon|^2\,dxdz+I_2'+(M_1+M_2)'+\int_\Omega g\left|\overline{\mathrm{div}_x u_\varepsilon}(\phi_H-u_\varepsilon)\partial_z\phi_H\right|\,dxdz\\
&\le CE\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)+\lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}+\|\partial_z u_\varepsilon-\partial_z\phi_H\|^2_{L^2(\Omega)}+N.\end{aligned}\tag{4.27}$$

Furthermore, using Cauchy's and Hölder inequalities, we have

$$\begin{aligned}N&=\int_\Omega g\left|\overline{\mathrm{div}_x u_\varepsilon}(\phi_H-u_\varepsilon)\partial_z\phi_H\right|\,dxdz=\int_\Omega g\left|(\phi_H-u_\varepsilon)\partial_z\phi_H\int_0^1\mathrm{div}_x u_\varepsilon\,dz\right|\,dxdz\\
&\le g\int_\Omega|u_\varepsilon\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)|dxdz+\int_\Omega|u_\varepsilon\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)|dxdz\\
&\le g\int_\Omega|\partial_z\nabla_x\phi_H|\cdot|\phi_H-u_\varepsilon|^2dxdz+g\int_\Omega|\phi_H\partial_z\nabla_x\phi_H\cdot(\phi_H-u_\varepsilon)|dxdz\\
&\quad+\int_\Omega|(u_\varepsilon-\phi_H)\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)|dxdz+\int_\Omega|\phi_H\partial_z\phi_H\cdot\nabla_x(\phi_H-u_\varepsilon)|dxdz\end{aligned}$$

$$
\begin{aligned}
&\leq\ gC\int_\Omega|\phi_H-u_\varepsilon|^2dxdz+g\|\phi_H\partial_z\nabla_x\phi_H\|_{L^\infty(\Omega)}\left(\int_\Omega 1^2dxdz+\int_\Omega|\phi_H-u_\varepsilon|^2dxdz\right)\\
&\quad+g\|\partial_z\phi_H\|_{L^\infty(\Omega)}\left(\int_\Omega|\phi_H-u_\varepsilon|^2dxdz+\int_\Omega|\nabla_x(\phi_H-u_\varepsilon)|^2dxdz\right)\\
&\quad+g\|\phi_H\partial_z\phi_H\|_{L^2(\Omega)}\left(\int_\Omega 1^2dxdz+\int_\Omega|\nabla_x\phi_H-\nabla_x u_\varepsilon|^2dxdz\right)\\
&\leq\ \lambda\|\nabla_x\phi_H-\nabla_x u_\varepsilon\|^2_{L^2(\Omega)}+\lambda\|\partial_z\phi_H-\partial_z u_\varepsilon\|^2_{L^2(\Omega)}+C.
\end{aligned}\tag{4.28}
$$

In summary, here is what we have demonstrated so far.

$$
\begin{aligned}
&E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\Big(\nabla u_\varepsilon:\nabla(u_\varepsilon-\phi_H)+\varepsilon^2|\nabla v_\varepsilon|^2\Big)\,dxdzdt\\
&\leq\int_0^\tau h(t)E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t)dt+\lambda\int_0^\tau\Big(\|\nabla_x u_\varepsilon-\nabla_x\phi_H\|^2_{L^2(\Omega)}+\|\partial_z u_\varepsilon-\partial_z\phi_H\|^2_{L^2(\Omega)}\Big)\,dt\\
&\quad+\int_0^\tau\int_\Omega\Big((r-\rho_\varepsilon)\partial_tP'(r)+(p(r)-p(\rho_\varepsilon))\mathrm{div}\phi+(r\phi-\rho_\varepsilon U_\varepsilon)\cdot\nabla P'(r)\Big)dxdzdt\\
&\quad-\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}(\phi_H-u_\varepsilon)\nabla_x p(r)dxdzdt-\int_0^\tau\int_\Omega g\rho_\varepsilon(\phi_3-v_\varepsilon)\,dxdzdt\\
&\qquad+\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H)\Big)(\phi_H-u_\varepsilon)dxdzdt+o(\varepsilon^2).
\end{aligned}\tag{4.29}
$$

Then we deduce that (4.9) can be rewritten as

$$
\begin{aligned}
&E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\Big(\nabla_x(u_\varepsilon-\phi_H):\nabla_x(u_\varepsilon-\phi_H)+|\partial_z(u_\varepsilon-\phi_H)|^2+\varepsilon^2|\nabla v_\varepsilon|^2\Big)\,dxdzdt\\
&\leq\int_0^\tau h(t)E\Big([\rho_\varepsilon,U_\varepsilon]|[r,\phi]\Big)(t)dt+\lambda\int_0^\tau\Big(\|\nabla_x u_\varepsilon-\nabla_x\phi_H\|^2_{L^2(\Omega)}+\|\partial_z u_\varepsilon-\partial_z\phi_H\|^2_{L^2(\Omega)}\Big)\,dt\\
&\quad+\int_0^\tau\int_\Omega(\frac{\rho_\varepsilon}{r}-1)\Big(\Delta_x\phi_H+\partial_z(\partial_z\phi_H\Big)(\phi_H-u_\varepsilon)dxdzdt\\
&\quad-\int_0^\tau\int_\Omega\Big((\rho_\varepsilon-r)\partial_tP'(r)+(p(\rho_\varepsilon)-p(r))\mathrm{div}\phi+(\rho_\varepsilon U_\varepsilon-r\phi)\cdot\nabla P'(r)\Big)dxdzdt\\
&\quad-\int_0^\tau\int_\Omega g\rho_\varepsilon(\phi_3-v_\varepsilon)\,dxdzdt-\int_0^\tau\int_\Omega\frac{\rho_\varepsilon}{r}(\phi_H-u_\varepsilon)\nabla_x p(r)dxdzdt+o(\varepsilon^2).
\end{aligned}\tag{4.30}
$$

Step 3: Estimate the remaining quantities in the relative energy inequality (4.30) Here, we begin by the decomposition of the term $\int_\Omega(\frac{\rho_\varepsilon}{r}-1)\Big(\Delta_x\phi_H+\partial_{zz}\phi_H\Big)(\phi_H-u_\varepsilon)dxdzdt$ of (4.30) into two parts and we use the regularity of the strong solution to get

$$
\begin{aligned}
&\left|\int_\Omega\frac{1}{r}(\rho_\varepsilon-r)\Big(\Delta_x\phi_H+\partial_{zz}\phi_H\Big)(\phi_H-u_\varepsilon)dxdzdt\right|\\
&\leq C\int_\Omega|\rho_\varepsilon-r|_{ess}|\phi_H-u_\varepsilon|dxdz+C\int_\Omega|\rho_\varepsilon-r|_{res}|\phi_H-u_\varepsilon|dxdz\\
&\quad=K_1+K_2.
\end{aligned}\tag{4.31}
$$

According to (3.9), we have by using the inequality of Cauchy

$$\begin{aligned} K_1 &\leq C\int_\Omega \frac{1}{\sqrt{\rho_\varepsilon}}|\rho_\varepsilon - r|_{ess}\sqrt{\rho_\varepsilon}|\phi_H - u_\varepsilon| dxdz \\ &\leq C\int_\Omega \left(|\rho_\varepsilon - r|_{ess}^2 + \rho_\varepsilon|\phi_H - u_\varepsilon|^2\right) dxdz \\ &\leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big). \end{aligned} \tag{4.32}$$

We split K_2 as follows

$$K_2 = C\int_{\rho_\varepsilon > 2r} |\rho_\varepsilon - r||\phi_H - u_\varepsilon| dxdz + C\int_{\rho_\varepsilon < \frac{1}{2}r} |\rho_\varepsilon - r||\phi_H - u_\varepsilon| dxdz,$$

where the first integral may be treated in the same way as K_1 and the second integral is estimated with the help of the inequality of Poincaré as follows

$$\begin{aligned} C\int_{\rho_\varepsilon < \frac{1}{2}r} |\rho_\varepsilon - r||\phi_H - u_\varepsilon| dxdz &\leq C\int_\Omega |\phi_H - u_\varepsilon| dxdz \\ &\leq C\int_\Omega 1_{res}^2 dxdz + \lambda\int_\Omega |\phi_H - u_\varepsilon|^2 dxdz \\ &\leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big) + \lambda\|\nabla_x u_\varepsilon - \nabla_x \phi_H\|_{L^2(\Omega)}^2 \\ &\quad + \lambda\|\partial_z u_\varepsilon - \partial_z\phi_H\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.33}$$

Furthermore, we have

$$\begin{aligned} &-\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x p(r) + (\rho_\varepsilon - r)\partial_t P'(r) + (p(\rho_\varepsilon) - p(r))\mathrm{div}\phi + (\rho_\varepsilon U_\varepsilon - r\phi)\cdot\nabla P'(r) dxdzdt \\ &= -\int_0^\tau\int_\Omega \frac{\rho_\varepsilon}{r}(\phi_H - u_\varepsilon)\nabla_x p(r) + (\rho_\varepsilon - r)P''(r)\partial_t r + \phi_H\nabla_x p(r) + \phi_3\partial_z p(r) dxdzdt \\ &\quad - \int_0^\tau\int_\Omega \rho_\varepsilon u_\varepsilon\nabla_x P'(r) + \rho_\varepsilon v_\varepsilon\partial_z P'(r) - r\phi_H\nabla_x P'(r) - r\phi_3\partial_z P'(r) + p(\rho_\varepsilon)\mathrm{div}\phi\, dxdzdt \\ &= -\int_0^\tau\int_\Omega \rho_\varepsilon P''(r)\left(\phi_H\nabla_x r + \partial_t r\right) - rP''(r)\partial_t r + p(\rho_\varepsilon)\mathrm{div}\phi\, dxdzdt \\ &\quad - \int_0^\tau\int_\Omega \phi_3\partial_z p(r) + \rho_\varepsilon v_\varepsilon\partial_z P'(r) - r\phi_3\partial_z P'(r) dxdzdt \\ &= -\int_0^\tau\int_\Omega \rho_\varepsilon P''(r)\left(-r\mathrm{div}_x\phi_H - r\partial_z\phi_3 - \phi_3\partial_z r\right)\, dxdzdt \\ &\quad - \int_0^\tau\int_\Omega rP''(r)\left(r\mathrm{div}_x\phi_H + \phi_H\nabla_x r + r\partial_z\phi_3 + \phi_3\partial_z r\right) dxdzdt \\ &= -\int_0^\tau\int_\Omega \Big(p(\rho_\varepsilon) - p(r) - p'(r)(\rho_\varepsilon - r)\Big)\mathrm{div}\phi\, dxdzdt \\ &\quad - \int_0^\tau\int_\Omega \rho_\varepsilon v_\varepsilon\partial_z P'(r) - r\phi_3\partial_z P'(r) - (\rho_\varepsilon - r)\,\phi_3\partial_z p'(r)\, dxdzdt, \end{aligned} \tag{4.34}$$

where we used the fact that $r\nabla_x P'(r) = \nabla_x p(r) = p'(r)\nabla_x r$, $\partial_t P'(r) = P''(r)\partial_t r$, $\nabla_x P'(r) = P''(r)\nabla_x r$ and $\partial_t r + r\mathrm{div}_x\phi_H + \phi_H\nabla_x r + r\partial_z\phi_3 + \phi_3\partial_z r = 0$.

Using the analogous argument as in [15] Section 2.2.5, we can carry out the following estimate:

$$\begin{aligned}
&\left|\int_0^\tau \int_\Omega -\Big(p(\rho_\varepsilon) - p(r) - p'(r)(\rho_\varepsilon - r)\Big)\mathrm{div}\phi \, dxdzdt\right| \\
&\qquad \leq C\int_0^\tau h(t)E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)dt. \qquad (4.35)
\end{aligned}$$

Indeed, as far as the first term is concerned of (4.34) we use the Taylor formula together with the regularity C^2 of the pressure p, to estimate the essential part

$$\begin{aligned}
&-\int_0^\tau \int_\Omega \Big(p(\rho_\varepsilon) - p(r) - p'(r)(\rho_\varepsilon - r)\Big)_{ess}\mathrm{div}\phi \, dxdzdt \\
&\quad \leq C\int_0^\tau \|\mathrm{div}\phi\|_{L^\infty(\Omega)}\|(\rho_\varepsilon - r)_{ess}\|^2_{L^2(\Omega)}dt \\
&\quad \leq C\int_0^\tau h(t)E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)dt, \qquad (4.36)
\end{aligned}$$

where $h(t) = \|\mathrm{div}\phi\|_{L^\infty(\Omega)}$.
For the residual part, we show that

$$\left|\Big(p(\rho_\varepsilon) - p(r) - p'(r)(\rho_\varepsilon - r)\Big)_{res}\right| \leq C\left(1_{res} + (\rho_\varepsilon^2)_{res}\right). \qquad (4.37)$$

Thus, we have

$$\begin{aligned}
&-\int_0^\tau \int_\Omega \Big(p(\rho_\varepsilon) - p(r) - p'(r)(\rho_\varepsilon - r)\Big)_{res}\mathrm{div}\phi \, dxdzdt \\
&\quad \leq C\int_0^\tau \|\mathrm{div}\phi\|_{L^\infty(\Omega)}\int_\Omega \left(1_{res} + (\rho_\varepsilon^2)_{res}\right)dxdzdt \\
&\quad \leq C\int_0^\tau h(t)E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)dt. \qquad (4.38)
\end{aligned}$$

In (4.34), $r\phi_3\partial_z P'(r)$ is a bounded function then $\int_0^\tau\int_\Omega r\phi_3\partial_z P'(r)dxdzdt \leq C$.
We show as in (4.12) that

$$\begin{aligned}
\int_\Omega \rho_\varepsilon v_\varepsilon \partial_z P'(r)dxdz &= \int_\Omega \left[-\int_0^z \Big(\mathrm{div}_x(\xi_\varepsilon\tilde{u}_\varepsilon) + \tfrac{1}{2}gz\widetilde{\mathrm{div}_x u_\varepsilon}\Big)dz\right]\partial_z P'(r)dxdz \\
&= \int_\Omega\int_0^z \xi\tilde{u}_\varepsilon\partial_z\nabla_x P'(r)\,dzdxdz - \int_\Omega\int_0^z \tfrac{1}{2}gz\widetilde{\mathrm{div}_x u_\varepsilon}\,\partial_z P'(r)\,dzdxdz \\
&\leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big). \qquad (4.39)
\end{aligned}$$

For the remainder term of (4.34), we have, similarly to (4.31)

$$\begin{aligned}\int_\Omega (\rho_\varepsilon - r)\,\phi_3\partial_z p'(r)\,dxdz &\leq \|\phi_3\partial_z p'(r)\|_{L^\infty(\Omega)}\left(\int_\Omega |\rho_\varepsilon - r|_{ess}\,dxdz + \int_\Omega |\rho_\varepsilon - r|_{res}\,dxdz\right)\\ &\leq CE\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big). \qquad (4.40)\end{aligned}$$

Moreover, using the inequalities of Cauchy and Hölder and the fact that ρ_ε is bounded, we get from (3.10)

$$\begin{aligned}\int_\Omega \rho_\varepsilon(\phi_3 - v_\varepsilon)\,dxdz &\leq \int_\Omega \tfrac{1}{2}\rho_\varepsilon^2\,dxdz + \int_\Omega \tfrac{1}{2}|\phi_3 - v_\varepsilon|^2\,dxdz\\ &\leq C + C\|\nabla v_\varepsilon - \nabla\phi_3\|_{L^2}^2 dt. \qquad (4.41)\end{aligned}$$

Therefore, the relative entropy inequality can be written as

$$\begin{aligned}&E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)(\tau) + C\int_0^\tau \left(\|\nabla u_\varepsilon - \nabla\phi_H\|_{L^2}^2 + \varepsilon^2\|\nabla v_\varepsilon\|_{L^2}^2\right)dt\\ &\leq C\int_0^\tau h(t)E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)(t)dt + E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)(0)\\ &\quad +\lambda\int_0^\tau \|\nabla u_\varepsilon - \nabla\phi_H\|_{L^2}^2 dt + o(\varepsilon^2). \qquad (4.42)\end{aligned}$$

We deduce that when ε tends towards 0 in (4.42),

$$E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)(\tau) \leq C\int_0^\tau h(t)E\Big([\rho_\varepsilon, U_\varepsilon]|[r,\phi]\Big)(t)dt. \qquad (4.43)$$

Then applying Gronwall's inequality, the proof of Theorem 3.4 is complete.

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