POSTULATION IN PROJECTIVE SPACES OF A UNION OF A LOW DIMENSIONAL SCHEME AND GENERAL RATIONAL CURVES OR GENERAL UNIONS OF LINES

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ABSTRACT. Let $Y \subset \mathbb{P}^r$, $r \geq 3$, be a scheme with $\dim(Y) \leq r - 2$. We prove the existence of an integer $d_0$ such that for all $d \geq d_0$ a general union $X$ of $Y$ and either a general degree $d$ rational curve or (if $r \neq 3$) $d$ lines has maximal rank, i.e. for each $t \in \mathbb{N}$ $X$ gives the expected number of conditions to the set of all degree $t$ hypersurfaces of $\mathbb{P}^r$. We also consider unions of $Y$ and a general curve with a prescribed genus and high degree.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a closed subscheme. We recall that $X$ is said to have maximal rank if for all integers $t \geq 0$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, i.e. if for all integers $t \geq 0$ the restriction map $\rho_{X,t} : H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \rightarrow H^0(\mathcal{O}_X(t))$ is either injective or surjective. In many interesting cases the integer $h^0(\mathcal{O}_X(t))$ is known. Hence if we also know that $X$ has maximal rank, then we know that $h^0(\mathcal{I}_X(t)) = \max\{0, (-\frac{r+t}{r}) - h^0(\mathcal{O}_X(t))\}$, i.e. we know the number of conditions that $X$ imposes to the set of all degree $t$ hypersurfaces of $\mathbb{P}^r$ and this number is the “expected” one.

For all closed subschemes $Y \subset \mathbb{P}^r$ with $\dim(Y) \leq r - 2$ and all integers $d \geq 0$ let $R(r,Y,d)$ (resp. $L(r,Y,d)$) be the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoint union of $Y$ and a smooth rational $d$ curve (resp. $Y$ and the disjoint union of $d$ lines); our convention says that the $\emptyset$ is the only rational curve of degree 0. For all integers $g \geq 0$, $r \geq 3$ and $d \geq 2g + r$ let $Z(r,Y,d,g)$ denote the set of all disjoint unions $X \subset \mathbb{P}^r$ of $Y$ and a smooth curve of degree $d$ and genus $g$. Set $Z(r,d) := R(r,\emptyset,d)$, $L(r,d) := L(r,\emptyset,d)$ and $Z(r,d,g) := Z(r,\emptyset,d,g)$. The conditions “$r \geq 3$ and $d \geq 2g + r$” implies that the set $Z(r,d,g)$ of all smooth curves $E \subset \mathbb{P}^r$ with degree $d$ and genus $g$ is parametrized by an irreducible variety and hence we are allowed to say the sentence “the general element of $E \in Z(r,Y,d,g)$”. Fix any $E \in Z(r,d,g)$. Since $d \geq 2g + r$, we have $h^1(\mathcal{O}_E(1)) = 0$ and $h^0(\mathcal{O}_E(t)) = td + 1 - g$ for all $t > 0$. If either $E$ is general or $r = 3$, then $E$ spans $\mathbb{P}^r$. We even know the Hilbert function of a general $E \in Z(r,d,g)$ ([16], [5], [6], [7]) ($E$ has maximal rank), but we will not need it.

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because we will always have $d \gg 2g + r$ and for these triples $(r, d, g)$ the maximal rank of $E$ is easily proved.

Let $\text{Hilb}^p(\mathbb{P}^r)$ be the set of all closed subschemes $Y \subset \mathbb{P}^r$ with $p$ as Hilbert polynomial. Grothendieck’s proved that $\text{Hilb}^p(\mathbb{P}^r)$ is (in a natural way) a projective algebraic scheme. The regularity of all $Y \in \text{Hilb}^p(\mathbb{P}^r)$ is bounded ([18, Theorem at page 101]). G. Gotzmann found the optimal upper bound for the regularity. By Gotzmann’s Regularity theorem ([13, 78, Satz (2.9)] and [17, Lemma C.23]), there exists a number $x$ only depending on $p$ and easily obtained from the coefficients of $p$, called the Gotzmann number of $p$, for which the ideal sheaf $\mathcal{I}_Y$ of each scheme $Y \in \text{Hilb}^p(\mathbb{P}^r)$ is $x$-regular (in the sense of Castelnuovo-Mumford regularity [18, Definition at page 99]). In particular there is an integer $d_1$ such that $h^1(\mathcal{I}_Y(t)) = 0$ and $h^i(\mathcal{O}_Y(t)) = 0$ for all $i > 0$, all $t \geq d_1$ and all $Y \in \text{Hilb}^p(\mathbb{P}^r)$. In particular we have $h^0(\mathcal{O}_Y(t)) = p(t)$ for every integer $t \geq d_1$. Hence for each $Y \in \text{Hilb}^p(\mathbb{P}^r)$, each integer $d > 0$, each integer $t \geq d_1$, each $A \in \mathcal{Z}(r, Y, d)$ and $B \in \mathcal{L}(r, Y, d)$ we have $h^0(\mathcal{O}_A(t)) = p(t) + td + 1$ and $h^0(\mathcal{O}_B(t)) = p(t) + (t + 1)d$.

In this paper we prove the following results (the case $\deg(p) = 0$ of Theorem 1.1 improves [3]).

**Theorem 1.1.** Fix integers $r, m$ such that $m \geq 0$, $r \geq 4$ and $r \geq m + 2$. Let $p$ be a degree $m$ admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with $p$ as its Hilbert polynomial. Then there is an integer $d_0$ (depending only on $r$ and $p$) with the following property. Fix any integer $d \geq d_0$. Let $X \subset \mathbb{P}^r$ be a general union of $Y$ and $d$ lines. Then $X$ has maximal rank.

**Theorem 1.2.** Fix integers $r, m$ such that $m \geq 0$, $r \geq 3$ and $r \geq m + 2$. Let $p$ be a degree $m$ admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with $p$ as its Hilbert polynomial. Then there is an integer $d_0$ (depending only on $r$ and $p$) with the following property. Fix any integer $d \geq d_0$. Let $X \subset \mathbb{P}^r$ be a general union of $Y$ and a general degree $d$ rational curve of $\mathbb{P}^r$. Then $X$ has maximal rank.

**Theorem 1.3.** Fix integers $r, m, g$ such that $m \geq 0$, $r \geq 3$, $g \geq 0$, and $r \geq m + 2$. Let $p$ be a degree $m$ admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with $p(t)$ as its Hilbert polynomial. Then there is an integer $d_0 \geq 2g + r$ (depending only on $r$, $g$ and $p$) with the following property. Fix any integer $d \geq d_0$. Let $X \subset \mathbb{P}^r$ be a general union of $Y$ and a general degree $d$ smooth curve $E \subset \mathbb{P}^r$ with genus $g$. Then $X$ has maximal rank.

Of course, Theorem 1.1 is a particular case of Theorem 1.3. We point out that statements like Theorem 1.3 may easily proved using statements like Theorem 1.1 if in Theorem 1.3 we don’t look for a “good” upper bound for the integer $d_0$.

Non-asymptotic results are harder (see [1], [2] and [4] for some of them).

2. **Preliminaries**

We use the convention that $\emptyset$ is the only scheme with dimension $-1$. Let $\mathbb{K}$ be an algebraically closed base field. For any integral curve $E \subset \mathbb{P}^n$, $n \geq 3$, with $\deg(E) \geq 2$ (resp. $\deg(E) \geq 1$) a 2-secant line (resp. a 1-secant line) $D$ of $E$
is a line \( D \subset \mathbb{P}^n \) such that \( \mathcal{H}(E \cap D) = 2 \) (resp. \( \mathcal{H}(E \cap D) = 1 \)), \( D \cap E \subset E_{\text{reg}} \) and at each \( P \in E \cap D \) the tangent line \( T_P E \) of \( E \) at \( P \) is different from \( D \). If \( E \) is irreducible, then the set of all 2-secant lines of \( E \) is parametrized by an irreducible variety of dimension 2. This variety is non-empty in characteristic zero and in most cases in positive characteristic (i.e. if \( E \) is not very strange in the sense of [21]; this is always true for smooth curves and in particular for the general rational curve of degree \( d \) of \( \mathbb{P}^n \)). Let \( p'' \) be the Hilbert polynomial of the scheme \( Y \cap H \cap M \) in \( M \); we have \( p''(x) = p(x) - 2p(x - 1) + p(x - 2) \) and in particular \( p'' \equiv 0 \) if \( \dim(Y) \leq 1 \).

We usually write “rational curve” instead of “smooth rational curve”. Each \( Z(r, d), d > 0, \) is an irreducible variety. If \( d \geq r \), then a general \( X \in Z(r, d) \) is non-degenerate. Let \( Z'(r, d) \) be the closure of \( Z'(r, d) \) in the Hilbert scheme of \( \mathbb{P}^r \). For any integer \( e > 0 \), any closed subscheme \( W \subset \mathbb{P}^r \) and any \( T \subset |O_{\mathbb{P}^r}(e)| \) let \( \text{Res}_T(W) \) denote the residual scheme of \( W \) with respect to \( T \), i.e. the closed subscheme of \( \mathbb{P}^r \) with \( \mathcal{I}_W : \mathcal{I}_T \) as its ideal sheaf. For each \( t \in \mathbb{Z} \) we have the following exact sequence, which is often called the Castelnuovo’s sequence:

\[
0 \rightarrow \mathcal{I}_{\text{Res}_T(W)}(t - e) \rightarrow \mathcal{I}_W(t) \rightarrow \mathcal{I}_{W \cap T, T}(t) \rightarrow 0
\]

**Remark 2.1.** Let \( Y \subset \mathbb{P}^r \) be a closed subscheme with \( \dim(Y) < r \). Fix an integer \( e \in \{1, 2\} \) and take a general \( T \subset |O_{\mathbb{P}^r}(e)| \). Since \( O_{\mathbb{P}^r}(e) \) is very ample and \( T \) is general, \( T \) does not contain the support of any component of \( Y \), not even any embedded component, by the prime avoidance lemma. Hence \( \text{Res}_T(Y) = Y \).

**Remark 2.2.** Let \( U \subset \mathbb{P}^n, n \geq 3, \) be a reducible conic. Call \( P \) the singular point of \( U \). Let \( V \subset \mathbb{P}^n \) be any 3-dimensional linear subspace containing \( U \). Let \( Z \) be the closed subscheme of \( V \) with \( (\mathcal{I}_{P, V})^2 \) as its ideal sheaf. The scheme \( U \cup Z \) is called a sundial ([10]). \( U \) is the support of \( U \cup Z \), because \( U = (U \cup Z)_{\text{red}} \). \( P \) is the support of the nilradical of the sheaf \( O_{U \cup Z} \). The scheme \( U \cup Z \) is a flat limit of a family of pairs of disjoint lines ([15, Example 2.1.1], [10]).

For all integer-valued polynomials \( p \in \mathbb{Q}[x] \) and all positive integers \( r, t \) define the integers \( a_{r, p, t}, b_{r, p, t}, c_{r, p, t}, \) and \( d_{r, p, t} \) by the relations

\[
p(t) + ta_{r, p, t} + 1 + b_{r, p, t} = \binom{r + t}{t}, 0 \leq b_{r, h, t} \leq t - 1 \tag{2.1}
\]

\[
p(t) + (t + 1)c_{r, p, t} + d_{r, p, t} = \binom{r + t}{t}, 0 \leq d_{r, b, t} \leq t \tag{2.2}
\]

From (2.1) and (2.2) we get the following relations.

\[
p'(t) + a_{r, p, t - 1} + t(a_{r, p, t} - a_{r, p, t - 1}) + b_{r, p, t} - b_{r, p, t - 1} = \binom{r + t - 1}{r - 1} \tag{2.3}
\]

\[
p'(t) + c_{r, p, t - 1} + (t + 1)(c_{r, p, t} - c_{r, p, t - 1}) + d_{r, p, t} - d_{r, p, t - 1} = \binom{r + t - 1}{r - 1} \tag{2.4}
\]

If \( t \) is at least the Gotzmann’s number, \( d_1, \) of \( p \), then \( p(t) \geq 0 \) and equality holds if and only if \( p(t) \equiv 0 \). Hence \( c_{r, p, t} \leq c_{r, 0, t} \) and \( a_{r, p, t} \leq a_{r, 0, t} \) if \( t \geq d_1 \). We also
have $p'(t) = p(t) - p(t - 1) \geq 0$ if $t > d_1$. From (2.3) and (2.4) we also get

$$a_{r,p,t} - a_{r,p,t-1} \geq a_{r,0,t} - a_{r,0,t-1} - 1 \quad \text{and} \quad c_{r,p,t} - c_{r,p,t-1} \geq a_{r,0,t} - a_{r,0,t-1} - 1 \quad \text{if} \quad t > d_1,$$

because $b_{r,p,t} - b_{r,p,t-1} - b_{r,0,t} + b_{r,p,t-1} < 2t$ and $d_{r,p,t} - d_{r,p,t-1} - d_{r,0,t} + d_{r,p,t-1} < 2t + 2$.

**Remark 2.3.** Fix integers $d \geq r \geq 3$. Fix a hyperplane $H \subset \mathbb{P}^r$. If $r = 3$, then fix a smooth quadric surface $Q$. Let $C \subset \mathbb{P}^r$ be a smooth rational curve of degree $d$. Let $N_C$ be the normal bundle of $C$ in $\mathbb{P}^r$. $N_C$ is a rank $r - 1$ vector bundle on $C$ and $\deg(N_C) = (r + 1)d - 2$. Since $C \cong \mathbb{P}^1$, $N_C$ is isomorphic to a direct sum of $r - 1$ line bundles of degree $a_1 \geq \cdots \geq a_{r-1}$. If $C$ is general, then $N_C$ is rigid, i.e., $a_1 - a_r \leq 1$, i.e., $a_1 = \left[(r+1)d - 2)/(r-1)\right]$ and $a_{r-1} = \left[((r+1)d - 2)/(r-1)\right]$, etc. Hence $h^1(C, N_C(-1)) = 0$. Hence for a general $S \subset H$ with $\sharp(S) = d$ there is $A \in Z(r, d)$ with $S = A \cap H$ ([19, Theorem 1.5]). Now assume $r = 3$. In this case we get $a_1 = a_2 = 2d - 1$ and hence $h^1(C, N_C(-2)) = h^0(C, N_C(-2)) = 0$ for a general $C \subset Z(3,d)$. Hence for a general $B \subset Q$ with $\sharp(B) = 2d$ there is $A \in Z(3,d)$ such that $B = A \cap C$ ([19, Theorem 1.5]).

### 3. A reformulation of the theorems

It is easy to check that to prove Theorem 1.1 (resp. Theorem 1.2, resp. Theorem 1.3) for the integer $r$ and the scheme $Y$ with Hilbert polynomial $p$ it is sufficient to prove the following statements.

**Lemma 3.1.** Fix $r$ and the admissible polynomial $p$. There is an integer $t_0$ with the following property. Fix any integer $t \geq t_0$ and any integer $d \geq 0$. Let $E \subset \mathbb{P}^r$ be a general union of $d$ lines. Then either $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y \cup E}(t)) = 0$.

**Lemma 3.2.** Fix $r$ and the admissible polynomial $p$. There is an integer $t_0$ with the following property. Fix any integer $t \geq t_0$ and any integer $d \geq 0$. Let $E \subset \mathbb{P}^r$ be a general rational curve of degree $d$. Then either $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y \cup E}(t)) = 0$.

**Lemma 3.3.** Fix $r$, the admissible polynomial $p$ and $g \in \mathbb{N}$. There is an integer $t_0$ with the following property. Fix any integer $t \geq t_0$ and any integer $d \geq 2g + r$. Let $E \subset \mathbb{P}^r$ be a general smooth curve of degree $d$ and genus $g$. Then either $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y \cup E}(t)) = 0$.

**Remark 3.4.** Fix $r$ and the admissible polynomial $p$. In the set-up of Theorem 1.3 also fix an integer $g \geq 0$. Suppose you want to prove Lemma 3.1 (resp. Lemma 3.2, resp. Lemma 3.3) for the integer $t$. It is sufficient to find a disjoint union $E \subset \mathbb{P}^r$ of $c_{r,p,t}$ lines (resp. a rational curve $E \subset \mathbb{P}^r$ of degree $a_{r,p,t}$, resp. a smooth genus curve $E \subset \mathbb{P}^r$ of degree $a_{r,p,t,g}$) and a disjoint union $E' \subset \mathbb{P}^r$ of $c_{r,p,t+1}$ lines (resp. a rational curve $E' \subset \mathbb{P}^r$ of degree $a_{r,p,t+1}$, resp. a smooth genus curve $E' \subset \mathbb{P}^r$ of degree $a_{r,p,t+1} + 1$) such that $E \cap Y = E' \cap Y = \emptyset$, $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$ and $h^0(\mathcal{I}_{Y \cup E}(t)) = 0$. If $d_{r,p,t} = 0$ (resp. $b_{r,b,t} = 0$, resp. $b_{r,p,t,g} = 0$), then we do not need to find $E'$. By the semicontinuity theorem for the cohomology ([14, III.12.8]) instead of finding $E$ or $E'$ it is sufficient to find a scheme $G$ or $G'$ in the closure of $L(r, d)$ or of $Z(r, d)$ or of $Z(r, d, g)$ in the Hilbert scheme of $\mathbb{P}^r$ such that $Y \cap G = Y \cap G' = \emptyset$, $h^1(\mathcal{I}_{Y \cup G}(t)) = 0$ and $h^0(\mathcal{I}_{Y \cup G'}(t)) = 0$.
4. PROOFS OF THEOREMS 1.2 AND 1.3 IN $\mathbb{P}^3$

Fix a polynomial $p \in \mathbb{Z}[x]$ with $\deg(p) \leq 1$ and write $p(x) = a_0x + a_1$. Since a general rational curve has maximal rank ([16]), we may assume $p \neq 0$, i.e. $(a_0,a_1) \neq (0,0)$. If $a_0 = 0$, then $p$ is admissible if and only if $a_1 \geq 0$; $a_1$ is the Gotzmann’s number of the constant polynomial $p$. If $a_1 \neq 0$, then $p$ is admissible if and only if $a_0 > 0$ and $a_1 \geq -a_0(a_0-3)/2$; in this case $a_0(a_0-1)/2 + a_1$ is the Gotzmann’s number of $p$ ([12, Example 1.2]). Fix $Y \in \text{Hilb}^p(\mathbb{P}^3)$.

From (2.1) and (2.2) we get

$$2a_0 + 2a_{3,p,t-2} + t(a_{3,p,t} - a_{3,p,t-2}) + b_{3,p,t} - b_{3,p,t-2} = (t + 1)^2 \tag{4.1}$$

From (4.1) we get

$$2a_0 + 2c_{3,p,t-2} + (t+1)(c_{3,p,t} - c_{3,p,t-2}) + d_{3,p,t} - d_{3,p,t-2} = (t + 1)^2 \tag{4.2}$$

Fix an integer $t \geq d_1 + 2$.

**Remark 4.1.** We have $c_{3,0,t} = (t + 3)(t + 2)/6$, $d_{3,0,t} = 0$ if $t \equiv 0,1$ (mod 3) and $c_{3,0,t} = (t^2 + 5t + 4)/6$, $d_{3,0,t} = (t + 1)/3$ if $t \equiv 2$ (mod 3). First assume $a_0 \geq a_1$. In this case if $t \gg 0$, say $t > 3(a_0 - a_1)$, we have $c_{3,p,t} = (t + 3)(t + 2)/6 - a_0$, $d_{3,p,t} = a_0 - a_1$ if $t \equiv 0,1$ (mod 3) and $c_{3,p,t} = (t^2 + 5t + 4)/6 - a_0$, $d_{3,p,t} = (t + 1)/3 + a_0 - a_1$ if $t = 0,1$ (mod 3). Now assume $a_0 < a_1$. In this case if $t \gg 0$, say $t > 3(a_1 - a_0)$, we have $c_{3,p,t} = (t + 3)(t + 2)/6 - a_0 - 1$, $d_{3,p,t} = t + 1 + a_0 - a_1$ if $t \equiv 0,1$ (mod 3) and $c_{3,p,t} = (t^2 + 5t + 4)/6 - a_0$, $d_{3,p,t} = (t + 1)/3 + a_0 - a_1$ if $t \equiv 2$ (mod 3).

**Remark 4.2.** Write $t = 6k + e$ with $0 \leq e \leq 5$. We have $a_{3,0,6k} = 6k^2 + 6k + 1$, $b_{3,0,6k} = 5k$, $a_{3,0,6k+1} = 6k^2 + 8k + 4$, $b_{3,0,6k+1} = 0$, $a_{3,0,6k+2} = 6k^2 + 10k + 6$, $b_{3,0,6k+2} = 3k + 1$, $a_{3,0,6k+3} = 6k^2 + 12k + 6$, $b_{3,0,6k+3} = 2k + 1$, $a_{3,0,6k+4} = 6k^2 + 14k + 8$, $b_{3,0,6k+4} = 3k + 2$, $a_{3,0,6k+5} = 6k^2 + 16k + 11$, $b_{3,0,6k+5} = 0$ ([16]). First assume $a_1 > 0$ and $t$ very large, say $t > 7a_1$. We have $a_{3,0,6k} = 6k^2 + 6k + 1 - a_0$, $b_{3,0,6k} = 5k - a_1$, $a_{3,0,6k+1} = 6k^2 + 8k + 3 - a_0$, $b_{3,0,6k+1} = 6k + 1 - a_1$, $a_{3,0,6k+2} = 6k^2 + 10k + 6 - a_0$, $b_{3,0,6k+2} = 3k + 1 - a_1$, $a_{3,0,6k+3} = 6k^2 + 12k + 6 - a_0$, $b_{3,0,6k+3} = 2k + 1 - a_1$, $a_{3,0,6k+4} = 6k^2 + 14k + 8 - a_0$, $b_{3,0,6k+4} = 3k + 2 - a_1$, $a_{3,0,6k+5} = 6k^2 + 16k + 10 - a_0$, $b_{3,0,6k+5} = 6k + 5 - a_1$. If $a_1 \leq 0$, then $a_{3,0,t} = a_{3,0,t} - a_0$ and $b_{3,0,t} = b_{3,0,t} + a_1$. Hence $a_{3,0,t} - a_{3,0,t} \geq 2 + t/2$ if, say, $t \geq 12 + \max\{d_1,12a_0\}$. In the set-up of Theorem 1.3 we get the same values just taking $a_1 - g$ instead of $a_1$.

Fix an integer $\rho$ such that $(\rho - 2)/4$ is at least the Gotzmann’s number of $Y$ and $t \equiv \rho$ (mod 2).

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface such that $Q \cap Y$ has dimension zero if $\dim(Y) = 1$ and it is empty if $\dim(Y) = 0$.

**Lemma 4.3.** Fix an integer $t \geq \rho$ such that $t - \rho$ is even. Let $E \subset \mathbb{P}^3$ be a general rational curve of degree $a_{3,0,t} - a_{3,0,\rho}$. Then $h^1(Y \cup E(t)) = 0$.

**Proof.** If $t = \rho$, then $E = \emptyset$ and hence the lemma is true. Hence we may assume $t \geq \rho + 2$ and that the lemma is true for the integer $t' := t - 2$. Let $E' \subset Q$ be a general rational curve of degree $a_{3,0,t-2} - a_{3,0,\rho}$. If $t \neq \rho + 2$, then $\deg(E') > 2$. Hence $E'$ spans $\mathbb{P}^3$ and $E' \cap Q$ is a general union of $2(a_{3,0,t-2} -$
Let \( \eta \) be an integer such that \( \eta \geq \rho \). If \( a_0 > 0 \) we also assume \( \eta > 12a_1 + \rho \) and \( \eta > 12a_0 + \rho \). Set \( a_{\eta, \eta} := a_{3,0,\eta} - a_{3,0,\rho} \) and \( b_{3,p,\eta,\eta} := 0 \). For all integers \( x \geq \eta + 2 \) such that \( x \equiv \eta \pmod{2} \) set \( a_{x,\eta} := \max\{0,(x - \eta)/2\} \). For all integers \( x \geq \eta \) the integers \( a_{3,p,x,\eta} \) and \( b_{3,p,x,\eta} \) by the relations

\[
p(x) + x a_{3,p,x,\eta} + 1 + b_{3,p,x,\eta} + a_{x,\eta} = \left( \frac{x + 3}{3} \right), \quad 0 \leq b_{3,p,x,\eta} \leq x - 1
\]

We have

\[
p(x) - p(x - 2) + 2a_{3,p,x-2,\eta} + x(a_{3,p,x,\eta} - a_{3,p,x-2,\eta}) + b_{3,p,x,\eta} - b_{3,p,x,\eta} + a_{x,\eta} - a_{x-2,\eta} = (x + 1)^2
\]

and \( -2 \leq a_{x,\eta} - a_{x-2,\eta} \leq 0 \) (more precisely, we have \( a_{x-2,\eta} \geq 0, a_{x,\eta} = 0 \) if \( a_{x-2,\eta} \leq 1 \) and \( a_{x,\eta} = a_{x-2,\eta} - 2 \) if \( a_{x-2,\eta} \geq 2 \).

**Lemma 4.4.** For all integers \( x \geq \eta \) with \( x \equiv \eta \pmod{2} \) we have \( a_{3,p,x,\eta} \leq a_{3,0,x} \). We have \( a_{3,p,x,\eta} = a_{3,p,x} \) and \( b_{3,p,x,\eta} = b_{3,p,x} \) for all \( x \geq \eta + 2a_{\eta,\eta} \).

**Proof.** The first inequality is true, because \( a_{x,\eta} \geq 0 \). The second inequality is true, because \( p(x) \geq 0 \), since \( x \geq \rho \). The last assertion is true by the definitions of the integers \( a_{3,p,x,\eta} \), \( a_{3,p,x} \), \( b_{3,p,x,\eta} \) and \( b_{3,p,x} \), because \( a_{x,\eta} = 0 \) for all \( x \geq \eta + 2a_{\eta,\eta} \).

**Lemma 4.5.** Assume \( x \geq \eta + 2 \). Then:

(a) \( a_{3,0,x} - a_{3,0,\rho} - (x - \eta)/2 \leq a_{3,p,x,\eta} \).

(b) \( a_{3,p,x,\eta} - a_{3,p,x-2,\eta} \geq a_{3,0,x} - a_{3,0,x-2} - 1 \geq 4 + x/2 \).

(c) \( a_{3,p,x,\eta} - a_{3,p,x-2,\eta} \leq x - 2 \).

(d) Assume \( a_0 > 0 \), i.e. assume \( \dim(Y) = 1 \). Then \( a_{3,p,x,\eta} - a_{3,p,x-2,\eta} \leq x - 2a_0 - 2 \).

**Proof.** Since \( a_{3,p,\eta,\eta} = a_{3,0,x} - a_{3,0,\rho} \), the equality in part (a) is true if \( x = \eta \). Now assume \( x \geq \eta + 2 \). Since \( x \geq 10a_0 \) and \( a_{3,0,x-2,\eta} \leq a_{3,0,x} \), (4.2) first with \( p \) and \( \eta \) and then with \( p \equiv 0 \) and \( \eta = 0 \) gives \( a_{3,p,x,\eta} - a_{3,p,x-2,\eta} \geq a_{3,0,x} - a_{3,0,x-2} - 1 \), concluding the proof of part (a) and proving the first inequality of part (b). The
second inequality in part (b) is true by the explicit values of the integers \( a_{3,x,0} \) and \( a_{3,x,-2} \) ([5, III.2]).

Part (c) follows from (4.2) and the relations \( a_{3,p,x-2,\eta} \geq x, p(x) - p(x - 2) = 2a_0, a_{x,\eta} - a_{x-2,\eta} \geq -1, b_{3,p,x-2,\eta} \geq 0 \) and \( b_{3,p,x,\eta} \leq x - 1 \).

Now assume \( a_0 > 0 \). From (a) we get \( a_{3,p,x-2,\eta} \geq 2xa_0 + 2 \), by our restrictions on \( \eta \). Hence part (d) follows from (4.2), because \( p(x) - p(x - 2) = 2a_0 \). □

Among the many papers concerning the postulation of curves in \( \mathbb{P}^3 \) there is [5] which contains one statement (called \( R(n, g) \) in [5]), which we are able to adapt to our need (see [5, IV.2 and Lemmas VI.14, VI.5]). For all integers \( z > 0, r \geq 3 \) let \( Z'(r, z) \) be the closure in the Hilbert scheme of \( \mathbb{P}^r \) of the set of all rational curves of degree \( z \). It is well-known that if \( A \in Z'(r, z) \) and \( D \) is a 1-secant line of \( A \), then \( A \cup D \in Z'(r, z + 1) \) ([16], [5], [23]).

We have \( a_{3,p,0,\eta} = a_{3,0,\eta} - a_{3,0,p} \) and \( b_{3,0,\eta,\eta} = 0 \). For all integers \( x \geq \eta + 2 \) with \( x \equiv \eta \) (mod 2) consider the following assertion \( R_x \):

\( R_x \), \( x \geq \eta + 2 \), \( x \equiv \eta \) (mod 2): For every integer \( y \) such that \( 0 \leq 2y \leq b_{3,p,x,\eta} \) there is a quintuple \((A, D, D', S, S')\) with the following properties:

1. \( A \in Z'(3, a_{3,p,x,\eta}) \) and \( A \) intersects transversally \( Q \);
2. \( D \) and \( D' \) are disjoint lines 1-secant to \( A \) and contained in \( Q \), \( Y \cap (D \cup D') = \emptyset \);
3. \( S \) and \( S' \) are finite sets, \( S \subset D \setminus D \cap Y \), \( S' \subset D' \setminus D' \cap Y \); \( \sharp(S) = b_{3,p,x,\eta} - y \), \( \sharp(S') = y \); if \( \pi : Q \to D \) is the natural projection, then \( \pi(S') \) is contained in \( S \cap (Y \cap (Q \setminus D \cup D')) \);
4. \( A = C \cup D_1 \cup \cdots \cup D_{2z} \), \( z := a_{3,p,x,\eta} - a_{3,p,x-2,\eta} \), with \( C \) a smooth rational curve of degree \( a_{3,p,x-2} \); each \( D_i \) a 1-secant line of \( C \) and \( D_i \cap D_j = \emptyset \) for all \( i \neq j \);
5. \( h^1(T_{Y \cup A \cup S \cup S'}(x)) = 0 \).

Since \( p(x) + xa_{3,p,x,\eta} + 1 + b_{3,p,x,\eta} + a_{x,\eta} = \binom{x+3}{3} \) and \( \sharp(S \cup S') = b_{3,p,x,\eta} \), the last condition of \( R_x \) is equivalent to the condition \( h^0(T_{Y \cup A \cup S \cup S'}(x)) = a_{x,\eta} \).

**Lemma 4.6.** \( R_{q+2} \) is true.

**Proof.** Fix an integer \( y \) such that \( 0 \leq 2y \leq b_{3,p,x+2,\eta} \). By part (a) of Lemma 4.5 there is a smooth rational curve \( B \subset \mathbb{P}^3 \) such that \( \deg(B) = a_{3,p,0,\eta} \geq \eta \), \( B \cap Y = \emptyset \) and \( h^1(T_{Y \cup B}(\eta)) = 0 \), i.e. \( h^0(T_{Y \cap B}(\eta)) = a_{\eta,\eta} \). Hence \( h^1(T_{Y \cup B \cup U}(\eta)) = 0 \) for a general \( U \subset \mathbb{P}^3 \) such that \( \sharp(U) = a_{\eta,\eta} \). Since \( U \) is general, we have \( (Y \cap B) \cap U = \emptyset \). Fix \( P \in U \). For a general set \( U \) we may assume the existence of a line \( R \) such that \( P \in R \), \( R \cap U = \{ P \} \) and \( R \) is 1-secant to \( B \). Hence the general line through \( P \) and intersecting \( B \) is 1-secant to \( B \). Take a general such a line \( R' \). A general line of \( \mathbb{P}^3 \) is disjoint from \( Y \). Hence for a general \( B \) and a general \( U \) we may also assume that \( (R \cup R') \cap Y = \emptyset \). There is a smooth quadric surface \( Q' \) such that \( R \cup R' \subset Q' \). Call \( Q \) a general smooth quadric containing \( R \cup R' \). For general \( U \) we may also assume that either \( Y \cap Q = \emptyset \) (case \( \dim(Y) = 0 \)) or that \( \dim(Y \cap Q) = 0 \) (case \( \dim(Y) = 1 \)). First deforming \( U \) and then deforming \( B \) among the rational curves containing the points \( R \cap B \) and \( R' \cap B \) (both with fixed \( R, R', Q \)) we may assume that \( B \) is transversal to \( Q \) and that \( B \cap (Q \setminus (R \cup R')) \) is formed by \( 2a_{3,p,0,\eta} - 2 \) general points of \( Q \) (Remark 2.3). Call \( |O_Q(0,1)| \) the
ruling of $Q$ containing $R$. Set $Z := 2P \subset \mathbb{P}^3$. The scheme $R \cup R' \cup Z$ is a sundial (Remark 2.2). Since $R \cup R' \cup Z$ is a flat limit of a family of two disjoint lines, one of them containing the point $R \cap B$ and the other one the point $R' \cap B$, we have $B \cup R \cup R' \cup Z \in Z'(3, a_{3,p,\eta,\eta} + 2)$. Part (a) of Lemma 4.5 gives $a_{3,p,\eta,\eta,\eta} \geq 2 \Leftrightarrow a_{3,p,\eta,\eta,\eta} \geq 2$. Let $R_1, \ldots, R_z, z := a_{3,p,\eta,\eta,\eta} - 1$, be general lines of type $(0,1)$ on $Q$. Since $R_i \cap R = \emptyset$ for all $i \neq j, R_i$ is 1-secant to $R$ and $R_i \cap B = \emptyset$ for all $i$, we have $W := B \cup R \cup R' \cup Z \cup R_1 \cup \cdots \cup R_z \in Z'(3, a_{3,p,\eta,\eta,\eta})$. Set $F := R \cup R' \cup Z \cup R_1 \cup \cdots \cup R_z$. For general lines $R_i$ we have $W \cap Y = \emptyset$. Fix lines $D, D'$ of $Q$ of type $(0,1)$ containing exactly a point of $B \cap (Q \setminus R \cup R')$, $D \neq D'$, and call $\pi : Q \to D$ the projection. For general $B$ we may assume that $\pi_{B \cap Q}$ is injective and that $\pi(B \cap Q) \cap \pi(Q \cap Y) = \emptyset$. Fix $S' \subset D'$ such that $\pi(S') \subset \pi(B \cap (Q \cap R))$ and $\pi(S') = \emptyset$ (this is possible, because $\pi(B \cap Q) = 2 \deg(B) \geq y + 2$. Fix $S \subset D$ such that $\pi(S) = b_{i,\eta,\eta,\eta} - 2$ and $\pi(S') \subseteq S$ (notice that $y$ of the points of $S$ are fixed; we may assume that the other points of $S$ are general in $D$). Since $B \cup W$ is a flat limit of a family of curves which are unions of $B$ and $a_{3,p,\eta,\eta,\eta}$ lines, to prove $R_{\eta,\eta,\eta}$ it is sufficient to prove that $h^1(\mathcal{I}_{Y \cup B \cup W \cup S}((\eta,2)) = 0$. Since Res$_Q(Y \cup B \cup W \cup S) = Y \cup B \cup \{P\}$, we have $h^1(\mathcal{I}_{\text{Res}_Q(Y \cup B \cup W \cup S)}((\eta,2)) = 0$, it is sufficient to check that $h^1(Q, \mathcal{I}_{Y \cap (Y \cup B \cup W \cup S)}((\eta,2))) = 0$. By (4.2) we have $h^0(\mathcal{O}_{Q \cap (Y \cup B \cup W \cup S)})((\eta + 2)) = (\eta + 3)^2 = h^0(\mathcal{O}_Q((\eta,2)))^2$. Hence it is sufficient to prove that $h^1(Q, \mathcal{I}_{Y \cap Q \cap Y \cup Q \cup Q \cup Q \cap W}((\eta,2))) = 0$. We first need to check that $h^1(Q, \mathcal{I}_{Y \cap (Y \cap Q)}((\eta,2))) = 0$. This true by part (d) of Lemma 4.5, because $\deg(Y \cap Q) = 2a_\eta$. We also need to check that $h^1(Q, \mathcal{I}_{Y \cap Q \cap Q \cup Q \cup Q \cap W}((\eta,2))) = 0$; this is true because $\pi(S) \leq \eta + 1$ and in parts (c) and (d) of Lemma 4.5 we have the term $-2$ in the right hand sides of the inequalities. \hfill \Box

Lemma 4.7. $R_x$ is true for all integers $x \geq \eta + 2$

Proof. By Lemma 4.6 we may assume $x \geq \eta + 4$ and that $R_{x-2}$ is true. Fix an integer $y$ such that $0 \leq 2y \leq b_{3,p,\eta,\eta}$.

(a) In this step we assume $b_{3,p,\eta,\eta,\eta} \geq a_{3,p,\eta,\eta} - a_{3,p,\eta,\eta}$. This is the case handled in [5, Lemma VI.4]; here we use part (b) of Lemma 4.5 (which substitutes [5, Lemma VII.5] and (when $\dim(Y) = 1$), part (d) Lemma 4.5 (which substitutes [5, Lemma VII.6]).

(b) Now assume $b_{3,p,\eta,\eta} < a_{3,p,\eta,\eta} - a_{3,p,\eta,\eta}$. This is the case handled in [5, Lemma VI.5] (it is easier than case (a)). \hfill \Box

Proof of Theorem 1.2 for $r = 3$ and the admissible polynomial $p$: We need to check Lemma 3.2 and Remark 3.4. We have $a_{x,\eta} = 0$ for all $x \geq \eta + 2a_{\eta,\eta}$. As quite standard in this set-up (e.g. as in [5, §3]), we may take as $t_0$ the maximum of the integers $\eta + 2a_{\eta,\eta}$ for the two congruence classes modulo 2 of the integer $\eta$. In this way we get the $h^0$-part of Remark 3.4. Now we check the $h^0$-part of Remark 3.4. If $b_{3,p,\eta} = 0$, then $h^i(\mathcal{I}_{Y \cup W}(t)) = 0, i = 0, 1$, for a general $W \in Z(3, a_{3,p,\eta})$ and hence the $h^0$-part of Remark 3.4 is true in this case. Now assume $b_{3,p,\eta} > 0$. We need to prove that $h^0(\mathcal{I}_{Y \cup U}(t)) = 0$ for a general $U \in Z(3, a_{3,p,\eta} + 1)$. First assume $b_{3,p,\eta} = a_{3,p,\eta} - a_{3,p,\eta} + 1$. Take $(A, D, D', S, S')$ with $y = 0$ and use that we get an $h^0(Q, \mathcal{I}_T(t)) = 0$ for a suitable $t$ whose one-dimensional part is the
union of a curve of type $(1,0)$ and $a_{3,p,t} - a_{3,p,t-2}$ lines of type $(0,1)$. Now assume $b_{3,p,t-2} > a_{3,p,t-2} + 2$. Take $(A, D, D', S, S')$ for $y = b_{3,p,t-2} - a_{3,p,t-2} + 1$. Then we add in $Q$ two lines of type $(1, 0)$ ($D$ and $D'$) and $a_{3,p,t} - a_{3,p,t-2} - 1$ lines of type $(0,1)$.

**Proof of Theorem 1.3 in $\mathbb{P}^3$:** We may assume $g > 0$. Fix $Y \subset \mathbb{P}^3$ with $p$ as its Hilbert polynomial. It is sufficient to prove Lemma 3.4 for $r = 3$ and the scheme $Y$. Take $\eta$ as in the proof of Theorem 1.2 for $Y$. Let $z_0$ be a positive integer such that $z_0 \geq \eta + 2 + 2g$, $a_{z_0, \eta} = 0$ and Lemma 3.3 is true for $Y$ and all $t \geq z_0$. Hence $a_{z, \eta} = 0$ for all $z \geq z_0$. We fix a large integer $t$ for which we want to prove Lemma 3.4 (we need at least $t \geq z_0 + 2g$). Increasing by one if necessary $z_0$ we may assume $t \equiv z_0 \pmod{2}$. Set $g(z_0) := 0$. For all integers $z \geq z_0 + 2$ such that $z \equiv z_0 \pmod{2}$ set $g(z) := \min\{g, g(z_0) + (z - z_0)/2\}$. Define the integers $a_{3,p,g,z,z_0}$ and $b_{3,p,g,z,z_0}$ by the relations

$$p(z) + a_{3,p,g,z,z_0} + 1 - g(z) + b_{3,p,g,z,z_0} = \left(\frac{z + 3}{3}\right)$$

If $z \geq z_0 + 2$ we have $0 \leq g(z) - g(z - 2) \leq 1$ and

$$2a_0 + 2a_{3,p,g,z,z_0} + z(a_{3,p,g,z,z_0} - a_{3,p,g,z-2,z_0}) + g(z - 2) - g(z) + b_{3,p,g,z,z_0} - b_{3,p,g,z,z_0}$$

Call $R_2(g)$ the following assertion:

$R_2(g)$, $z \geq z_0$, $z \equiv z_0 \pmod{2}$: For all integer $y$ such that $0 \leq 2y \leq b_{3,p,g,z,z_0}$ there is a quintuple $(A, D, D', S, S')$ with the following properties:

1. $A$ is a curve of degree $a_{3,p,g,z,z_0}$ and arithmetic genus $g(y)$, $A \cap Y = \emptyset$ and $A$ intersects transversally $Q$;
2. $D$ is a 2-secant line of $A$, $D'$ is a 1-secant to $A$, $D \cup D' \subset Q$, $D \cap D' = \emptyset$, $Y \cap (D \cup D') = \emptyset$;
3. $S$ and $S'$ are finite sets, $S \subset D \setminus D \cap Y$, $S' \subset D' \cap D' \cap Y$, $\sharp(S) = b_{3,p,g,z,z_0} - y$, $\sharp(S') = y$; if $\pi : Q \to D$ is the natural projection, then $\pi(S')$ is contained in $S \cap \pi(Y \cap (Q \setminus D \cup D'))$;
4. $A = C \cup D_1 \cup \cdots \cup D_w, w := a_{3,p,g,z,z_0} - a_{3,p,g,z-2,z_0}$, with $C$ a smooth curve of degree $a_{3,p,g,z-2,z_0}$ and genus $g(z)$, each $D_i$ a 1-secant line of $C$ and $D_i \cap D_j = \emptyset$ for all $i \neq j$;
5. $h^i(\mathcal{I}_{Y \cup \cup_{i \neq j}S_i}(z)) = 0, i = 0, 1$.

Since $a_{z,\eta} = a_{\eta, z} = 0$ and $0 \leq g(z) - g(z - 2) \leq 1$, it is easy to see that Lemma 4.5 holds for the integers $a_{3,p,g,z,z_0}$ and $b_{3,p,g,z,z_0}$; alternatively one Remark 2.3. Since $R_2(g)$ is true and the proof that $R_2(g) \implies R_2(g)$ is similar to the proof of Lemma 4.7, we get $R_2(g)$ for all $z \geq z_0$, $z \equiv z_0 \pmod{2}$ and then Lemma 3.3 using Remark 3.4. In [5, Lemma VI.1] the authors described how to increase at each step the arithmetic genus by the maximal amount possible for non-special curves: in [5, Lemmas VI.2 and VI.3] the authors described how to increases the genus by an unknown (non-maximal) value.

The next lemma will be used to prove the case $r = 4$ of Theorem 1.1. We will also use it in the proofs of Theorem 1.2 and 1.3 (Lemma 5.1), although for them a weaker form would be sufficient.
Lemma 4.8. Fix an admissible polynomial $p$ with $\deg(p) \leq 1$. Then there is an integer $\psi > 0$ with the following property. Fix $Y \subset \mathbb{P}^3$ with $p$ as its Hilbert polynomial and fix any integer $t \geq \psi$. Let $A \subset \mathbb{P}^3$ be a general union of $c_{3,0,\psi}$ disjoint lines. Then $h^1(\mathcal{I}_{Y \cup A}(t)) = 0$.

Proof. Take a large integer $\psi$ bigger than $4$ times the Gotzmann's number of $p$ and such that for which for all integers $y \geq \psi - 2$ the integers $c_{3,p,y}$ and $d_{3,p,y}$ are as in Remark 5.3. Notice that for all integers $t \geq \psi - 1$ we get a statement similar to Lemma 4.5 for the integers $c_{3,p,t}$ and $d_{3,p,t}$. For all integers $t \geq \psi - 1$ call $A(t)$ the statement of the lemma if $t \equiv \psi \pmod{2}$. Notice that if $A(t)$ is true for some $t \geq \psi$ with $t - \psi$ odd, then Lemma 4.8 is true for the integer $t$, because $c_{3,0,t} - c_{3,0,\psi} \geq c_{3,0,t} - c_{3,0,\psi - 1}$. Fix $a \in \{\psi - 1, \psi\}$ and set $\alpha(t) := \left(\frac{t+3}{3}\right) - [p(t) + (t + 1)(c_{3,0,t} - c_{3,0,a})] = (t + 1)c_{3,0,a} + d_{3,0,t} - p(t)$. For large $a$ we have $\alpha(t) \geq \alpha(t - 2)$, because $p(t) - p(t - 2) + |d_{3,0,t} - d_{3,0,t-2}| \leq 2a_0 + t$. Hence we may prove that $A(t)$ implies $A(t + 2)$ just adding $c_{3,0,t+2} - c_{3,0,t-2}$ lines of type $(0,1)$ on $Q$, without even having to check the different case for the congruence classes of $t$ modulo 3 as in [15, §2].

5. Proofs in $\mathbb{P}^r$, $r \geq 4$

5.1. Proof of Theorem 1.2. In this subsection we also assume that Theorem 1.2 is true in $H \cong \mathbb{P}^{r-1}$ for the Hilbert polynomial $p'$ of $Y \cap H$ (this is possible, because the case $r = 3$ is proved in section 4). It is sufficient to prove Lemma 3.2 for $Y$ and the integer $r$. We assume that Lemma 3.2 is true in $H$ for the scheme $Y \cap H$ and call $t_1$ an integer such that for all integers $t \geq t_1$ and $d \geq 0$ either $h^0(\mathcal{I}_{Y \cap H}(t)) = 0$ or $h^1(\mathcal{I}_{Y \cap H}(t)) = 0$ for a general rational curve $W \subset H$ such that $\deg(W) = d$. If $r \geq 5$ we also assume that $t_1$ works in $\mathbb{P}^{r-2}$ for the admissible polynomial $p''$.

Lemma 5.1. Fix an integer $r \geq 4$ and an integer $t$. Assume $t \geq r - 3 + \rho$ and $t \geq \psi + 2$, where $\psi$ is the integer appearing on Lemma 4.8. Let $E \subset \mathbb{P}^r$ be a general union of $t$ lines. Then $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$.

Proof. Let $N \subset \mathbb{P}^r$ be a general 3-plane. Let $q$ be the Hilbert polynomial of the scheme $Y \cap N$. Since $t \geq \psi + 2$ and $c_{3,0,x} = \lfloor (x + 3)(x + 2)/6 \rfloor$ for all $x$, we have $t \leq c_{3,0,t} - c_{3,0,\psi}$. Hence $h^1(N, \mathcal{I}_{Y \cap N \cap E}(t)) = 0$ (Lemma 4.8). Take a general flag $H_0 \supset H_1 \supset \cdots \supset H_{r-3}$ with $\dim(H_i) = r - i$ for all $i$ and $N = H_{r-3}$. Each scheme $Y \cap H_i$ has as Hilbert polynomial the $i$-th difference function of $p$. Our assumption on $t$ implies $h^i(H_i, \mathcal{I}_{Y \cap H_i}(t - 1 - i)) = 0$. Using $r - 3$ Castelnuovo's sequences we
get $h^1(I_{Y\cup E}(t)) = 0$. Hence the lemma follows from the semicontinuity theorem for cohomology.

**Lemma 5.2.** Assume $r = 4$. Fix integers $t \geq t_1 + 4$ and $d,e$ such that $d > 0$, $0 \leq e \leq t - 1$ and $p'(t) + td + 1 + (t + 1)e + 3t \leq \binom{t+2}{3}$. Let $A \subset H$ be a general rational curve of degree $d$, $B \subset H$ be a general union of $e$ lines and $L \subset H$ a general 2-secant line of $A$. Then $h^1(H, I_{Y \cap H \cup A \cup B \cup L}(t)) = 0$.

**Proof.** Write $p'(x) = a_0x + a_1$.

(a) Assume that $d \geq a_3p', t-4$. Let $U \subset H$ be a general rational curve of degree $a_3p', t-4$. By the inductive assumption we have $h^0(H, I_{Y \cap H \cup U}(t - 4)) = 0$ and $h^0(I_{Y \cap H \cup U}(t - 4)) = b_{3p', t-4}$. Fix a general 2-secant line $D \subset H$ and let $Q \subset H$ be a general quadratic surface containing $D$. By the inductive assumption we have $h^0(Q, I_{Y \cap Q \cup U})(t - 4)) = 0$. Hence $h^1(I_{Y \cap Q \cup U}(t - 2)) = 0$. By Remark 2.3 we may assume that $U \cap Q$ is a general union of $2a_3p', t-4$ points of $Q$. As in [5, §VIII], we may control the postulation of $(U \cup D) \cap Q$. Hence $h^1(Q, I_{Y \cap Q}(t - 2)(-E)) = 0$. A Castelnuovo’s sequence gives $h^1(H, I_{Y \cap H \cup D \cup U \cup E}(t - 2)) = 0$. Then we deform $E$ to a general union $E' \subset H$ of $a_3p', t-2 - a_3p', t-4 - 2$. By the semicontinuity theorem we have $h^1(H, I_{Y \cap H \cup U \cup D \cup E}(t - 2)) = 0$. Take a general smooth quadric $Q'$. Set \[ e' := e - (a_3p', t-2 - a_3p', t-4 - 2) \text{ and } d' := d - a_3p', t-4. \] Since $a_3p', t-2 - a_3p', t-4 \geq t-3$ (part (b) Lemma 4.5), we have $e' \leq (a_3p', t-2 - a_3p', t-4 - 2) - t - 1 = a_3p', t-2 - a_3p', t-4 - 1$. Hence $h^0(Q, I_{Y \cap Q}(t - 2)) = 0$. We have $h^0(Q', I_{Y \cap Q \cup E}(t - 2)) = 0$. Hence $h^1(Q', I_{Y \cap Q \cup E}(t - 2)) = 0$. Since $E' \cap Q'$ is general in $Q'$ and $h^0(Q', I_{Y \cap Q}(t)(-E)) = 0$. We have $h^1(Q', I_{Y \cap Q \cup E}(t)) = 0$.

Now assume $e \leq a_3p', t-2 - a_3p', t-4 - 3$. We add in $Q$ the line $D$, general lines of type (0, 1) and \[ \min\{a_3p', t-2 - a_3p', t-4 - 2 - e, d - a_3p', t-4\} \] lines of type (0, 1) each of them meeting $U$ at one point (call $T$ the union of the latter $a_3p', t-2 - a_3p', t-4 - 2 - e$ lines). Before making the last step we smooth $U \cup T$ to a smooth rational curve $U_1$, and move $D$ during the smoothing to a 2-secant line of $U_1$.

(b) Now assume $d < a_3p', t-4$. Instead of $U$ we take a general rational curve $U' \subset H$ with degree $U' = d$. We again take as $D$ a general 2-secant of $U'$ and as $Q$ a general smooth quadric containing $D$. In $Q$ and $Q'$ we add no line intersecting $U'$, except $D$.

**Lemma 5.3.** Assume $r \geq 5$. Fix integers $t > t_1$ and $d,e$ such that $d > 0$, $0 \leq e \leq t - 1$ and $p'(t) + td + 1 + (t + 1)e + 3t \leq \binom{t+2}{3}$. Let $A \subset H$ be a general rational curve of degree $d$, $B \subset H$ be a general union of $e$ lines and $L \subset H$ a general 2-secant line of $A$. Then $h^1(H, I_{Y \cap H \cup A \cup B \cup L}(t)) = 0$. 

\[ \square \]
Proof. Let $M \subset H$ be a general hyperplane.

(a) Assume $d \geq a_{r-1,p',t-1}$. Let $W \subset H$ be a general rational curve of degree $a_{r-1,p',t-1}$. The inductive assumption gives $h^1(H, \mathcal{I}_{Y \cap H \cup W}(t-1)) = 0$ and $h^0(\mathcal{I}_{Y \cap H \cup W}(t-1)) = b_{r-1,p',t-1}$. For a general $W$ the set $W \cap M$ is a general subset of $M$ with cardinality $a_{r-1,p',t-1}$ (Remark 2.3). Let $E \subset M$ be a general union of a smooth rational curve $E_1$ of degree $d - a_{r-1,p',t-1}$ (if $d > a_{r-1,p',t-1}$, i.e. if $E_1 \neq \emptyset$, then impose that $E_1$ is general among the rational curves of $M$ containing a prescribed point, $P$, of $W \cap M$), a general union $E_2$ of $e$ lines of $M$ and a line $D$ spanned by two of the points of $W \cap M$. Since $W \cap M$ is general, $E$ may be considered as a general union of a general rational curve of degree $d - a_{r-1,p',t-1}$ and $e + 1$ disjoint lines. Recall that $p''(x) = p'(x) - p'(x - 1)$. Since

$$p''(t) + a_{r-1,p',t-1} + t(a_{r-1,p',t} - a_{r-1,p',t-1}) + b_{r-1,p',t} - b_{r-1,p',t-1} = \left(\frac{r + t - 2}{r - 2}\right),$$

we may use induction on $r$ to get first $h^1(M, \mathcal{I}_{Y \cap M \cup W}(t)) = 0$ (here the starting point of the induction is given by Lemma 5.2, at each inductive step we use that $a_{r-1,p',t} - a_{r-1,p',t-1} < a_{r-2,p',t}$ for large $t$) and then $h^1(M, \mathcal{I}_{Y \cap M \cup W \cap M \cup E}(t)) = 0$. A Castelnuovo’s sequence gives $h^1(H, \mathcal{I}_{Y \cap H \cup W \cup E}(t)) = 0$.

(b) Now assume $d < a_{r-1,p',t-1}$. Instead of $W$ we take a general rational curve $W' \subset H$ with $\deg(W') = d$ and take $E = E_2 \cup D \subset M$ with $E_2$ union of $e$ general lines and $D$ the line spanned by two of the points of $W' \cap M$. \hfill \Box

(i) To prove Theorem 1.2 we need to prove Lemma 3.2 and use Remark 3.4. We use induction on the integer $r$, the starting point of the induction being the case $r = 3$ proved in section 4. We easily see (as in the case $p = 0$, i.e. $Y = \emptyset$) the existence of an integer $t_2 \geq t_1$ such that for every $t \geq t_2$ we have $h^1(\mathcal{I}_Y \cup W(t)) = 0$ for a general $W \in Z(r, a_{r,0,t} - a_{r,0,t_1})$ (we use Remark 2.3 and Lemmas 5.2 and 5.3 ignoring the secant line $D$). Set $\eta := t_2$ and $a_{r,0,\eta} := \left(\frac{r + \eta}{r}\right) - p(\eta) - \eta(a_{r,0,t} - a_{r,0,t_1}) - 1$. We have $a_{r,\eta} = \eta a_{r,0,\eta} + b_{r,0,\eta} - p(\eta)$. For all integers $t \geq \eta$ set $a_{r,t} := \max\{0, a_{r,\eta} - t + \eta\}$. For all integers $t \geq \eta$ define the integers $a_{r,p,t,\eta}$ and $b_{r,p,t,\eta}$ by the relations

$$p(t) + t a_{r,p,t,\eta} + 1 + b_{r,p,t,\eta} + a_{r,p,t,\eta} = \left(\frac{r + t}{r}\right), \quad 0 \leq b_{r,p,t,\eta} \leq r - 1 \quad (5.1)$$

For all $t > \eta$ from (5.1) we get

$$p'(t) + a_{r,p,t-1,\eta} + t(a_{r,p,t,\eta} - a_{r,p,t-1,\eta}) + b_{r,p,t,\eta} - b_{r,p,t-1,\eta} = \left(\frac{r + t - 1}{r - 1}\right)$$

with either $a_{r,p,t-1,\eta} > 0$ and $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} = -1$ or $a_{r,p,t,\eta} = a_{r,p,t-1,\eta} = 0$.

Call $E_t$, $t > t_1$, the following assertion:

$E_t$, $t > t_1$: Let $W \subset \mathbb{P}^r$ be a general union of $b_{r,p,t}$ lines and a general rational curve of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Then $h^1(\mathcal{I}_Y \cup W(t)) = 0$.

Claim 1: For all $t > t_1$, $E_t$ is true.

Proof of Claim 1: We first need to check that $E_t$ is meaningful, i.e. that

$$a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1} \geq 0.$$
from $p$), because $b_{r,p,t_1} < t_1$ and $a_{r,0,t} \sim t^{-1}/r!$ if $t \gg 0$ and hence, $a_{r,0,t_1+1} - a_{r,0,t_1} \sim t^{-2}/r \cdot (r - 2)!$ if $t_1 \gg 0$. Now we check $E_{t_1+1}$. Let $U \subset \mathbb{P}^r$ be a general union of $t_1$ lines. By Lemma 5.1 we have $h^1(\mathcal{I}_{Y \cup U}(t_1)) = 0$. Let $F \subset H$ be a general rational curve of degree $a_{r,0,t_1+1} - a_{r,0,t_1} - b_{r,p,t_1+1}$. The inductive assumption on Theorem 1.2 (or a weaker statement like Lemma 4.8 for a rational curve) gives $h^1(H, \mathcal{I}_{Y \cap H \cup F}(t_1 + 1)) = 0$. Since $t a_{r,0,t_1} \geq 2 t_1$, (5.2) gives $p'(t_1 + 1) + t \left( a_{r,0,t_1+1} - a_{r,0,t_1} - b_{r,p,t_1} \right) + 1 \leq \left( \frac{r + t_1}{r - 1} \right)$. Since $E \cap H$ is general in $H$ and $b_{r,p,t_1+1} \leq t_1$, we get $h^1(H, \mathcal{I}_{Y \cap H \cup F}(t_1 + 1)) = 0$. Hence a Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y \cup U \cup F}(t_1 + 1)) = 0$.

Now assume $t \geq t_1 + 2$ and that $E_{t-1}$ is true. Lemmas 5.2 (if $r = 4$) and 5.3 (if $r > 4$) with $e = b_{r,p,t}$ show that $h^1(H, \mathcal{I}_{Y \cap H \cup F}(t)) = 0$ for a general union $F \subset H$ of $b_{r,p,t}$ lines and a rational curve $F_1$ of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Let $G \subset \mathbb{P}^r$ be a general union of $b_{r,p,t}$ lines and a general rational curve $G_1$ of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Set $G_2 := G \setminus G_1$. By Remark 2.3 we may assume that $G \cap H$ is a general union of $a_{r,0,t} - a_{r,0,t_1}$ points of $H$. Increasing if necessary $t_1$ we may assume that $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1} \geq r t$ (indeed $a_{r,0,t} \cong t^{-1}/r!$ for $t \gg 0$).

Since $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1} \geq r t$ and $E_1$ is general, it is easy to check that $E_1$ passes through at least $t$ general points of $H$ (alternatively use Remark 2.3 in the projective space $H$ and, calling $N_{E_1,H}$, the normal bundle of $E_1$ in $H$, that $h^1(E_1, N_{E_1,H}(\ell - G)) = 0$ for a general $G \subset E_1$ with $\ell(G) = t$). Hence without losing the generality of $F$ we may assume that $G_2 \cap H \subset F_1$, that $F_2 \cap G = \emptyset$ and that $G_1 \cap F_1$ is a single point. Hence $G \cup F$ is a union of an element of $Z'(r, a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t})$ and $b_{r,p,t}$ lines ($[6]$, [7]). Since $h^0(\mathcal{O}_{Y \cap H \cup F_1}(G \cap (H \setminus F))(t)) = p'(t) + t (a_{r,0,t} - a_{r,0,t_1} - b_{r,0,t_1} - 1) = p'(t) + t (a_{r,0,t} - a_{r,0,t_1} - b_{r,0,t_1} - 1)$.

Call $B_t$, $t \geq \eta$, the following assertion: $B_t$, $t \geq \eta$: Let $W \subset H$ be a general union of $b_{r,p,t,\eta}$ general lines and the general rational curve of degree $a_{r,p,t,\eta} - b_{r,p,t,\eta}$. Then $h^1(\mathcal{I}_{Y \cup W}(t)) = 0$.

**Claim 2:** For all $t \geq \eta$, $B_t$ is true.

**Proof of Claim 2:** $B_\eta$ is true, because $E_\eta$ is true. Assume $t > \eta$ and that $B_{t-1}$ is true. Take a general union $E \subset \mathbb{P}^r$ of $b_{r,p,t-1,\eta}$ general lines and the general rational curve $E_1$ of degree $a_{r,p,t-1,\eta} - b_{r,p,t-1,\eta}$. Since $B_{t-1}$ is true, we have $h^1(\mathcal{I}_{Y \cup E}(t-1)) = 0$ i.e. $h^0(\mathcal{I}_{Y \cup E}(t-1)) = a_{t-1,\eta}$. Assume for the moment $a_{t-1,\eta} > 0$. Since we may take as $H$ a general hyperplane, we have $h^1(\mathcal{I}_{Y \cup E \cup \{P\}}(t-1)) = 0$ for a general $P \in H$.

Let $A \subset H$ be a general smooth rational curve of degree $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} - 1 - b_{r,p,t,\eta}$, $L$ a general 2-secant line of $A$ and $B \subset H$ be a general union of $b_{r,p,t,\eta}$ lines. By Lemma 5.2 (case $r = 4$) and Lemma 5.3 (case $r > 4$), we have $h^1(H, \mathcal{I}_{Y \cap H \cup A \cup B \cup D}(t)) = 0$. Since $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} - 1 - b_{r,p,t,\eta} \geq r (t - 1)$ we may assume that $A$ contains exactly one point of $E_1 \cap H$, that $(E \setminus E_1) \cap H \subset A$, and that one of the points, $P$, of $D \cap A$, is a general point of $H$. Let $V \subset \mathbb{P}^r$ be any 3-dimensional linear space containing the plane $V' \subset H$ spanned by $D$ and the tangent line to $A$ at $P$. Let $Z \subset V$ be the 2-point of $V$ with $P$ as its reduction, i.e., the closed subscheme of $V$ with $(\mathcal{I}_{P,V})^2$ as its ideal sheaf. Set $W := E \cup A \cup D \cup Z \cup B$. Since $W \cap Y = \emptyset$ and $W$ is a flat limit of a family...
of disjoint unions of $b_{r,p,t,η}$ lines and a rational curve of degree $a_{r,p,t,η}$ ([6], [7]), to prove $B_t$ it is sufficient to prove that $h^1(I_{Y∪W}(t)) = 0$. Since $Res_Y(Y ∪ W) = Y ∪ E ∪ \{P\}$, it is sufficient to prove that $h^1(H, I_{Y∩H∪U∪J∪D}(t)) = 0, i = 0, 1$. This is true, because $h^1(H, I_{Y∩H∪U∪J∪D}(t)) = 0$, $E ∩ H$ is general in $H$ and $h^0(C∩H∪U∪J∪D) = p(t) + a_{r,p,t-1,η} + t(a_{r,p,t,η} - a_{r,p,t-1,η}) + b_{r,p,t,η} = b_{r,p,t,η} - 2 - b_{r,p,t,η} = (r+t-1)$ (by (5.2), because $a_{t,η} = a_{t,η} - 1$).

Now assume $a_{t,η} = 0$, i.e., assume $t ≥ a_{t,η} + 2$. We make the construction just given, taking a general rational curve $A′ ⊂ H$ of degree $a_{r,p,t,η} - a_{r,p,t-1,η} - b_{r,p,t,η}$ instead of $A ∪ D ∪ Z$.

(ii) To conclude the proof of Theorem 1.2 it is sufficient to check the $h^0$-part of Remark 3.4, i.e. it is sufficient to prove that $h^0(I_{Y∩W}(t)) = 0$ for a general rational curve of degree $a_{r,p,t} + 1$. Let $W ⊂ H$ be a general union of $b_{r,p,t-1,η}$ general lines and the general rational curve of degree $a_{r,p,t-1,η}$. Since $t - 1 ≥ η + a_{t,η},$ we have $b_{r,p,t-1,η} = b_{r,p,t-1}$ and $a_{r,p,t-1,η} = a_{r,p,t-1}$. Hence $E_{t-1}$ gives $h^i(I_W(t-1)) = 0, i = 0, 1$. Let $E ⊂ H$ be a general rational curve of degree $a_{r,p,t} - a_{r,p,t-1} + 1$ with the only restriction that it contains exactly one point of each connected component of $W$. Since $a_{r,p,t} - a_{r,p,t-1} + 1 ≥ rt$, $E$ may be seen as a general rational curve of $H$ with degree $a_{r,p,t} - a_{r,p,t-1} + 1$ (as seen in the proof of $B_t$). Hence for large $t$ and the inductive assumption either $h^0(H, I_{Y∩H∪E}(t)) = 0$ or $h^1(H, I_{Y∩H∪E}(t)) = 0$. Since $W ∩ H$ is general in $H$, we get $h^0(H, I_{Y∩H∪W∩H∪E}(t)) = 0$. A Castelnuovo’s sequence gives $h^0(I_{Y∪W∪E}(t)) = 0$.

### 5.2. Proofs of Theorems 1.1 and 1.3.

We recall the following easy lemma ([3], Lemma 6).

**Lemma 5.4.** Fix integers $n ≥ 3, t > 0, d ≥ 0, e ≥ 0$ such that $(t+1)(d+2e) ≤ (n+t)$. Let $X ⊂ \mathbb{P}^n$ be a general union of $d$ lines and $e$ reducible conics. Then $h^1(I_X(t)) = 0$.

**Proof of Theorem 1.1:** We use Lemma 3.1 and Remark 3.4. We first check the inductive step, i.e. we assume that the statement of Theorem 1.1 is true in $\mathbb{P}^{r-1}$ for the admissible polynomial $p'$ and we prove Theorem 1.1 in $\mathbb{P}^r$ for the Hilbert polynomial $p$. We use that $c_{r,p,t} ∼ c_{r,0,t} ∼ a_{r,p,t} ∼ t^{r-1}/r!$ for $t > 0$ and hence $c_{r,p,t} - c_{r,p,t-1} ∼ \frac{t^{r-2}}{r(r-2)!} ≥ 3t^r$ for all $r ≥ 4$. For all integers $n ≥ 3$ and $t ≥ 0$ and all polynomials $f$ define the integers $x_{n,f,t}$ and $y_{n,f,t}$ by the following relations

$$f(t) + (t+1)x_{n,f,t} + (2t + 1)y_{n,f,t} = \left(\frac{r + t}{r}\right), \quad 0 ≤ y_{n,f,t} ≤ t \quad (5.3)$$

For all $t > 0$ from (5.3) we get

$$f(t) - f(t-1) + x_{n,f,t-1} + (t+1)(x_{n,f,t} - x_{n,f,t-1}) + 2y_{n,f,t} - 2y_{n,f,t-1} = \left(\frac{r + t - 1}{r - 1}\right)$$

Notice that $x_{n,f,t} ≤ c_{n,f,t} ≤ x_{n,f,t} + 2y_{n,f,t} ≤ x_{n,f,t} + 2t$. Fix an integer $t_1$ larger that the Gotzmann’s number of $p$.

Instead of $E_t$ we use the following statement $E_t′$.
$E'_t$: Let $W \subset \mathbb{P}^r$ be a general union of $y_{r,p,t}$ reducible conics with singular point contained in $H$ and $x_{r,0,t} - x_{r,0,t_1}$ lines. Then $h^1(I_{Y \cup W}(t)) = 0$.

We prove $E'_t$ for all $t > t_1$ using Lemma 5.4 instead of Lemmas 5.2 and 5.3 (see the proof of Claim 1 below for more details). Then we fix an integer $\eta > t_1$ as in the proof of Theorem 1.2 and set $a'_{t,\eta} := \binom{r+t}{r} - (\eta+1)(x_{r,0,0} - x_{r,0,t_1}) - (2\eta+1)y_{r,p,n}$. For all integers $t > \eta$ set $\alpha_{r,p,t,\eta} := \max\{0, \alpha_{r,p,t,\eta} + \eta - t\}$. For all integer $t \geq \eta$ define the integers $x_{r,p,t,\eta}$ and $y_{r,p,t,\eta}$ by the relations

$$p(t) + (t + 1)x_{r,p,t,\eta} + (2t + 1)y_{r,p,t,\eta} + \alpha'_{t,\eta} = \binom{r + t}{r}, \quad 0 \leq y_{r,p,t,\eta} \leq t$$

Notice that $x_{r,p,t,\eta} = x_{r,0,0} - x_{r,0,t_1}$ and $y_{r,p,t,\eta} = y_{r,p,t}$. For all $t > \eta$ we have

$$p(t) - p(t - 1) = x_{r,p,t-1,\eta} + (t + 1)(x_{r,p,t,\eta} - x_{r,p,t-1,\eta}) +$$

$$2y_{r,p,t,\eta} - 2y_{r,p,t-1,\eta} + \alpha_{t,\eta} - \alpha'_{t-1,\eta} = \binom{r + t - 1}{r - 1}$$

Since $\alpha_{t,\eta} = 0$ for all $t \geq \eta + \alpha_{r,p,t,\eta}$ we have $x_{r,p,t,\eta} = x_{r,p,t}$ and $y_{r,p,t,\eta} = y_{r,p,t}$ for all $t \geq \eta + \alpha_{r,p,t,\eta}$. For all $t \geq \eta$ call $B'_t$ the following assertion:

$B'_t, \ t \geq \eta$: Let $W \subset \mathbb{P}^r$ be a general union of $x_{r,p,t,\eta}$ lines and $y_{r,p,t,\eta}$ reducible conics whose singular point is contained in $H$. Then $h^1(I_{Y \cup W}(t)) = 0$ (or, equivalently, $h^0(I_{E}(t)) = 0$).

Claim 1: $B'_t$ is true for all $t \geq \eta$.

Proof of Claim 1: $B'_t$ is true, because it is equivalent to $E'_t$. Assume $t \geq \eta + 2$ and that $B'_{t-1}$ is true. Take $W$ satisfying $B'_{t-1}$. For a general $W$ no irreducible component of $W$ is contained in $H$ and hence for each conic $T \subset W$ the scheme $T \cap H$ is a tangent vector of $H$. Hence for a general $W$ the scheme $W \cap H$ is a general union of $y_{r,p,t-1,\eta}$ tangent vectors of $H$ and $x_{r,p,t-1,\eta}$ general points of $H$. First assume $y_{r,p,t,\eta} \geq y_{r,p,t-1,\eta}$. Let $E \subset H$ be a general union of $x_{r,p,t,\eta} + 2y_{r,p,t,\eta} - x_{r,p,t-1,\eta} - 2y_{r,p,t-1,\eta}$ lines with the only restriction that $y_{r,p,t,\eta} - y_{r,p,t-1,\eta}$ of these lines contain a point of a different line of $W$; this is possible, because for large $\eta$ we have $x_{r,p,t,\eta} - x_{r,p,t-1,\eta} \sim x^{r-2}/r \cdot (r - 2)! > 2t$. We need to check that $h^1(H, I_{Y \cap W \cap H \cup E}(t)) = 0$. Hence $W \cap H$ is a general union of $y_{r,p,t-1,\eta}$ tangent vectors of $H$ and $x_{r,p,t-1,\eta}$ general points of $H$. In characteristic zero a general tangent vector gives independent conditions to any non-trivial linear system ([11], [9, Lemma 1.5]); in arbitrary characteristic (as in [15] or [3]) we may use that the number of tangent vectors is very small. A Castelnuovo’s sequence proves that $h^1(I_{Y \cup W \cup E}(t)) = 0$. By the semicontinuity theorem $B'_t$ is true. Now assume $y_{r,p,t,\eta} < y_{r,p,t-1,\eta}$. Fix $S \subseteq \text{Sing}(W)$ with $\sharp(S) = y_{r,p,t-1,\eta} - y_{r,p,t,\eta}$. For each $P \in S$ call $U_P$ the connected component of $W$ containing $P$. For each $P \in S$ fix any 3-dimensional linear subspace of $\mathbb{P}^r$ containing $U_P$, but not contained in $H$. Let $Z_P \subset V_P$ be the closed subscheme of $V_P$ with $(I_{P,Y})^2$ as its ideal sheaf. Notice that $C_P := U_P \cup Z_P$ is a sundial. Let $E \subset H$ be a general union of $x_{r,p,t,\eta} + 2y_{r,p,t,\eta} - x_{r,p,t-1,\eta} + 2y_{r,p,t-1,\eta}$. Set $G := W \cup E \cup \cup P(S \cup Z_P$. Since each $C_P$ is a sundial, $G$ is a flat limit of a family of disjoint unions of $x_{r,p,t,\eta}$ lines and $y_{r,p,t,\eta}$ reducible conics whose singular point is contained in $H$. To get $h^1(I_{Y \cup G}(t)) = 0$ we need to control the union of $Y \cap H$, $W \cap H \setminus S$ and $Z_P \cap H$, $P \in S$. To control $W \cap H \setminus S$ we only need to control the postulation of a general union of $y_{r,p,t,\eta}$.
tangent vectors on \( H \). The scheme \( Z_p \cap H \) is a degree 3 scheme supported by \( P \) and spanning the plane \( V_p \cap H \) (it is called a triple point in [15]), since the sum of the numbers of tangent vectors and triple points is small, this is easy (e.g., [15, Assertion \( H''_{r,N} \) in §3]); this sum is so small that we could take 2-points of \( H \) instead of tangent vectors and triple points in the sense of [15] and still get an \( h^1 \)-vanishing using the Alexander-Hirschowitz theorem.

From \( B'_t \) we easily get Theorem 1.1.

To start the induction we need to do the case \( r = 4 \) only using Lemma 4.8 with \( p' \) as the Hilbert polynomial. Take \( \psi \) as in the statement of Lemma 4.8 with respect to the admissible polynomial \( p' \). We have \( c_{4,p,t} - c_{4,p,t-1} \sim c_{4,0,t} - c_{4,p,t-1} \sim t^2/8 \gg (c_{3,0,4} + 4)(t + 1) \) Hence we may use Lemma 4.8 instead of the case \( r = 3 \) of Theorem 1.1 (which we do not claim, although we believe that it is true). \( \square \)

Proof of Theorem 1.3 in \( \mathbb{P}^r \), \( r \geq 4 \): We use Lemma 3.3 and Remark 3.4. Fix \( p, r \geq 4, Y \) and an integer \( g > 0 \). Let \( y_0 \) be an integer such that Theorem 1.2 is true for all integers \( t \geq y_0 \). We also assume \( y_0 > 2g + r \) and \( a_{r,p,g,t} \geq 2gr + r^2 \) for all \( y \geq y_0 \) set \( g(y) = \min \{ g, y - y_0 \} \). For all integer \( t \geq y_0 \) define the integers \( a_{r,p,g,t,y_0} \) and \( b_{r,p,g,t,y_0} \) by the relations

\[
p(t) + ta_{r,p,g,t,y_0} + 1 - g(t) + b_{r,p,g,t,y_0} = \binom{r + t}{r} - 1 \leq b_{r,p,g,t,y_0} \leq t - 1.
\]

Since \( y_0 > g \geq g(t) \) if \( b_{r,p,t} \leq t - g(y) - 1 \), then \( a_{r,p,g,t} = a_{r,p,t} \) and \( b_{r,p,g,t} = b_{r,p,t} + g(t) \), while \( a_{r,p,g,t} = a_{r,p,t} + 1 \) and \( b_{r,p,g,t} = b_{r,p,t} + g(t) - t \) if \( b_{r,p,t} \geq t - g(t) \). Let \( \Psi(t) \) denote the following assertion:

\[
\Psi(t), t \geq y_0: \text{Let } W \subset \mathbb{P}^r \text{ be a general union of a general curve of genus } g(t) \text{ and degree } a_{r,p,g,t} - b_{r,p,g,t} \text{ lines. Then } h^1(\mathcal{I}_{Y \cup W}(t)) = 0 \text{ (equivalently, } h^0(\mathcal{I}_{Y \cup W}(t)) = 0).
\]

\( \Psi(t) \), \( t \geq y_0 \). Let \( W \subset \mathbb{P}^r \) be a general union of a general curve of genus \( g(t) \) and degree \( a_{r,p,g,t} \) and degree \( a_{r,p,g,t} \). Let \( N_W \) be the normal bundle of \( W \) in \( \mathbb{P}^r \). Since \( a_{r,p,g,t} \geq 2gr + r^2 \) and \( W \) is a general element of \( (\mathcal{Z}(r,a_{r,p,g,t},g(t)), \text{it is easy to check that} (8, \text{part (2) of Théorème 5}) \). Hence for general \( W \) the set \( W \cap H \) is a general subset of \( H \) with cardinality \( a_{r,p,g,t} \) (Remark 2.3).

(b) \( \Psi(y_0) \) is true, because \( g(y_0) = 0 \), \( y_0 \) is large (say \( y_0 \geq \eta \) with \( \eta \) as in the proof of Theorem 1.2) and hence it is true the assertion \( W \cap H = \emptyset \) introduced during the proof of Theorem 1.2.

Claim 1: \( \Psi(t) \) for all \( t > y_0 \)

Proof of Claim 1: First assume \( y_0 \leq t \leq y_0 + g \). Let \( W \subset \mathbb{P}^r \) be a general union of a general curve \( W_1 \) of genus \( g(t - 1) \) and degree \( a_{r,p,g,t-1} - b_{r,p,g,t-1} \) and \( b_{r,p,g,t-1} \) lines. Since \( \Psi(t-1) \) is true, we have \( h^1(\mathcal{I}_{Y \cup W_1}(t-1)) = 0 \), \( i = 0, 1 \). We have \( a_{r,p,g,t-1} - a_{r,p,g,t-1} \geq (r + 2) t \) for large \( y_0 \). By Lemmas 5.2 and 5.3 and the semicontinuity theorem for cohomology we have \( h^1(\mathcal{I}_{Y \cup W_1}(t-1)) = 0 \), where \( A' \) is a general elliptic curve of degree \( a_{r,p,g,t} - a_{r,p,g,t-1} - b_{r,p,g,t} \) and \( B \) is a general union of \( b_{r,p,g,t} \) lines (\( A' \) is a flat limit of a family \( A \cup D \) with \( A \) rational and \( D \) 2-secant line of \( A \)). By Remark 2.3 and [8, part (2) of Théorème 5 for the genus 1]) \( A' \) contains \( b_{r,p,g,t-1} + 1 \) general points of \( H \). Hence we may assume that \( A' \) contains exactly one point of \( W_1 \cap H \) and all points of \( (W \setminus W_1) \cap H \). Since \( W \cap (H \setminus A') \) is general in \( H \) by part (a), we get \( h^i(\mathcal{H}_Y \cap H, (W \cap H) \cup A') = 0 \),
\[ i = 0, 1, \text{ by (}5.2\text{). A Castelnuovo’s sequence gives } h^i(\mathcal{I}_{Y \cup W \cup A \cup B}(t)) = 0, \ i = 0, 1. \]

Since \( W \cup A' \cup B \) is a flat limit of a family of disjoint unions of \( b_{r,p,g,t} \) lines and a smooth curve of genus \( g(t) \) and degree \( a_{r,p,g,t} - b_{r,p,g,t} \), \( \Psi(t) \) is true in this case. Now assume \( t > y_0 \). In this range we only add \( b_{r,p,g,t} \) lines and a rational curve of degree \( a_{r,p,g,t} - a_{r,p,g,t-1} - b_{r,p,g,t} \) containing one point of \( W \cap H \).

To conclude the proof of Theorem 1.2 it is sufficient to use \( \Psi(t) \) instead of \( B_t \) as in the proof of Theorem 1.2. \( \square \)

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**References**

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