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CONSTRUCTION OF SEMI - FREE S^3 ACTIONS ON HOMOTOPY SPHERE WITH UNTWISTED FIXED POINT SET

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Abstract. In his paper "Surgery and the theory of differentiable transformation groups", William Browder developed surgery techniques to study semi-free actions of S^1 on homotopy spheres, under the additional assumption that the fixed point set is a homotopy sphere. He used this surgery to show how to construct such actions. In this paper, I discuss a similar theory about semi-free actions of S^3 on homotopy spheres. An open problem is raised at the end of the paper.

1. Introduction and preliminaries

Throughout this paper, R^n denotes the Euclidean $n-$space, S^n denotes the unit $n-$sphere in R^{n+1} and $QP(n)$ the quaterionic projective space, with the usual differentiable structures. A homotopy $n-$sphere is denoted by $\sum^n$, it means a closed differentiable $n-$manifold having the homotopy type of S^n. A homotopy quaterionic projective n-space is denoted by $HQP(n)$, it means a closed differentiable $4n-$manifold having the homotopy type of $QP(n)$.

$\pi_n(M)$ denotes the $n^{th}-$homotopy group of M, $H_i(M,G)$ and $H^i(M,G)$ denote the homology and cohomology of a space M, with coefficients in the group G under the assumption that it satisfies the Eeilenberg$-$Steenrod axioms, See [[12],page 6]. If $G = Z$, we write $H^i(M)$ and $H_i(M)$ for $H^i(M,Z)$ and $H_i(M,Z)$ respectively. It is well known that S^1 and S^3 are the only compact connected Lie groups which have free differentiable actions on homotopy spheres [11]. It follows from Gleason's lemma [5] that such an action is always a principal fibration which is homotopically equivalent to the classical Hopf fibration.

In fact, there are always infinitely many differentiably distinct free actions of S^3 on $\sum^{4n+3}$ for $n \geq 2$, see [7].

2. Construction of semi-free actions:

An action (G, M, ϕ) is called semi$-$free, if it is free outside the fixed point set. That is, for an action $\phi : G \times M \to M$, let F be the fixed point set of the action,

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ϕ is semi−free if $\phi(g,x)=x$, for some $x\in M\ -\ F$ then $g=e$ which is the identity of G. Notice that there are only two types of orbits, fixed points and G.

Lemma 2.1. *([16]) Let $\phi: S^i\times M\to M$ $(i=1,3)$, be a semi free differentiable action. let F^k denote the union of the $k-$dimensional components of the set of all fixed points of ϕ. Then the normal bundle of an imbedding $F^k\subset M$ has naturally a complex strucure for $i=1$ and a quaterionic structure for $i=3$ and that the induced S^i-action on the normal bundle is a scalar multiplication.*

It follows from Lemma 2.1 that the codimension of each component of F in M is even for $i=1$ and is divisible by 4 for $i=3$. We shall study the situation where $(S^3,\sum^m,\phi)$ is a semi−free differentiable action on a homotopy sphere $\sum^m$ and the fixed point set is a homotopy sphere $\sum^r$. Let $(S^3,\sum^m,\phi)$ be a semi−free action with fixed point set $\sum^r\subset\sum^m$, then S^3 acts freely outside $\sum^r$ and S^3 acts freely and linearly on the normal space to $\sum^r$ at each point of $\sum^r$. See [[3],page 58]. By Lemma 2.1, the normal bundle of $\sum^r$ has a quaterionic structure and $m-r=4k,\ k\geq 1$. Let μ be the (quaterionic) bundle over $\sum^r$ defined by the action. We prove the following :

Theorem 2.2. *If $(S^3,\sum^m,\phi_1)$ and $(S^3,\sum^m,\phi_2)$ are equivalent, then F_1 is diffeomorphic to F_2 and μ_1 is equivalent to μ_2, where F_1 and F_2 are the fixed point sets of ϕ_1 and ϕ_2 respectively, μ_1 and μ_2 are the normal bundles of F_1 andF_2 respectively.*

Proof. Since $(S^3,\sum^m,\phi_1)$ and $(S^3,\sum^m,\phi_2)$ are equivalent then there exists an equivariant diffeomorphism f such that the following diagram commutes :

$$\begin{array}{ccc} S^3\times\sum^m & \xrightarrow{\phi_1} & \sum^m \\ I\times f\downarrow & & f\downarrow \\ S^3\times\sum^m & \xrightarrow{\phi_2} & \sum^m \end{array}$$

Let $x\in F_1$, i.e, $\phi_1(q,x)=x\ \forall\ q\in S^3$ then $f\circ\phi_1(q,x)=\phi_2\circ(I\times f)(q,x)\ \forall\ q\in S^3$, thus $f(x)=\phi_2(q,f(x))\ \forall\ q\in S^3$ implies $f(x)\in F_2$ hence $f(F_1)\subseteq F_2$.
Now, assume that $y\in F_2$, i.e., $\phi_2(q,y)=y\ \forall\ q\in S^3$, but f is an equivariant diffeomorphism, then $\exists\ x\in\sum^m$ such that $y=f(x)$ and $f\circ\phi_1(q,x)=\phi_2\circ(I\times f)(q,x)\ \forall\ q\in S^3$, then $f(\phi_1(q,x))=\phi_2(q,f(x))=f(x)$, but f is $1-1$, we get $\phi_1(q,x)=x\ \forall\ q\in S^3$, hence $x\in F_1$ and $f(x)\in f(F_1)$, therefore $f(F_1)=F_2$. Moreover, the equivalence $f:\sum^m\to\sum^m$ defines a quaterionic map of the normal bundles μ_1 and μ_2 of F_1 and F_2 respectively, so they are equivalent. □

Now, $\sum^r$ is a closed submanifold of $\sum^m$ and is invariant under the action of S^3, therefore there exists a tubular neighbourhood E of $\sum^r$ which is invariant under the action of S^3. So we may consider $\mu:E\to\sum^r$, the normal bundle to $\sum^r$ in $\sum^m$ and S^3 acts differentiably on E, see [[3],page 58]. Let S^{4k-1} be the boundary of a fibre of E, then we can prove the following:

Lemma 2.3. *$S^{4k-1}\subset\sum^m-\sum^r$ is a homotopy equivalence.*

Proof. consider the exact cohomology sequence of the pair $(\sum^m, \sum^r)$:

$\to H^{i-1}(\sum^r) \to H^i(\sum^m, \sum^r) \to H^i(\sum^m) \to H^i(\sum^r) \to H^{i+1}(\sum^m, \sum^r) \to H^{i+1}(\sum^m) \to$
At $i = m :\to 0 \to H^m(\sum^m, \sum^r) \to Z \to 0$, then $H^m(\sum^m, \sum^r) \cong Z$,
At $i = r : 0 \to Z \to H^{r+1}(\sum^m, \sum^r) \to 0$, hence $H^{r+1}(\sum^m, \sum^r) \cong Z$,
And for $i \neq m, r+1 : 0 \to 0 \to H^i(\sum^m, \sum^r) \to 0$, thus $H^i(\sum^m, \sum^r) \cong 0$ for $i \neq m,\ r+1$. Finally, we obtain

$$H^i(\sum^m, \sum^r) = \begin{cases} Z \text{ for } i = m,\ r+1 \\ 0 \text{ otherwise} \end{cases}$$

It is clear that $\sum^m - \sum^r$ is simply connected [[9],page 3] and using Lefshetz duality [[12],page 32],
$H_i(\sum^m - \sum^r) \cong H^{m-i}(\sum^m, \sum^r)$, we finally deduce that

$$H_i(\sum^m - \sum^r) = \begin{cases} Z \text{ for } i = 0, 4k-1 \\ 0 \text{ otherwise} \end{cases}$$

hence, the inclusion map $i : S^{4k-1} \to \sum^m - \sum^r$ induces
$i_* : \pi_j(S^{4k-1}) \to \pi_j(\sum^m - \sum^r)$ an isomorphism $\forall\ j$, see[[14],page 283]. Therefore $S^{4k-1} \subset \sum^m - \sum^r$ is a homotopy equivalence. □

Now, let $N = \sum^m - E_0$ where E_0 is the interior of an equivariant tubular neighbourhood of $\sum^r$ with $\overline{E_0} \subset int(E)$. Then S^3 acts freely on N and $S^{4k-1} \subset N$. Notice that S^{4k-1} is a homotopoy equivalence to N, it follows from the exact homotopy sequence of the fibre maps, using the diagram :

$$\begin{array}{ccccc} \to S^3 \to & & S^{4k-1} & & \to S^{4k-1}/S^3 \to \\ \downarrow & & \downarrow & & \downarrow \\ \to S^3 \to & & N & & \to N/S^3 \to \end{array}$$

that is $S^{4k-1}/S^3 \to N/S^3$ is a homotopy equivalence.
Set $N^1 = N/S^3$ and $S^{4k-1}/S^3 = QP(k-1)$. Notice that the region between ∂N^1 and $QP(k-1) \times S^r$ is an h−cobordism, so if $m \geq 6$, then by the h−cobordism theorem of Smale [15], N^1 is diffeomorphic to $QP(k-1) \times D^{r+1}$ and $N \to N^1$ is equivalent to
$h \times I : S^{4k-1} \times D^{r+1} \to QP(k-1) \times D^{r+1}$, where $h : S^{4k-1} \to QP(k-1)$ is the hopf map. Hence, we have proved the following theorem:

Theorem 2.4. *Let $(S^3, \sum^m, \phi)$ be a semi−free S^3 action on $\sum^m$ with fixed point set $\sum^r$, $m - r = 4k$, $k \geq 1$, $m \geq 6$. If N is the complement of an open tubular neighbourhood of $\sum^r$ in $\sum^m$,then N is equivariantly diffeomorphic to $S^{4k-1} \times D^{r+1}$, with the standard action on S^{4k-1} and trivial action on D^{r+1}.*

Now, we will describe how to construct smooth semi−free S^3 actions on a homotopy $m-$sphere $\sum^m$. Let $\sum^r$ be a homotopy $r-$sphere and μ a (quaterionic) normal bundle over $\sum^r$ given by $\mu : E(\mu) \to \sum^r$ where $E(\mu)$ is the total space of μ such that $E(\mu) \cong D^{4k} \times \sum^r$, i.e., $E(\mu)$ is the a trivial bundle and suppose that $h : S^{4k-1} \times \sum^r \to S^{4k-1} \times S^r$ is an equivariant diffeomorphism .

Theorem 2.5. *There is a semi free action $(S^3, \sum^m, \phi)$ with a fixed point set $\sum^r$ and $\sum^m = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ where $\cup_h$ means that we identify $S^{4k-1} \times \sum^r \subset E(\mu)$ with $S^{4k-1} \times S^r \subset S^{4k-1} \times D^{r+1}$ via the diffeomorphism h.*

Proof. Consider the semi−free action on the total space of $\mu : E(\mu) \to \sum^r$ defined by the quaterionic structure and the free S^3- action on $S^{4k-1} \times D^{r+1}$ defined by the free action on S^{4k-1}, i.e., the standard action and $h : S^{4k-1} \times \sum^r \to S^{4k-1} \times S^r$ is an equivariant diffeomorphism, then $M = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ has a semi−free action of S^3 with fixed point set $\sum^r$ and normal bundle μ, it is enough to show that M is a homotopy sphere.
It is clear that

$$\pi_1(\partial E(\mu)) \cong \pi_1(S^{4k-1} \times S^r) \cong \pi_1(S^{4k-1}) \oplus (S^r) \cong \pi_1(\sum^r) \text{ and } \pi_0(\partial E(\mu)) \cong \pi_0(\sum^r) \cong 0,$$

i.e., $E(\mu)$ and $S^{4k-1} \times D^{r+1}$ are simply connected and $E(\mu) \cap S^{4k-1} \times D^{r+1}$ is simply connected, hence by VanKampen's theorem [2], M is simply connected.
Now, we consider the Mayer -Vietoris sequence for M [4]:

$$\to H_{s+1}(M) \to H_s(\partial E(\mu)) \to H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1}) \to H_s(M) \to$$

By the Künneth formula [[6], page 98], since $H_s(\partial E(\mu)),\ H_s(E(\mu)),$
$H_s(S^{4k-1} \times D^{r+1})$ are torsion free for, $0 < s < 4k + r - 1$, we obtain

$$H_s(\partial E(\mu)) = H_s(S^{4k-1} \times \sum^r) \cong \oplus_{i=0}^{s} H_i(S^{4k-1}) \otimes H_{s-i}(\sum^r).$$

If $i = 0$, then $H_0(S^{4k-1}) \otimes H_s(S^r) \cong Z \otimes H_s(\sum^r)$.
If $i = s$, then $H_s(S^{4k-1}) \otimes H_0(S^r) \cong H_s(S^{4k-1}) \otimes Z$ and for $i \neq 0$
$H_i(S^{4k-1}) \otimes H_{n-i}(S^r) \cong 0$ therefore $H_s(\partial E(\mu)) \cong Z \otimes H_s(S^r) \oplus H_s(S^{4k-1}) \otimes Z$.
Again, we compute
$H_s(S^{4k-1} \times D^{r+1}) = \oplus_{i=0}^{s}\ H_i(S^{4k-1}) \otimes H_{s-i}(D^{r+1}) \cong H_s(S^{4k-1}) \otimes Z$.
similarly $H_s(E(\mu)) = H_s(D^{4k} \times \sum^r) \cong Z \otimes H_s(\sum^r)$.
therefore $H_s(\partial E(\mu)) \cong H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1})$,
hence $H_s(M) \cong H_{s+1}(M) \cong 0,\ \forall\ 0 < s < 4k + r - 1$.
For, $s = 4k + r - 1$: $H_{4k+r-1}(E(\mu)) \cong 0,\ H_{4k+r-1}(S^{4k-1} \times D^{r+1}) \cong 0$ and
$H_{4k+r-1}(\partial E(\mu)) \cong Z \otimes Z \cong Z$.

Substituting in the Mayer -Vietoris sequence for $M : 0 \to H_{4k+r}(M) \to Z \to 0$.
Finally, we obtain

$$H_s(M) \cong \begin{cases} z \text{ for } s = 0,\ 4k + r, \\ 0 \ otherwise \end{cases}$$

hence M is a homology sphere.
But M is simply connected, closed without boundary, therefore M is a homotopy sphere. □

3. Applying surgery to construct semi free S^3- actions:

In this section, we used surgery techniques as Browder [1] to create a diffeomorphism of $QP(k-1)\times\sum^r$ with $QP(k-1)\times S^r$, then we apply Theorem 2.5

Theorem 3.1. *Let $\sum^{4n-1}$ be a homotopy sphere which bounds a parallelizable manifold, $n\geq 1$. Then for each even $k\geq 2$, there is a semi−free action of S^3 on a homotopy sphere $\sum^{4(n+k)-1}$ with $\sum^{4n-1}$ as untwisted fixed point set.*

Proof. Let $\sum^{4n-1}=\partial W^{4n}$, W is a parallelizable manifold. We may consider $W_0=W-int(D^{4n})$ as a parallelizable cobordism between $\sum^{4n-1}$ and S^{4n-1} thus we may define a normal map

$$f:(W_0,\ \sum^{4n-1}\cup S^{4n-1})\to(S^{4n-1}\times I,\ S^{4n-1}\times\{0\}\cup S^{4n-1}\times\{1\})$$

with $f|_{S^{4n-1}}=$ Identity. Since W is a parallelizable manifold [[10], page 514], we may assume that W_0 is $(2n-1)$ connected.
Multiplying by $QP(k-1)$, we get $I\times f$:

$$Qp(k-1)\times(W_0,\sum^{4n-1}\cup S^{4n-1})\to Qp(k-1)\times(S^{4n-1}\times I,\ S^{4n-1}\times\{0\}\cup S^{4n-1}\times\{1\}),$$

with $I\times f|_{QP(k-1)\times S^{4n-1}}=$Identity.
The remainder of the proof is computing the obstruction σ for this map to be a cobordism and using this to determine if $Qp(k-1)\times\sum^{4n-1}$ is diffeomorphic to $Qp(k-1)\times S^{4n-1}$.

Claim: $Ker(I\times f)_*=H_*(QP(k-1))\times Ker(f_*)$.

By the Künneth formula, since $H_*(QP(k-1))$ is torsion free then,

$$H_*(QP(k-1)\times(W_0,\sum^{4n-1}\cup S^{4n-1}))\cong H_*(QP(k-1))\otimes H_*(W_0,\sum^{4n-1}\cup S^{4n-1})$$

and $(I\times f)_*=I\otimes f_*$

therefore, $Ker(I\times f)_*=H_*(QP(k-1))\times Ker(f_*)$.

Now, consider the commutative diagram induced by f:

$$\begin{array}{ccccc}
\longrightarrow H_i(\partial W_0) & \longrightarrow H_i(W_0) & \longrightarrow & H_i(W_0,\partial W_0) & \longrightarrow \\
\downarrow & \downarrow & & f_*\downarrow & \\
\to H_i(\partial S^{4n-1}\times I) & \to H_i(S^{4n-1}\times I) & \to & H_i(S^{4n-1}\times I,\partial S^{4n-1}\times I) & \to
\end{array}$$

Notice that, $H_i(\partial S^{4n-1}\times I)\cong H_i(\partial W_0)$ and $H_i(W_0)\cong 0$ for $i\neq 0,\ 2n$. We get

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{2n}(W_0) & \xrightarrow{\cong} & H_{2n}(W_0,\partial W_0) & \longrightarrow & \\ & & & & f_*\downarrow & & \\ & 0 & \longrightarrow\ 0 & \longrightarrow & 0 & \longrightarrow\ 0 & \end{array}$$

Hence $Ker(f_*)\cong H_{2n}(W_0)$.

But $Ker(I\times f)_*=Ker(I\otimes f_*)=H_*(QP(k-1))\otimes Ker(f_*)\cong H_*(QP(k-1))\otimes H_{2n}(W_0)$, and $Ker(I\times f)_{*2n+2k-2}=H_{2k-2}(QP(k-1))\otimes H_{2n}(W_0)$,
since $2k-2=2(2s)-2\neq 0 \bmod (4)$, k is even, then $H_{2k-2}(QP(k-1))\cong 0$ and $Ker(I\times f)_{*2n+2k-2}\cong 0$.
Therefore, $\sigma(I\times f)=0$ and there exists an $h-$ cobrdism between $QP(k-1)\times\sum^{4n-1}$ and $QP(k-1)\times S^{4n-1}$, but $k\geq 2$ and $n\geq 1$, then $(4k-4)+(4n-1)=4(k+n)-5\geq 7$. Hence, Smale's $h-$cobordism theorem can be applied and $QP(k-1)\times\sum^{4n-1}$ is diffeomorphic to $QP(k-1)\times S^{4n-1}$. Applying Theorem 2.5, it follows that there is a semi$-$free action of S^3 on some homotopy sphere $\sum^m$ with $\sum^{4n-1}$ as untwisted fixed point set, where $m=4(n+k)-1$. □

Open Problem

Browder [1] showed how to construct semi$-$free S^1 actions, with$\sum^r$ as untwisted fixed point set, i.e., its normal bundle is trivial. He stated that he did not know of any action with a twisted fixed point set. However Schultz [14] used complicated computations of homotopy groups, proved the following Theorem:
Let $k\geq 2$ be a positive integer. Then there exist infinitely many values of n for which S^{2n} has a semi$-$free S^1 action with $S^{2(n-k+1)}$ as twisted fixed point set. In this work, as in the work of Browder, we consider$\sum^r$ as untwisted fixed point set and I raise the following question: Does there exist smooth semi free S^3 actions on homotopy spheres for which the fixed point set is twisted?

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