Gulf Journal of Mathematics
Vol 2, Issue 3 (2014) 30-37

ON n-STRONGLY GORENSTEIN RINGS

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Abstract. This paper introduces and studies a particular subclasses of the class of commutative rings with finite (resp. weak) Gorenstein global dimensions.

1. Introduction and preliminaries

Throughout the paper, all rings are commutative with identity, and all modules are unitary.
Let R be a ring, and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [9, 10] introduced the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [12] introduced the Gorenstein flat dimension. Some references are [3, 7, 8, 9, 10, 12, 13].

Recall that an R-module M is called Gorenstein projective, if there exists an exact sequence of projective R-modules:

$$\mathbf{P} : ... \to P_1 \to P_0 \to P^0 \to P^1 \to ...$$

such that $M \cong Im(P_0 \to P^0)$ and such that the functor $Hom_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The complex $\mathbf{P}$ is called a complete

Date: Received: Feb 2, 2014; Accepted: Mar 15, 2014.
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2010 *Mathematics Subject Classification.* 13D05, 13D02.
Key words and phrases. Strongly (n-)Gorenstein projective, injective and flat modules, Gorenstein global and weak dimensions.

projective resolution.

The Gorenstein injective R-modules are defined dually.

An R-module M is called Gorenstein flat, if there exists an exact sequence of flat R-modules:

$$\mathbf{F} : ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...$$

such that $M \cong Im(F_0 \to F^0)$ and such that the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever I is an injective R-module. The complex $\mathbf{F}$ is called a complete flat resolution.

The Gorenstein projective, injective, and flat dimensions are defined in terms of resolutions and denoted by $Gpd(-)$, $Gid(-)$, and $Gfd(-)$, respectively ([7, 11, 13]).

In [3], the authors proved, for any associative ring R, the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid \text{M is a (left) R-module}\} = \sup\{\mathrm{Gid}_R(M) \mid \text{M is a (left) R-module}\}.$$

They called the common value of the above quantities the left Gorenstein global dimension of R and denoted it by $l.Ggldim(R)$. Similarly, they set

$$l.wGgldim(R) = \sup\{Gfd_R(M) \mid \text{M is a (left) R-module}\}$$

which they called the left Gorenstein weak dimension of R. Since in this paper all rings are commutative, we drop the letter l.

Recently, in [16], particular modules of finite Gorenstein projective, injective, and flat dimensions are defined as follows:

Definition 1.1. Let n be a positive integer.

(1) An R-module M is said to be strongly n-Gorenstein projective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ where $\mathrm{pd}_R(P) \leq n$ and $Ext_R^{n+1}(M, Q) = 0$ whenever Q is projective.
(2) An R-module M is said to be strongly n-Gorenstein injective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ where $\mathrm{id}_R(I) \leq n$ and $Ext_R^{n+1}(E, M) = 0$ whenever E is injective.
(3) An R-module M is said to be strongly n-Gorenstein flat, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow F \longrightarrow M \to 0$ where $fd_R(F) \leq n$ and $Tor_R^{n+1}(M, I) = 0$ whenever I is injective.

Clearly, strongly 0-Gorenstein projective, injective, and flat modules are just the strongly Gorenstein projective, injective, and flat modules, respectively ([3, Propositions 2.9 and 3.6]).

In this paper, we investigate these modules to characterize a new classes of rings with finite (resp. weak) Gorenstein global dimensions, which we call n-strongly Gorenstein rings and weakly n-strongly Gorenstein rings.

2. n-strongly Gorenstein rings and weakly n-strongly Gorenstein rings

In [16], the authors proved the following proposition:

Proposition 2.1 (Proposition 2.16, [16])**.** *Let R be a ring. The following statements are equivalent:*

(1) *Every module is strongly n-Gorenstein projective.*
(2) *Every module is strongly n-Gorenstein injective.*

Thus, we give the following definitions:

Definition 2.2. Let n be a positive integer.

(1) A ring R is called n-strongly Gorenstein (n-SG ring for short), if R satisfies one of the equivalent conditions of Proposition 2.1.
(2) A ring R is called weakly n-strongly Gorenstein (n-wSG ring for short), if every R-module is strongly n-Gorenstein flat.

The 0-SG rings, 1-SG rings, 0-wSG rings, and 1-wSG rings are already studied in [6, 15] and they are called strongly Gorenstein semi-simple rings, strongly Gorenstein hereditary rings, strongly Gorenstein Von Neumann regular rings, and strongly Gorenstein semihereditary rings, respectively. Clearly, by definition, every n-SG (resp. n-wSG) ring is m-SG (resp. m-wSG) whenever $n \leq m$.

Our first result gives a characterization of Strongly n-Gorenstein rings.

Proposition 2.3. *For a ring R and a positive integer n, the following statements are equivalent:*

(1) *R is an n-SG ring.*
(2) $\mathrm{Ggldim}(R) \leq n$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$
where $\mathrm{pd}_R(P) < \infty$.
(3) $\mathrm{Ggldim}(R) < \infty$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$
where $\mathrm{pd}_R(P) \leq n$.

Proof. (1 $\Rightarrow$ 2) Clear since for every n-SG ring R we have $\mathrm{Ggldim}(R) \leq n$ (by [16, Proposition 2.2(1)]).
(2 $\Rightarrow$ 3) Follows directly from [3, Corollary 2.7].
(3 $\Rightarrow$ 1) Follows from [16, Proposition 2.10]. □

Similarly, for the n-wSG rings, we have the following result:

Proposition 2.4. *Let R be a ring and n be a positive integer. The following conditions are equivalent.*

(1) *R is an n-wSG ring.*
(2) $\mathrm{wGgldim}(R) \leq n$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) < \infty$.
(3) $\mathrm{wGgldim}(R) < \infty$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) \leq n$.
(4) *For every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) \leq n$.

We need the following lemma.

Lemma 2.5 (Theorem 2.2, [4])**.** *For an R-module M,* $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ *with equality if* $\mathrm{fd}_R(M) < \infty$.

Proof of Proposition 2.4. $(1 \Rightarrow 2)$ Follows from the fact that every strongly n-Gorenstein flat module has a Gorenstein flat dimension $\leq n$ (by [16, Proposition 3.2(1)]).

$(2 \Rightarrow 3)$ Let F be an R-module such that $\mathrm{fd}_R(F) < \infty$. Clearly, $\mathrm{Gfd}_R(F) \leq n$ since $\mathrm{wGgldim}(R) \leq n$. Then, by Lemma 2.5, $\mathrm{fd}_R(F) \leq n$.

$(3 \Rightarrow 4)$ Obvious.

$(4 \Rightarrow 1)$ Let I be an injective R-module. By hypothesis, there is an exact sequence of R-modules $0 \to I \to F \to I \to 0$ where $\mathrm{fd}_R(F) \leq n$. Clearly this exact sequence splits. Thus, $I \oplus I \cong F$. Hence, $\mathrm{fd}_R(I) \leq n$. Thus, for each R-module M, $\mathrm{Tor}_R^{n+1}(M, I) = 0$. Hence, each R-module is strongly n-Gorenstein flat. Consequently, R is an n-wSG ring.□

Recall that a ring R is called perfect if every flat R-module is projective.

Theorem 2.6. *Every n-SG ring is n-wSG with equivalence in the following two cases:*

(1) *R is Noetherian.*
(2) *R is perfect with finite Gorenstein global dimension.*

We need the following lemma.

Lemma 2.7. *For any arbitrary ring R we have:* $\mathrm{wGgldim}(R) \leq \mathrm{Ggldim}(R)$ *with equality in the following tow cases:*

(1) *R is Noetherian.*
(2) *R is perfect with finite Gorenstein global dimension.*

Proof. The desired inequality follows from [3, Corollary 1.2 (1)]. While, if R is Noetherian, the converse inequality follows from the equivalence [11, Theorem 12.3.1($3 \Leftrightarrow 4$)].
If R is perfect with $n := \mathrm{Ggldim}(R) < \infty$, we claim that every Gorenstein flat R-module is Gorenstein projective. Let M be an arbitrary Gorenstein flat R-module. By the definition of Gorenstein flat modules, we can pick an n-step right flat resolution as follows:

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow F_1 \longrightarrow ... \longrightarrow F_n \longrightarrow G$$

where all F_i are flats and so projectives since R is perfect. But $\mathrm{Gpd}_R(G) \leq n$. Thus, using [13, Theorem 2.20($i \Leftrightarrow iv$)], we conclude that M is Gorenstein projective, as desired. Consequently, $\mathrm{wGgldim}(R) \leq \mathrm{Ggldim}(R)$ and this finishes the proof. □

Proof of of Theorem 2.6. Using Proposition 2.3 and Proposition 2.4, it is clear that every n-SG ring is an n-wSG ring. Now, if R is an n-wSG ring which is Noetherian or perfect with finite Gorenstein global dimension then $\mathrm{wGgldim}(R) = \mathrm{Ggldim}(R) \leq n$. Thus, Using [3, Corollary 2.7], we have $pd_R(F) \leq n$ for each flat R-module with finite flat dimension. Thus, by Proposition 2.3 and Proposition 2.4, R is an n-SG ring. □

Corollary 2.8. *Let R be a Noetherian ring. The following are equivalent.*

(1) *R is an n-SG ring.*
(2) *For each R-module M there exists an exact sequence of R-module*

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$

where $\mathrm{pd}_R(P) \leq n$.
(3) *For each R-module M there exists an exact sequence of R-module*

$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$

where $\mathrm{fd}_R(F) \leq n$.

Proof. The unique no obvious implication ($3 \Rightarrow 1$) follows from Theorem 2.6 and Proposition 2.4. □

The next result studies the direct product of n-SG rings and n-wSG rings.

Theorem 2.9. *Let $\{R_i\}_{i=1}^m$ be a family of rings and set $R := \prod_{i=1}^{m} R_i$. Then, R is an n-SG ring (resp. n-wSG ring) if and only if R_i is an n-SG ring (resp. n-wSG ring) for each $i = 1, .., m$.*

Proof. By induction on m it suffices to prove the assertion for $m = 2$. First suppose that $R_1 \times R_2$ is an n-SG ring. We claim that R_1 is an n-SG ring. Let M be an arbitrary R_1 module. $M \times 0$ can be viewed as an $R_1 \times R_2$-module. For such module and since $R_1 \times R_2$ is an n-SG ring, there is an exact sequence $0 \longrightarrow M \times 0 \longrightarrow P \longrightarrow M \times 0 \longrightarrow 0$ where $\mathrm{pd}_{R_1\times R_2}(P) \leq n$. Thus, since R_1 is a projective $R_1 \times R_2$ module, by applying $- \otimes_{R_1\times R_2} R_1$ to the sequence above, we find the short exact sequence of R-modules: $0 \longrightarrow M \times 0 \otimes_{R_1\times R_2} R_1 \longrightarrow P \otimes_{R_1\times R_2} R_1 \longrightarrow M \times 0 \otimes_{R_1\times R_2} R_1 \longrightarrow 0$. Clearly $\mathrm{pd}_{R_1}(P \otimes_{R_1\times R_2} R_1) \leq \mathrm{pd}_{R_1\times R_2}(P) \leq n$. Moreover, we have the isomorphism of R-modules:

$$M \times 0 \otimes_{R_1\times R_2} R_1 \cong M \times 0 \otimes_{R_1\times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M.$$

Thus, we obtain an exact sequence of R-module with the form: $0 \longrightarrow M \longrightarrow P \otimes_{R_1\times R_2} R_1 \longrightarrow M \longrightarrow 0$. On the other hand, by [5, Theorem 3.1], we have $\mathrm{Ggldim}(R_1) \leq \mathrm{Ggldim}(R_1 \times R_2) \leq n$. Thus, using Proposition 2.3, R_1 is an n-SG ring, as desired. By the same argument, R_2 is also an n-SG ring.
Now, suppose that R_1 and R_2 are an n-SG rings and we claim that $R_1 \times R_2$ is an n-SG ring. Let M be an $R_1 \times R_2$-module. We have

$$M \cong M \otimes_{R_1\times R_2} (R_1 \times R_2) \cong M \otimes_{R_1\times R_2} ((R_1 \times 0) \oplus (R_2 \times 0)) \cong M_1 \times M_2$$

where $M_i = M \otimes_{R_1\times R_2} R_i$ for $i = 1, 2$. For each $i = 1, 2$, there is an exact sequence $0 \longrightarrow M_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0$ where $\mathrm{pd}_{R_i}(P_i) \leq n$ since R_i is an n-SG ring. Thus, we have the exact sequence of $R_1 \times R_2$-modules:

$$0 \longrightarrow M_1 \times M_2 \longrightarrow P_1 \times P_2 \longrightarrow M_1 \times M_2 \longrightarrow 0.$$

On the other hand, $\mathrm{pd}_{R_1\times R_2}(P_1 \times P_2) = \sup\{\mathrm{pd}_{R_i}(P_i)\}_{1,2} \leq n$ (by [14, Lemma 2.5 (2)]). Moreover, by [5, Theorem 3.1], $\mathrm{Ggldim}(R_1 \times R_2) = \sup\{\mathrm{Ggldim}(R_i)\}_{1,2} \leq n$. Thus, from Proposition 2.3, $R_1 \times R_2$ is an n-SG ring, as desired.

Using [17, Theorem 2.12], [5, Lemma 3.7], and Proposition 2.4, we prove the n-wSG case by the same reasoning as above. □

Let $T := R[X_1, X_2, ..., X_n]$ be the polynomial ring in n indeterminates over R. If we suppose that T is an m-SG ring (resp. m-wSG ring), it is easy, by [5, Theorem 2.1] (rep. [17, Theorem 2.13]), to see that $n \leq m$.

Theorem 2.10. *If $R[X_1, X_2, ..., X_n]$ is an m-SG ring (resp. m-wSG ring) then R is an $(m - n)$-SG ring (resp. $(m - n)$-wSG ring).*

Proof. By induction on n is suffices to prove the result for $n = 1$. So, suppose that $R[X]$ is an m-SG ring. Let M be an arbitrary R-module. For the $R[X]$-module $M[X] := M \otimes_R R[X]$ there is an exact sequence of $R[X]$-modules $0 \longrightarrow M[X] \longrightarrow P \longrightarrow M[X] \longrightarrow 0$ where $\mathrm{pd}_{R[X]}(P) \leq m$ (resp. $\mathrm{fd}_{R[X]}(P) \leq m$). Applying $- \otimes_{R[X]} R$ to the short exact sequence above and seeing that $M \cong_R M[X] \otimes_{R[X]} R$,

we obtain a short exact sequence of R-modules with the form $0 \longrightarrow M \longrightarrow P \otimes_{R[X]} R \longrightarrow M \longrightarrow 0$ (see that R is a projective $R[X]$-module). Moreover, $\mathrm{pd}_R(P \otimes_{R[X]} R) \leq \mathrm{pd}_{R[X]}(P) < \infty$ (resp. $\mathrm{fd}_R(P \otimes_{R[X]} R) \leq \mathrm{fd}_{R[X]}(P) < \infty$). On the other hand, by [5, Theorem 2.1] (resp. [17, Theorem 2.13]), $\mathrm{Ggldim}(R) = \mathrm{Ggldim}(R[X]) - 1 \leq m - 1$ (resp. $\mathrm{wGgldim}(R) = \mathrm{wGgldim}(R[X]) - 1 \leq m - 1$). Hence, by Proposition 2.3 (resp. Proposition 2.4), R is an $(m-1)$-SG ring (resp. $(m-1)$-wSG ring). □

Trivial examples of n-SG-ring (resp. n-wSG ring) are the rings with global dimension (resp. weak global dimension) $\leq n$. The following example gives a new family of n-SG rings (resp. n-wSG rings) with infinite weak global dimension.

Example 2.11. Consider the non semi-simple quasi-Frobenius rings $R_1 := K[X]/(X^2)$ and $R_2 := K[X]/(X^3)$ where K is a field, and let S be a non Noetherian ring such that $gldim(S) = n$. Then,

(1) $\mathrm{Ggldim}(R_1) = \mathrm{Ggldim}(R_2) = 0$ and R_1 is 0-SG ring but R_2 is not.
(2) $R_1 \times S$ is a non Noetherian n-SG (and so n-wSG) ring with infinite weak global dimension.
(3) $\mathrm{Ggldim}(R_2 \times S) = n$ but $R_2 \times S$ is not an n-SG ring.
(4) $\mathrm{wGgldim}(R_2[X_1, ..., X_n]) = \mathrm{Ggldim}(R_2[X_1, ..., X_n]) = n$ but $R_2[X_1, ..., X_n]$ is neither an n-SG ring nor an n-wSG ring.

Proof. From [6, Corollary 3.9] and [3, Proposition 2.6], $\mathrm{Ggldim}(R_1) = \mathrm{Ggldim}(R_2) = 0$ and R_1 is 0-SG ring but R_2 is not. So, (1) is clear. Moreover R_1 and R_2 have infinite weak global dimensions. By [5, Theorems 2.1 and 3.1] and Lemma 2.7 and the fact that R_2 is Noetherian, it is easy to see that,

- $\mathrm{Ggldim}(R_2 \times S) = n$, and
- $\mathrm{wGgldim}(R_2[X_1, ..., X_n]) = \mathrm{Ggldim}(R_2[X_1, ..., X_n]) = n$.

And using Theorems 2.9 and 2.10, we conclude that $R_1 \times S$ is an n-SG (and so n-wSG by Theorem 2.6) and that $R_2[X_1, ..., X_n]$ is neither n-SG ring nor n-wSG ring (by Theorem 2.6 since R_2 is Noetherian). Hence, (2), (3) and (4) holds. □

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