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HAHN-BANACH THEOREM-BASED ANALYTIC APPROACHES FOR THE ABSTRACT CAUCHY-EULER DIFFERENTIAL EQUATION

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Abstract. In this paper, we consider the following abstract Cauchy-Euler differential equation $t^2u''(t) + tu'(t) + u(t) = 0$, where $u : [0, t] \to X$ is an unknown function such that X is a Banach space. Utilizing the Hahn-Banach Theorem, two analytic approaches are presented.

1. Introduction

The classical Cauchy-Euler differential equation is a linear differential equation with variable coefficients of the form

$$\alpha_n t^n u^{(n)}(t) + \alpha_{n-1} t^{n-1} u^{(n-1)}(t) + \cdots + \alpha_1 t u'(t) + \alpha_\circ u(t) = f(t), \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is the unknown function, $f(t)$ is a given function, and $\alpha_n, \alpha_{n-1}, \cdots, \alpha_\circ$ are constants. If $f(t) = 0$, then the given (CEDE) is called homogeneous; otherwise it is said to be non-homogeneous. A distinguishing feature of (CEDEs) is that in each particular term of 1.1, the powers of the variable t match the order of the derivative of the unknown function $u(t)$ [1].

This type of equations often appear when dealing with mathematical models in both science and engineering [7], [8]. Problems concerning the analysis and design of computer algorithms often result in (CEDEs). Moreover, (CEDEs) come to light when solving partial differential equations by the method of Fourier. They also arise when using partial differential equations for modeling natural phenomena at which equilibrium characteristics are observed [2].

The solution of homogeneous (CEDEs) can be obtained by only assuming $u = t^m$ in (1.1) while for the non-homogeneous (CEDEs), the structure of the solution is of the form $u = u_h + u_p$, where u_h is the solution of the associated homogeneous (CEDE) and u_p is any particular solution to the original (CEDE) which can be determined by the mean of the variation of parameters method.

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In this paper, we are interested in developing two analytic approaches in order to solve the following homogeneous second order abstract Cauchy-Euler differential equation (ACEDE)

$$t^2u''(t) + tu'(t) + u(t) = 0, \tag{1.2}$$

where $u : [0, \infty] \to X$ is the unknown function such that X is a Banach space. To reach our goal, we employ a well known result in functional analysis namely, the Hahn-Banach theorem.

2. Preliminaries

It is a well-known fact that one can study the aspects of a given linear space by studying the features of the space of functions defined on that space [3]. In this section, we present some concepts and results that are needed for our main results.

The Hahn-Banach theorem is recognized as one of the most important results in functional analysis. This theorem enables us to extend bounded linear functionals that is defined on a subspace of a vector space to the whole space. Consequently, the dual space X^* is a non-trivial space associated with X.

Theorem 2.1. [2] *(Hahn-Banach Theorem) Let M be a linear subspace of a normed linear space N and let $f \in M^*$. Then f can be extended to a bounded linear functional $F \in N^*$ (defined on the whole linear space N) such that $F|_M = f$ and $\|f\|_{M^*} = \|F\|_{N^*}$.*

A consequence of Hahn-Banach theorem, is the following corollary.

Corollary 2.2. [2] *Let N be a normed linear space. Then for all $x \in N$ there exists $x^* \in N^*$ such that $\|x^*\|_{N^*} = 1$ and $x^*(x) = \|x\|_N$.*

Definition 2.3. [4] Let X and Y be any two Banach spaces and X^* is the dual space of X. For $x \in X$ and $y \in Y$, the operator $T : X^* \to Y$, defined by $T(x^*) = x^*(x)y = \langle x, x^* \rangle y$, is bounded one rank linear operator. We write $x \otimes y$ for T and such operators are called atoms.

Now, if we assume two atoms, namely, $x_1 \otimes y_1$ and $x_2 \otimes y_2$ such that $x_1 \otimes y_1 = x_2 \otimes y_2$, then for any $x^* \in X^*$ we have

$$\langle x_1, x^* \rangle y_1 = \langle x_2, x^* \rangle y_2.$$

Thus, $y_1 = y_2$. Similarly, one can prove $x_1 = x_2$.

It should be noted that any solution to an (ACEDE) that has the form. $x \otimes y = x^*(x)y = \langle x, x^* \rangle y$, is said to be an atomic solution.

3. Main Results

We present two novel analytical approaches for solving the (ACEDE) (1.2).

Approach 1.:

Theorem 3.1. *Consider the equation*

$$t^2u''(t) + tu'(t) + u(t) = 0, \tag{3.1}$$

where $u : [a, b] \to X$ is the unknown function such that X is a Banach space. Then

$$\langle u(t), x^* \rangle = c_1 \cos(\ln t) + c_2 \sin(\ln t),$$

where $x^* \in X^*$, $\|x^*\| = 1$, $\langle u(t), x^* \rangle \neq 0$, $\langle u'(t), x^* \rangle \neq 0$, and $\langle u''(t), x^* \rangle \neq 0$ for all $t \in [a, b]$ such that $u(t) \neq 0$, $u'(t) \neq 0$, and $u''(t) \neq 0$.

Proof. Let $x^* \in X^*$ with $\|x^*\| = 1$ such that for all $\langle u(t), x^* \rangle \neq 0$, $\langle u'(t), x^* \rangle \neq 0$, and $\langle u''(t), x^* \rangle \neq 0$. Apply x^* to both sides of (3.1) to get

$$t^2 \langle u''(t), x^* \rangle + t \langle u'(t), x^* \rangle + \langle u(t), x^* \rangle = 0. \tag{3.2}$$

Now, we rewrite $\langle u'(t), x^* \rangle$ as follows:

$$\begin{aligned} \langle u'(t), x^* \rangle &= \left\langle \frac{du(t)}{dt}, x^* \right\rangle \\ &= \left\langle \lim_{\triangle t \to 0} \frac{u(t + \triangle t) - u(t)}{\triangle t}, x^* \right\rangle \\ &= \lim_{\triangle t \to 0} \left(\frac{\langle u(t + \triangle t), x^* \rangle - \langle u(t), x^* \rangle}{\triangle t} \right) \\ &= \frac{d}{dt} \langle u(t), x^* \rangle . \end{aligned} \tag{3.3}$$

Similarly, one can prove that

$$\langle u''(t), x^* \rangle = \frac{d^2}{dt^2} \langle u(t), x^* \rangle . \tag{3.4}$$

Hence, by substituting (3.3) and (3.4) into (3.2), we get

$$t^2h''(t) + th'(t) + h(t) = 0, \tag{3.5}$$

where $h(t) = \langle u(t), x^* \rangle$. But, (3.5) is the classical (CEDE) with the corresponding characteristic equation $m(m-1)+m+1 = 0$, which can be obtained by assuming $h(t) = t^m$. Therefore, the solution to (3.1) is $h(t) = \langle u(t), x^* \rangle = c_1 \cos(\ln t) + c_2 \sin(\ln t)$. This ends the proof. □

Theorem 3.2. *Consider the equation*

$$t^2u''(t) + tu'(t) + u(t) = 0, \tag{3.6}$$

where $u : [a, b] \to X$ is the unknown function such that X is a Banach space. Then (3.6) has the following atomic solution

$$u(t) = [c_1 \cos(\ln t) + c_2 \sin(\ln t)]\, x,$$

for all $x \neq 0 \in X$.

Proof. Let $u(t) = v(t)\, x$ where $v : [a, b] \to \mathbb{R}$ and $x \neq 0 \in X$. By corollary 2.1, choose $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^*(x) = 1$. Thus, by applying x^* on both sides of (3.6), we have

$$t^2 \langle u''(t), x^* \rangle + t \langle u'(t), x^* \rangle + \langle u(t), x^* \rangle = 0. \tag{3.7}$$

Now, substitute $u(t) = v(t)x$ into (3.7) to get

$$t^2 v''(t) \langle x, x^* \rangle + tv'(t) \langle x, x^* \rangle + v(t) \langle x, x^* \rangle = 0. \tag{3.8}$$

But, $\langle x, x^* \rangle = x^*(x) = 1$. Therefore, (3.8), can be rewritten as

$$t^2 v''(t) + tv'(t) + v(t) = 0, \tag{3.9}$$

which is the classical (CEDE), and the corresponding characteristic equation is $m(m-1) + m + 1 = 0$, which can be obtained by assuming $v(t) = t^m$. Thus, the solution to (3.9) is $v(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t)$. Consequently, since $u(t) = v(t)x$, the equation (3.6) admits an atomic solution of the form

$$u(t) = [c_1 \cos(\ln t) + c_2 \sin(\ln t)]\, x,$$

for all $x \in X$. This ends the proof. □

Lemma 3.3. *A two rank atomic solution to* (3.6), *is a solution of the form*

$$u(t) = v_1(t)x_1 + v_2(t)x_2, \tag{3.10}$$

where v_1,$v_2 : [a, b] \to \mathbb{R}$ *and* x_1, $x_2 \in X$ *are linearly independent non-zero vectors.*

Proof. By Corollary 2.1, one can choose $x_1^*, x_2^* \in X^*$ with $\|x_1^*\| = \|x_2^*\| = 1$ together with the following two cases:

$$\begin{aligned} &\text{case (1):} \quad x_1^*(x_1) = 1, \;\; x_1^*(x_2) = 0, \\ &\text{case (2):} \quad x_2^*(x_1) = 0, \;\; x_2^*(x_2) = 1. \end{aligned} \tag{3.11}$$

Now, by substituting (3.10) in to (3.6), we have

$$t^2 \left[v_1^{''}(t)x_1 + v_2^{''}(t)x_2 \right] + t \left[v_1^{'}(t)x_1 + v_2^{'}(t)x_2 \right] + [v_1(t)x_1 + v_2(t)x_2] = 0. \tag{3.12}$$

□

Then, for case (1) of (3.11), if we apply x_1^* to both sides of (3.12), it can be reduced as

$$t^2 v_1^{''}(t) + tv_1^{'}(t) + v_1(t) = 0. \tag{3.13}$$

Thus,

$$v_1(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t). \tag{3.14}$$

Similarly, for case (2) of (3.11), if we apply x_2^* to both sides of (3.12), then

$$t^2 v_2^{''}(t) + tv_2^{'}(t) + v_2(t) = 0, \tag{3.15}$$

and hence,

$$v_2(t) = c_3 \cos(\ln t) + c_4 \sin(\ln t). \tag{3.16}$$

Consequently, by (3.10), (3.14), and (3.16) an atomic solution to (3.6) can be obtained as

$$\begin{aligned} u(t) &= v_1(t)x_1 + v_2(t)x_2 \\ &= [c_1 \cos(\ln t) + c_2 \sin(\ln t)]\, x_1 \\ &\quad + [c_3 \cos(\ln t) + c_4 \sin(\ln t)]\, x_2. \end{aligned} \tag{3.17}$$

In general, for (3.6), we can handle the finite rank atomic solutions case as

$$u(t) = \sum_{i=1}^{n} v_i(t)x_i,$$

where $v_i(t) = c_{i,1} \cos(\ln t) + c_{i,2} \sin(\ln t)$ with $x_i \neq 0 \in X$ are linearly independent vectors.

4. Conclusion

This paper successfully introduces two new analytic approaches for solving homogeneous second order abstract Cauchy-Euler differential equation (ACEDE). We utilize a consequence of the well-known Hahn-Banach theorem. In both approaches, we could reduce the original (ACEDE) into the classical (CEDE) and then we were able to formulate out two approaches for each particular case.

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