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ON COVERING NUMBER OF C^*-ALGEBRAS

SIRAJUDHEEN M. P.[1*] and RAVINDRAN K. T.[2]

Abstract. We introduce two new invariants for C^*-algebras: the approximate covering number, $ACov(\mathfrak{A}, m)$, and the unit approximate covering number, $UACov(\mathfrak{A}, m)$. Additionally, we define three tracial covering numbers: the tracial covering number, the tracial approximate covering number, and the tracial unit approximate covering number. We establish equivalent conditions for a number to be the approximate covering number or the unit approximate covering number. For unital $\mathcal{Z}$-stable C^*-algebras and simple, unital, separable AF-algebras, covering numbers are shown to be at most 2. Furthermore, we demonstrate that if a C^*-algebra $\mathfrak{A}$ is weakly purely infinite and $ACov(\mathfrak{A}, m) = 1$ for all $m \geq 2$, then $\mathfrak{A}$ is purely infinite. If the approximate covering number or unit approximate covering number of $\mathfrak{A}$ is 1, then $\mathfrak{A}$ has an abundance of soft elements, and $(Cu(\mathfrak{A}), \Sigma_{\mathfrak{A}})$ is ideal-filtered precisely when $\mathfrak{A}$ has an abundance of soft elements and $Cu(\mathfrak{A})$ is ideal-filtered. A similar result holds for unital $\mathfrak{A}$ with covering number 1. We prove that if Λ is a collection of unital nuclear C^*-algebras for which the tracial covering number does not exceed m, then, any simple unital C^*-algebra $\mathfrak{A}$ that falls within the corresponding collection of weakly tracially approximated C^*-algebras will also possess tracial covering number of at most m. Similar conclusions are derived for the tracial approximate covering number and the tracial unit approximate covering number, and finally, we explore the relationships among these three types of tracial covering numbers.

1. Introduction

Eberhard Kirchberg introduced the covering number of unital C^*-algebras in 2004 [13]. We extend this concept to non-unital C^*-algebras by using approximate units and defining a new invariant called the *approximate covering number.* Furthermore, we replace the approximate unit with the unit ball and introduce another invariant, the *unit approximate covering number.*

Elliott, Fan, and Fang developed the concept of *weakly tracially approximated* C^*-algebras, extending the ideas of tracial approximation and large sub-algebras. Their work studies tracial $\mathcal{Z}$-absorbing algebras, *tracial nuclear dimension, and m-almost divisibility* [8]. Inspired by these ideas, we also introduce three tracial

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* Corresponding author.
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covering numbers: the *tracial covering number*, the *tracial approximate covering number*, and the *tracial unit approximate covering number*.

The approximate covering number and the unit approximate covering number for C*-algebras are crucial invariants that provide significant theoretical insights and practical applications in operator algebras. The approximate covering number, defined for non-unital C*-algebras, incorporates approximate units and broadens the analysis to a wider class of algebras relevant in mathematical physics. In contrast, the unit approximate covering number specifically applies to unital C*-algebras. Both invariants refine the classification of C*-algebras based on properties such as simplicity, finiteness, and pure infiniteness, while also influencing the abundance of soft elements and the ideal-filtered nature of the Cuntz semigroup. Furthermore, these covering numbers are essential for identifying C*-algebras with infinite irreducible representations, distinguishing purely infinite and properly infinite C*-algebras that lack finite-dimensional irreducible representations. This understanding is vital for exploring infinite-dimensional behavior in *operator algebras* and *non-commutative geometry*. Ultimately, we aim to extend the theoretical frameworks and results of the covering number of unital C*-algebras to the non-unital case, enhancing classification schemes and facilitating the analysis of more complex algebraic structures across various applications.

This paper is structured in the following manner. In Section 2, we present frequently used notations and fundamental concepts. In Section 3, we introduce the *approximate covering number* $ACov(\mathfrak{A}, m)$ for a C^*-algebra $\mathfrak{A}$, establishing conditions for its finiteness and exploring its relationship with other invariants. Furthermore, we prove several related results.

In Section 4, We define the *unit approximate covering number* of a C^*-algebra and explore its relation to other invariants, such as the approximate covering number, covering number, and weak divisibility number. For any unital extension $0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{C} \longrightarrow 0$, we show that $UACov(\mathfrak{C}, m) \leq UACov(\mathfrak{B}, m) \leq UACov(\mathfrak{A}, m)$. If $UACov(\mathfrak{C}, m) = 1$, then $UACov(\mathfrak{A}, m) = UACov(\mathfrak{B}, m) = UACov(\mathfrak{C}, m) = 1$.

In Section 5, We show that both the *approximate covering number* and the *unit approximate covering number* of a properly infinite C^*-algebra are equal to 1. For unital $\mathcal{Z}$-stable C^*-algebras and simple, unital, separable AF-algebras, these numbers are at most 2. Furthermore, if $\mathfrak{A}$ is weakly purely infinite and $ACov(\mathfrak{A}, m) = 1$ for all $m \geq 2$, then $\mathfrak{A}$ is purely infinite. We also present equivalent conditions for a C^*-algebra $\mathfrak{A}$ to have an abundance of soft elements, ideal-filtered Cuntz semigroups.

In Section 6, We define the *tracial covering number*, *tracial approximate covering number*, and *tracial unit approximate covering number*. If Λ denotes a collection of unital nuclear C^*-algebras with tracial covering number not exceeding m, then any simple unital C^*-algebra $\mathfrak{A}$ within the corresponding class of weakly tracially approximated C^*-algebras will also have tracial covering number that does not exceed m. Similar results hold for the tracial approximate covering number and the tracial unit approximate covering number. Finally, we establish the relationships among these three tracial covering numbers.

2. Preliminary

The set of all positive elements in $\mathfrak{A}$ is denoted by $\mathfrak{A}_+$. We say that $a, b \in \mathfrak{A}_+$ are equivalent, written as $a \sim' b$, if $a = x^*x$ and $b = xx^*$ for some $x \in \mathfrak{A}$.

Let $k \in \mathbb{Z}^+$, and let $\iota : [0,1] \to \mathbb{C}$ be defined by $\iota(t) = t$. An element $a \in \mathfrak{A}_+$ is called k-homogeneous if there is a $*$-homomorphism $\phi : C_0((0,1]) \otimes M_k(\mathbb{C}) \to \mathfrak{A}$ such that $\phi(\iota \otimes 1_{M_k(\mathbb{C})}) = a$. Let $m \in \mathbb{Z}^+$ and $\mathfrak{A}$ be unital. The covering number, denoted by $Cov(\mathfrak{A}, m)$, is the smallest positive integer n such that there exist $a_1, a_2, \dots, a_n \in \mathfrak{A}_+$ and $b_1, b_2, \dots, b_n \in \mathfrak{A}$ such that $1_{\mathfrak{A}} = \sum_{i=1}^{n} b_i^* a_i b_i$, and each a_i is the sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, r_i$. If no such n exists, we write $Cov(\mathfrak{A}, m) = \infty$. For more details, see [13]. Let $\mathfrak{A}$ be a C^*-algebra and let $a, b \in \mathfrak{A}_+$. We say that a is Cuntz subequivalent to b (see [1, 12, 23]) if and only if for any $\epsilon > 0$, there exists an element r such that $\|a - rbr^*\| < \epsilon$. In this case, we write $a \overset{\mathfrak{A}}{\precsim} b$. We have $a \overset{\mathfrak{A}}{\sim} b$ if and only if $a \overset{\mathfrak{A}}{\precsim} b$ and $b \overset{\mathfrak{A}}{\precsim} a$. The Cuntz semigroup of $\mathfrak{A}$ is defined as $Cu(\mathfrak{A}) = (\mathfrak{A} \otimes \mathcal{K})_+ / \overset{\mathfrak{A}}{\sim}$, where the addition $+$ on $Cu(\mathfrak{A})$ is given by

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right].$$

Next, we recall some useful definitions.

Definition 2.1. ([8, 10, 11]) Let Λ denote a collection of unital C^*-algebras. A unital C^*-algebra $\mathfrak{A}$ is classified as weakly tracially approximated (denoted $WTA\Lambda$) if the following conditions are satisfied for any $\epsilon > 0$, any finite subset $F \subset \mathfrak{A}$, and any non-zero element $a \in \mathfrak{A}_+$: There exists a projection $\mathfrak{p} \in \mathfrak{A}$, an element $h \in \mathfrak{A}$ with $0 \leq h \leq 1$, and a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $h \in \mathfrak{B}$, $1_{\mathfrak{B}} = \mathfrak{p}$, and $\mathfrak{B} \in \Lambda$. Moreover:

(1) For every $w \in F$, it holds that $(\mathfrak{p} - h)w \in_\epsilon \mathfrak{B}$ and $w(\mathfrak{p} - h) \in_\epsilon \mathfrak{B}$.
(2) For every $w \in F$, it holds that $\|(\mathfrak{p} - h)w - w(\mathfrak{p} - h)\| < \epsilon$.
(3) The relation $1 - (\mathfrak{p} - h) \overset{\mathfrak{A}}{\precsim} a$ holds.
(4) $\|(\mathfrak{p} - h)a(\mathfrak{p} - h)\| \geq \|a\| - \epsilon$.

Definition 2.2. ([8, 9]) Let Λ denote a collection of unital C^*-algebras. A unital C^*-algebra $\mathfrak{A}$ is classified as a tracially approximated C^*-algebra, denoted $TA\Lambda$, if the following conditions are satisfied for any $\epsilon > 0$, any finite subset $F \subset \mathfrak{A}$, and any non-zero element $a \in \mathfrak{A}_+$:

There exists a projection $\mathfrak{p} \in \mathfrak{A}$ and a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ with $1_{\mathfrak{B}} = \mathfrak{p}$ and $\mathfrak{B} \in \Lambda$ such that:

(1) For every $x \in F$, it holds that $\|\mathfrak{p}x - x\mathfrak{p}\| < \epsilon$.
(2) For every $x \in F$, it holds that $\mathfrak{p}x\mathfrak{p} \in_\epsilon \mathfrak{B}$.
(3) The relation $1 - \mathfrak{p} \overset{\mathfrak{A}}{\precsim} a$ is satisfied.

3. Approximate Covering Number

The idea of an approximation covering number of a C^*-algebra is introduced in this section. We provide equivalent conditions for a number to be an approximate covering number. Additionally, we explore the relationship between approximate covering number, covering number, and the weak divisibility numbers of a unital C^*-algebra.

Definition 3.1. Let $\mathfrak{A}$ be any C^*-algebra and $m \in \mathbb{Z}_+$. The approximate covering number of $\mathfrak{A}$, denoted by $ACov(\mathfrak{A}, m)$, is the least positive integer n such that, for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there exist elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum, $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots r_i$ satisfying $\|a(\sum_{i=1}^{n} w_i^* a_i w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^{n} w_i^* a_i w_i)a - a\| < \epsilon$.

Remark 3.2. If there is no $n \in \mathbb{Z}_+$ so that $ACov(\mathfrak{A}, m) = n$, then we set $ACov(\mathfrak{A}, m) = \infty$.

Proposition 3.3. .
If $\{a_\lambda\}$ is an approximate unit of $\mathfrak{A}$ satisfying $a_\lambda = \sum_{i=1}^{k} b_{i,\lambda}^* a_{i,\lambda} b_{i,\lambda}$, where each $a_{i,\lambda}$ is the sum of mutually orthogonal $k_{i,\lambda,j}$-homogeneous elements with $k_{i,\lambda,j} \geq m$, then $ACov(\mathfrak{A}, m) \leq k$.

Proof. The proof directly follows from the definition. □

Corollary 3.4. *If $ACov(\mathfrak{A}, m) < \infty$, then $\mathfrak{A}$ has a approximate unit $\{a_\lambda\}$ such that each a_λ is finite sum of positive elements which are cuntz subequivalent to mutually orthogonal $k_{\lambda,j}$-homogeneous elements with $k_{\lambda,j} \geq m$.*

Proof. The proof directly follows. □

Next, we establish the relationship between the approximate covering number, the covering number, and the weak divisibility number of a unital C^*-algebra.

Lemma 3.5. *If $\mathfrak{A}$ is unital and $m \in \mathbb{Z}^+$. Then, $ACov(\mathfrak{A}, m) \leq Cov(\mathfrak{A}, m)$.*

Proof. If $Cov(\mathfrak{A}, m) = n$, for some $n \in \mathbb{Z}^+$. Then, there are $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that $1_{\mathfrak{A}} = \sum_{i=1}^{n} w_i^* a_i w_i$ and a_i is the sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r_i$. Then, for any $\epsilon > 0$ and $a \in \mathfrak{A}$, we have $\|a(\sum_{i=1}^{n} w_i^* a_i w_i) - a\| = 0 < \epsilon$ and $\|(\sum_{i=1}^{n} w_i^* a_i w_i)a - a\| = 0 < \epsilon$. Therefore, $ACov(\mathfrak{A}, m) \leq n$. □

Proposition 3.6. *If $\mathfrak{A}$ is unital and $m \in \mathbb{Z}^+$, then $ACov(\mathfrak{A}, m) \leq Cov(\mathfrak{A}, m) \leq w - Div_m(\mathfrak{A}) \leq (2m-1)Cov(\mathfrak{A}, m)$.*

Proof. This follows from lemma 3.5 and by [[20], Proposition 3.17] □

Proposition 3.7. *If $m \leq l$ then, $ACov(\mathfrak{A}, m) \leq ACov(\mathfrak{A}, l)$.*

Proof. Suppose that $m \leq l$ and $ACov(\mathfrak{A}, l) = n$. Then, for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq l$ for each i ranging from 1 to n and for each j from 1 to r_i satisfying $\|a(\sum_{i=1}^{n} w_i^* a_i w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^{n} w_i^* a_i w_i)a - a\| < \epsilon$. Since $m \leq l$, $k_{i,j} \geq m$. Therefore, $ACov(\mathfrak{A}, m) \leq n$. □

Proposition 3.8. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is non zero $* -$ homomorphism, then $ACov(\phi[\mathfrak{A}], m) \leq ACov(\mathfrak{A}, m)$.*

Proof. This follows directly from the definition. $\square$

Definition 3.9. Let $\mathfrak{A}$ be a C^*-algebra. We define $ACov(\mathfrak{A}) = \sup_m ACov(\mathfrak{A}, m)$.

Proposition 3.10. *If $\mathfrak{A}$ is a unital, nuclear, separable C^*-algebra with finite decomposition rank $\mathrm{dr}(\mathfrak{A})$ that does not possess any finite-dimensional irreducible representations, then $ACov(\mathfrak{A}) \leq \mathrm{dr}(\mathfrak{A}) + 1$.*

Proof. By lemma 3.5, $ACov(\mathfrak{A}, m) \leq \sup_m Cov(\mathfrak{A}, m) = Cov(\mathfrak{A})$ for all m. Therefore, $ACov(\mathfrak{A}) \leq Cov(\mathfrak{A})$. Hence by [[13], Proposition 2.3], $ACov(\mathfrak{A}) \leq \mathrm{dr}(\mathfrak{A}) + 1$. $\square$

Next, we will prove the equivalent conditions for a number to be the approximate covering number.

Proposition 3.11. *For any C^*-algebra $\mathfrak{A}$ and positive integer m, the following conditions are equivalent:*

(1) $ACov(\mathfrak{A}, m) = n$.
(2) n is the least positive integer such that for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there are finite-dimensional C^*-algebras $\mathcal{F}_i$, $1 \leq i \leq n$, with $\mathrm{rank}(\mathcal{F}_i) \geq m$, $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$, and $b_1, b_2, \ldots, b_n \in \mathfrak{A}$ satisfying $\|a(\sum_{i=1}^n b_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^n b_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i) a - a\| < \epsilon$.

Proof. Suppose n is a positive integer such that for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $b_1, b_2, \ldots, b_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for each i ranging from 1 to n and for each j from 1 to r_i, satisfying $\|a(\sum_{i=1}^n b_i^* a_i b_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^n b_i^* a_i b_i) a - a\| < \epsilon$. Set $\mathcal{F}_i = \bigoplus_{j=1}^{r_i} M_{k_{i,j}}(\mathbb{C})$. Since $\{a_{i,j}\}_{j=1}^{r_i}$ are mutually orthogonal $k_{i,j}$-homogeneous elements, $\{a_{i,j}\}_{j=1}^{r_i}$ mutually orthogonal and pairwise equivalent. Then, by universal property of cone over matrix algebra, there are $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$ such that $\phi_i(\iota \otimes e^{i,j}) = a_{i,j}$, where $e^{i,j}$ is one dimensional projection in $M_{k_{i,j}}(\mathbb{C})$.

$$\begin{aligned}\|a \sum_{i=1}^n b_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i - a\| &= \|a \sum_{i=1}^n b_i^* \sum_{j=1}^{r_i} a_{i,j} b_i - a\| \\ &= \|a(\sum_{i=1}^n b_i^* a_i b_i) - a\| \\ &< \epsilon.\end{aligned}$$

Similarly we can show that $\|(\sum_{i=1}^n b_i \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i) a - a\| < \epsilon$.

Conversely, Suppose that n is a positive integer such that for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there are finite-dimensional C^*-algebras $\mathcal{F}_i$, $1 \leq i \leq n$, with $\mathrm{rank}(\mathcal{F}_i) \geq m$, $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$, and $b_1, b_2, \ldots, b_n \in \mathfrak{A}$ satisfying $\|a(\sum_{i=1}^n b_i \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i - a\| < \epsilon$ and $\|(\sum_{i=1}^n b_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) b_i) a - a\| < \epsilon$. Take $a_i = \phi_i(\iota \otimes 1_{\mathcal{F}_i}), a_{i,j} = \phi_i(\iota \otimes e^{i,j})$, then $a_i = \sum_{j=1}^{r_i} a_{i,j}$ and $a_{i,j}$ are mutually orthogonal $m_{i,j}$-homogeneous elements $\in \mathfrak{A}_+$ with $m_{i,j} \geq m$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots r_{i,k}$ satisfying $\|a(\sum_{i=1}^n b_i^* a_i b_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^n b_i^* a_i b_i)a - a\| < \epsilon$. □

Proposition 3.12. *If $\mathfrak{A}$ and $\mathfrak{B}$ are unital C^*-algebras, and one of $Cov(\mathfrak{A}, m)$ or $Cov(\mathfrak{B}, m)$ is finite, then $ACov(\mathfrak{A} \otimes \mathfrak{B}, m) < \infty$ for any tensor product $\otimes$.*

Proof. By [[13], Remark 2.2], $Cov(\mathfrak{A} \otimes \mathfrak{B}, m) \leq Cov(\mathfrak{A}, m)$. Therefore, by Proposition 3.6, $ACov(\mathfrak{A} \otimes \mathfrak{B}, m) \leq Cov(\mathfrak{A} \otimes \mathfrak{B}, m) < \infty$. □

4. Unit Approximate Covering Number

This section introduces the idea of the unit approximate covering number for a C^*-algebra. We present conditions that are equivalent for a number to be recognized as a unit approximate covering number. We also delve into the connections between the approximate covering number, unit approximate covering number, covering number, and weak divisibility numbers of an unital C^*-algebra.

Proposition 4.1. *Let $\mathfrak{A}$ be a unital C^*-algebra, where $m, n \in \mathbb{Z}_+$. The conditions listed below are equivalent:*

(1) For any $\epsilon > 0$, there are elements $a_1, a_2, \dots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \dots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots r_i$ satisfying $\|a(\sum_{i=1}^n w_i^* a_i w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^n w_i^* a_i w_i)a - a\| < \epsilon$ for all $\|a\| \leq 1$.
(2) For any $\epsilon > 0$, there are elements $a_1, a_2, \dots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \dots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots r_i$ satisfying $\|(\sum_{i=1}^n w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$.
(3) For any $\epsilon > 0$, there are finite-dimensional C^*-algebras $\mathcal{F}_i$, $1 \leq i \leq n$, with $\operatorname{rank}(\mathcal{F}_i) \geq m$, $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$, and $w_1, w_2, \dots, w_n \in \mathfrak{A}$ satisfying $\| \sum_{i=1}^n w_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) w_i - 1_{\mathfrak{A}}\| < \epsilon$.

Proof. (1) $\Rightarrow$ (2). Assume that (1) holds. Let $\epsilon > 0$, there are elements $a_1, a_2, \dots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \dots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for each i ranging from 1 to n and for each j from 1 to r_i, satisfying $\|a(\sum_{i=1}^n w_i^* a_i w_i) - a\| < \epsilon$ for all $\|a\| \leq 1$. Therefore, $\|(\sum_{i=1}^n w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $\epsilon > 0$, there are elements $a_1, a_2, \dots, a_{n,} \in \mathfrak{A}_+$ and $w_1, w_2, \dots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for each i ranging from 1 to n and for each j from 1 to r_i, satisfying

$\|(\sum_{i=1}^n w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$. Let $a \in \mathfrak{A}$ such that $\|a\| \leq 1$, then

$$\begin{aligned}\|a(\sum_{i=1}^n w_i^* a_i w_i) - a\| &= \|a(\sum_{i=1}^n w_i^* a_i w_i - 1_{\mathfrak{A}})\| \\ &\leq \|a\| \|\sum_{i=1}^n w_i^* a_i w_i - 1_{\mathfrak{A}}\| \\ &< \epsilon.\end{aligned}$$

(1) $\Rightarrow$ (3). Assume that (1) holds. Let $\epsilon > 0$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots r_i$ satisfying $\|a(\sum_{i=1}^n w_i^* a_i w_i) - a\| < \epsilon$ for all $\|a\| \leq 1$. Set $\mathcal{F}_i = \bigoplus_{j=1}^{r_i} M_{k_{i,j}}(\mathbb{C})$. Since $\{a_{i,j}\}_{j=1}^{r_i}$ are mutually orthogonal $k_{i,j}$-homogeneous elements, $\{a_{i,j}\}_{j=1}^{r_i}$ mutually orthogonal and pairwise equivalent. Then, by universal property of cone over matrix algebra, there are $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$ such that $\phi_i(\iota \otimes e^{i,j}) = a_{i,j}$, where $e^{i,j}$ is one-dimensional projection in $M_{k_{i,j}}(\mathbb{C})$. Therefore,

$$\begin{aligned}\|\sum_{i=1}^n w_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) w_i - 1_{\mathfrak{A}}\| &= \|\sum_{i=1}^n w_i^* \sum_{j=1}^{r_i} a_{i,j} w_i - 1_{\mathfrak{A}}\| \\ &\leq \|\sum_{i=1}^n w_i^* a_i w_i - 1_{\mathfrak{A}}\| \\ &< \epsilon.\end{aligned}$$

(3) $\Rightarrow$ (1). Assume that (3) holds. Take $a_i = \phi_i(\iota \otimes 1_{\mathcal{F}_i}), a_{i,j} = \phi_i(\iota \otimes e^{i,j})$. Then it can be seen that (1) holds. □

Now, we are ready to define the unit approximate covering Number.

Definition 4.2. Let $\mathfrak{A}$ be a C^*-algebra and $m \in \mathbb{Z}^+$. The unit approximate covering number of $\mathfrak{A}$, denoted by $UACov(\mathfrak{A}, m)$, is the smallest positive integer n such that for any $\epsilon > 0$, there exist elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r_i$, satisfying $\|a(\sum_{i=1}^n w_i^* a_i w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^n w_i^* a_i w_i)a - a\| < \epsilon$ for all $\|a\| \leq 1$. If no such n exists, we write $UACov(\mathfrak{A}, m) = \infty$.

Next, We provide equivalent conditions for a number to be a unit approximate covering number.

Proposition 4.3. *For a unital C^*-algebra $\mathfrak{A}$ and $m \in \mathbb{Z}^+$. Then, the conditions listed below are equivalent:*

(1) $UACov(\mathfrak{A}, m) = n$.

(2) n is least positive integer such that for any $\epsilon > 0$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$

with $k_{i,j} \geq m$ for each i ranging from 1 to n and for each j from 1 to r_i, satisfying $\|(\sum_{i=1}^{n} w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$.

(3) n is least positive integer such that for any $\epsilon > 0$, there are finite-dimensional C^*-algebras $\mathcal{F}_i$, $1 \leq i \leq n$, with $\operatorname{rank}(\mathcal{F}_i) \geq m$, $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$, and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ satisfying $\| \sum_{i=1}^{n} w_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) w_i - 1_{\mathfrak{A}}\| < \epsilon$.

Proof. The proof follows directly from Proposition 4.1 and the definition. □

Proposition 4.4. *For a C^*-algebra $\mathfrak{A}$ and $m \in \mathbb{Z}^+$. Then, The conditions listed below are equivalent:*

(1) $UACov(\mathfrak{A}, m) = n$.

(2) n is the least positive integer such that for any $\epsilon > 0$, there are finite-dimensional C^*-algebras $\mathcal{F}_i$, $1 \leq i \leq n$, with $\operatorname{rank}(\mathcal{F}_i) \geq m$, $*$-homomorphisms $\phi_i : C_0((0,1]) \otimes \mathcal{F}_i \to \mathfrak{A}$, and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ satisfying $\|a(\sum_{i=1}^{n} w_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^{n} w_i^* \phi_i(\iota \otimes 1_{\mathcal{F}_i}) w_i) a - a\| < \epsilon$ for all $\|a\| \leq 1$.

Proof. The justification is based on a framework similar to Proposition 3.11. □

Proposition 4.5. *If $m \leq l$ then, $UACov(\mathfrak{A}, m) \leq UACov(\mathfrak{A}, l)$.*

Proof. The proof follows directly from the definition. □

Definition 4.6. Let $\mathfrak{A}$ be a C^*-algebra. We define $UACov(\mathfrak{A}) = \sup_m UACov(\mathfrak{A}, m)$.

Lemma 4.7. *If $\mathfrak{A}$ is unital and $m \in \mathbb{Z}_+$. Then, $UACov(\mathfrak{A}, m) \leq ACov(\mathfrak{A}, m)$.*

Proof. If $n = ACov(\mathfrak{A}, m)$. Then, for any $\epsilon > 0$ and $a \in \mathfrak{A}$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogenous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for each i ranging from 1 to n and for each j from 1 to r_i, satisfying $\|a(\sum_{i=1}^{n} w_i^* a_i w_i) - a\| < \epsilon$ and $\|(\sum_{i=1}^{n} w_i^* a_i w_i)a - a\| < \epsilon$. Therefore, $\|(\sum_{i=1}^{n} w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$. Then, by Proposition 4.3, $UACov(\mathfrak{A}, m) \leq n$. □

Next, we establish the relationship between the approximate covering number, covering number, unit approximate covering number and weak divisibility number of a unital C^*-algebra.

Proposition 4.8. *If $\mathfrak{A}$ is unital and $m \in \mathbb{Z}^+$, then $UACov(\mathfrak{A}, m) \leq ACov(\mathfrak{A}, m) \leq Cov(\mathfrak{A}, m) \leq w - Div_m(\mathfrak{A}) \leq (2m-1)Cov(\mathfrak{A}, m)$.*

Proof. The proof follows directly from Proposition 3.6 and Lemma 4.7. □

Corollary 4.9. *If $\mathfrak{A}$ is a unital, nuclear, separable C^*-algebra with finite decomposition rank $\operatorname{dr}(\mathfrak{A})$ that does not possess any finite-dimensional irreducible representations, then $UACov(\mathfrak{A}) \leq \operatorname{dr}(\mathfrak{A}) + 1$.*

Proof. By lemma 4.7 and Proposition 3.10, $UACov(\mathfrak{A}) \leq \operatorname{dr}(\mathfrak{A}) + 1$. □

Proposition 4.10. *If $\mathfrak{A}$ and $\mathfrak{B}$ are unital, and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a non-zero unital $*$-homomorphism, then $UACov(\mathfrak{B}, m) \leq UACov(\mathfrak{A}, m)$.*

Proof. Assume that for some $n \in \mathbb{Z}^+$, we have $UACov(\mathfrak{A}, m) = n$. Then, for any $\epsilon > 0$, there are elements $a_1, a_2, \ldots, a_n \in \mathfrak{A}_+$ and $w_1, w_2, \ldots, w_n \in \mathfrak{A}$ such that each a_i is a finite sum $a_i = \sum_{j=1}^{r_i} a_{i,j}$ of mutually orthogonal $k_{i,j}$-homogeneous elements $a_{i,j} \in \mathfrak{A}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots r_i$ satisfying $\|(\sum_{i=1}^n w_i^* a_i w_i) - 1_{\mathfrak{A}}\| < \epsilon$. Since ϕ unital, $\|(\sum_{i=1}^n \phi(w_i)^* \phi(a_i)\phi(w_i)) - 1_{\mathfrak{B}}\| < \epsilon$ and $\phi(a_1), \phi(a_2), \ldots, \phi(a_n) \in \mathfrak{B}_+$ and $\phi(w_1), \phi(w_2), \ldots, \phi(w_n) \in \mathfrak{B}$ such that each $\phi(a_i)$ is a finite sum $\phi(a_i) = \sum_{j=1}^{r_i} \phi(a_{i,j})$ of mutually orthogonal $k_{i,j}$-homogeneous elements $\phi(a_{i,j}) \in \mathfrak{B}_+$ with $k_{i,j} \geq m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots r_i$. This implies that $UACov(\mathfrak{A}, m) \leq n$. □

Corollary 4.11. *If $\mathfrak{A}$ and $\mathfrak{B}$ are unital, and one of $UACov(\mathfrak{A}, m)$ or $UACov(\mathfrak{B}, m)$ is finite, then $UACov(\mathfrak{A} \otimes \mathfrak{B}, m) < \infty$, for any tensor product $\otimes$.*

Proof. Suppose that $UACov(\mathfrak{A}, m) < \infty$. Since the map $\phi : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{B}$ defined by $\phi(a) = a \otimes 1_{\mathfrak{B}}$, is unital, $UACov(\mathfrak{A} \otimes \mathfrak{B}, m) \leq UACov(\mathfrak{A}, m)$. Therefore, $UACov(\mathfrak{A} \otimes \mathfrak{B}, m) < \infty$. □

Corollary 4.12. *Let $\mathfrak{A}_1 \xrightarrow{\phi_1} \mathfrak{A}_2 \xrightarrow{\phi_2} \mathfrak{A}_3 \xrightarrow{\phi_3} \ldots$, where each $UACov(\mathfrak{A}_k, m)$ are finite and connecting maps are unital, be the inductive sequence and $(\mathfrak{A}, \{\mu_n\})$ be the inductive limit. Then, $UACov(\mathfrak{A}, m) \leq inf_k UACov(\mathfrak{A}_k, m)$. If $UACov(\mathfrak{A}_k, m) = 1$ for some k, then, $UACov(\mathfrak{A}, m) = 1$.*

Proof. The proof follows directly from the definition. □

Corollary 4.13. *If $0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{C} \longrightarrow 0$ is a unital extension, then $UACov(\mathfrak{C}, m) \leq UACov(\mathfrak{B}, m) \leq UACov(\mathfrak{A}, m)$. In particular, if $UACov(\mathfrak{C}, m) = 1$, then $UACov(\mathfrak{C}, m) = UACov(\mathfrak{B}, m) = UACov(\mathfrak{A}, m) = 1$. If $UACov(\mathfrak{C}, m) = UACov(\mathfrak{A}, m)$, then $UACov(\mathfrak{C}, m) = UACov(\mathfrak{B}, m) = UACov(\mathfrak{A}, m)$.*

Proof. The proof follows from Proposition 4.10. □

Corollary 4.14. *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is non zero $*$-homomorphism, then $UACov(\phi[\mathfrak{A}], m) \leq UACov(\mathfrak{A}, m)$.*

Proof. This conclusion follows straight from the definition. □

5. Application of Covering numbers

In this section, we explore the application of covering numbers to various classes of C^*-algebras. We establish that for key algebras like the Cuntz algebra $\mathcal{O}_2$ and properly infinite C^*-algebras, covering numbers are equal to one. Additionally, we show that for unital $\mathcal{Z}$-stable algebras and simple AF-algebras, these covering numbers are bounded by two. We also discuss the implications of covering numbers for purely infinite algebras and those with no finite-dimensional representations. The results highlight their connection to the abundance of soft elements.

Proposition 5.1. *For Cuntz algebra $\mathcal{O}_2$, $ACov(\mathcal{O}_2, m) = 1 = UACov(\mathcal{O}_2, m)$.*

Proof. By the proposition 4.8 and Remark 2.2 in [13], $ACov(\mathcal{O}_2, m) = 1 = UACov(\mathcal{O}_2, m)$. □

Corollary 5.2. *If $\mathfrak{A}$ is a unital C^*-algebra that allows a non-zero unital $*$-homomorphism from $\mathcal{O}_2$ into $\mathfrak{A}$, then $ACov(\mathfrak{A}, m) = 1 = UACov(\mathfrak{A}, m)$.*

Proof. By the Propositions 5.1 and 4.10. □

Proposition 5.3. *If $\mathfrak{A}$ is properly infinite. Then, $ACov(\mathfrak{A}, m) = 1 = UACov(\mathfrak{A}, m)$.*

Proof. If $\mathfrak{A}$ is properly infinite, then due to the existence of a unital $*$-homomorphism from $\mathcal{O}_2$ to $\mathcal{O}_\infty$, we have $ACov(\mathfrak{A}, m) = 1 = UACov(\mathfrak{A}, m)$. □

Proposition 5.4. *If a unital C^*-algebra $\mathfrak{A}$ is $\mathcal{Z}$-stable, then $ACov(\mathfrak{A}, k) \leq 2$ and $UACov(\mathfrak{A}, k) \leq 2$.*

Proof. Let $\mathfrak{A}$ be a unital $\mathcal{Z}$-stable C^*-algebra. Consider the function $f : [0, 1] \to M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ defined by $f(t) = t1_{mn}$. This f is a positive element in the dimension-drop algebra $\mathcal{I}(m, n)$. Let $a_1 = f^{\frac{1}{3}}$, which is n-homogeneous, and $a_2 = (1 - f)^{\frac{1}{3}}$, which is m-homogeneous. Define $b_1 = f^{\frac{1}{3}}$ and $b_2 = (1 - f)^{\frac{1}{3}}$. Then, we have $b_1^* a_1 b_1 + b_2^* a_2 b_2 = 1$.

Consequently, for any $k \leq \min(m, n)$, it follows that $\mathrm{Cov}(\mathcal{I}(m, n), k) \leq 2$. By Proposition 4.8, this also implies that $\mathrm{ACov}(\mathcal{I}(m, n), k) \leq 2$ and $\mathrm{UACov}(\mathcal{I}(m, n), k) \leq 2$.

Therefore, for any relatively prime integers m and n and for any $k \leq \min(m, n)$, we have $\mathrm{Cov}(\mathcal{I}(m, n), k) \leq 2$, and by Proposition 4.8, $\mathrm{ACov}(\mathcal{I}(m, n), k) \leq 2$ and $\mathrm{UACov}(\mathcal{I}(m, n), k) \leq 2$.

Since the Jiang-Su algebra $\mathcal{Z}$ can be viewed as an inductive limit of dimension-drop algebras $\mathcal{I}(m, n)$ with m and n being relatively prime, and the connecting maps are unital, it can be concluded from Proposition 4.10, Lemma 3.5, and Remark 2.2 from [13] that $\mathrm{ACov}(\mathcal{Z}, k) \leq 2$, $\mathrm{UACov}(\mathcal{Z}, k) \leq 2$, and $\mathrm{Cov}(\mathcal{Z}, k) \leq 2$.

Now, since $\mathfrak{A}$ is a unital $\mathcal{Z}$-stable C^*-algebra, there is a unital $*$-isomorphism from $\mathcal{Z} \otimes \mathfrak{A}$ to $\mathfrak{A}$. By Proposition 4.10, Lemma 3.5, and Remark 2.2 from [13], we conclude that $\mathrm{ACov}(\mathfrak{A}, k) \leq 2$ and $\mathrm{UACov}(\mathfrak{A}, k) \leq 2$. □

Corollary 5.5. *For a simple, separable, unital AF-algebra $\mathfrak{A}$, we have $ACov(\mathfrak{A}, k) \leq 2$ and $UACov(\mathfrak{A}, k) \leq 2$.*

Proof. Let $\mathfrak{A}$ be a simple, separable, unital AF-algebra. Then $\mathfrak{A}$ is a unital $\mathcal{Z}$-stable C^*-algebra. By Proposition 5.4, this implies that $\mathrm{ACov}(\mathfrak{A}, k) \leq 2$ and $\mathrm{UACov}(\mathfrak{A}, k) \leq 2$. □

Proposition 5.6. *Let $\mathfrak{A}$ be a separable, unital C^*-algebra with nuclear dimension $\dim_{nuc}(\mathfrak{A}) \leq n$, and assume that $\mathfrak{A}$ has no finite-dimensional irreducible representations. Then,* $\mathrm{UACov}(\mathfrak{A}, m) \leq n + 1$ *and* $\mathrm{ACov}(\mathfrak{A}, m) \leq n + 1$.

Proof. By Lemma 3.5, Proposition 4.8 and [[26], Proposition 4.4], $ACov(\mathfrak{A}, m) \leq n+1$ and $UACov(\mathfrak{A}, m) \leq n+1$. ☐

Proposition 5.7. *If $\mathfrak{A}$ is weakly purely infinite and satisfies the condition* $\mathrm{ACov}(\mathfrak{A}, m) = 1$ *for every $m \geq 2$, then $\mathfrak{A}$ must be purely infinite.*

Proof. Assume that $\mathfrak{A}$ is weakly purely infinite and that $\mathrm{ACov}(\mathfrak{A}, m) = 1$ for every $m \geq 2$. According to the definition, for any real number $\epsilon > 0$ and a positive element a, there exist mutually orthogonal and pairwise equivalent positive elements $a_1, a_2, \ldots, a_m$ in $\mathfrak{A}$ and an element b in $\mathfrak{A}$ such that $\|a(b^* \sum_{i=1}^{m} a_i b) - a\| < \epsilon$ and $\|(b^* \sum_{i=1}^{m} a_i b)a - a\| < \epsilon$.

This indicates that $\mathfrak{A}$ possesses the Global Glimm property. Thus, by Proposition 4.15 from [14], $\mathfrak{A}$ is purely infinite.

☐

Proposition 5.8. *Let $\mathfrak{A}$ be a C^*-algebra, with the condition that $ACov(\mathfrak{A}, m) = 1$ for every $m \geq 2$. The subsequent statements are equivalent:*

(1) $\mathfrak{A}$ possesses an abundance of soft elements, and $(Cu(\mathfrak{A}), \Sigma_{\mathfrak{A}})$ is ideal-filtered.
(2) $\mathfrak{A}$ possesses an abundance of soft elements, and $Cu(\mathfrak{A})$ is ideal-filtered.

Proof. Suppose that $ACov(\mathfrak{A}, m) = 1$ for all $m \geq 2$. Then, by definition, for $\epsilon > 0$ and $a \in \mathfrak{A}_+$, there are mutually orthogonal and pairwise equivalent elements $a_1, a_2, \ldots, a_m$ in $\mathfrak{A}_+$ and $b \in \mathfrak{A}$ such that $\|a(b^* \sum_{i=1}^{m} a_i b) - a\| < \epsilon$ and $\|(b^* \sum_{i=1}^{m} a_i b)a - a\| < \epsilon$. This implies that $\mathfrak{A}$ has Global Glimm property. Therefore, by [[24], Proposition 7.14], (1) $\iff$ (2). ☐

Proposition 5.9. *Let $\mathfrak{A}$ be a C^*-algebra, with the condition that $UACov(\mathfrak{A}, m) = 1$ for every $m \geq 2$. The subsequent statements are equivalent:*

(1) $\mathfrak{A}$ possesses an abundance of soft elements, and $(Cu(\mathfrak{A}), \Sigma_{\mathfrak{A}})$ is ideal-filtered.
(2) $\mathfrak{A}$ possesses an abundance of soft elements, and $Cu(\mathfrak{A})$ is ideal-filtered.

Proof. The proof follows a similar approach to that of Proposition 5.8.

☐

Proposition 5.10. *Let $\mathfrak{A}$ be a unital C^*-algebra, and assume that $Cov(\mathfrak{A}, m) = 1$ for all $m \geq 2$. The subsequent statements are equivalent:*

(1) $\mathfrak{A}$ possesses an abundance of soft elements, and $(Cu(\mathfrak{A}), \Sigma_{\mathfrak{A}})$ is ideal-filtered.
(2) $\mathfrak{A}$ possesses an abundance of soft elements, and $Cu(\mathfrak{A})$ is ideal-filtered.

Proof. The proof resembles that of Proposition 5.8.

☐

Corollary 5.11. *If $\mathfrak{A}$ is a unital separable C^*-algebra that meets any of the following criteria, then $UACov(\mathfrak{A}, m) = 1$ and $ACov(\mathfrak{A}, m) = 1$:*

(1) $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_\infty$.
(2) $\mathfrak{A}$ is purely infinite and nuclear, and it is either stable or has an approximate unit consisting of projections.
(3) $\mathfrak{A}$ is purely infinite and $RR(\mathfrak{A}) = 0$.
(4) $\mathfrak{A}$ is strongly purely infinite and $RR(\mathfrak{A}) = 0$.
(5) Every non-zero projection in $\mathfrak{A}$ is properly infinite.

Proof. The proof follows from Propositions 4.8, 4.10 and [[14], Proposition 9.1, Corollary 9.4]. □

Corollary 5.12. *For any unital C^*-algebra $\mathfrak{A}$, $ACov(\mathfrak{A} \otimes \mathcal{Z}, m) \leq 2$ and $UACov(\mathfrak{A} \otimes \mathcal{Z}, m) \leq 2$.*

Proof. The proof follows from Proposition 5.4. □

Corollary 5.13. *Let $\mathfrak{A}$ be a unital C^*-algebra. If $\otimes_{min}^{\infty}\mathfrak{A}$ contains a unital sub-homogeneous C^*-algebra without characters, then it follows that $ACov(\otimes_{min}^{\infty}\mathfrak{A}, m) \leq 2$ and $UACov(\otimes_{min}^{\infty}\mathfrak{A}, m) \leq 2$.*

Proof. This conclusion follows logically from Proposition 5.4 and [[15], Proposition 2.4].

□

Corollary 5.14. *If $\mathfrak{D}$ is strongly self-absorbing, then $ACov(\mathfrak{D}, m) \leq 2$ and $UACov(\mathfrak{D}, m) \leq 2$.*

Proof. Suppose that $\mathfrak{D}$ is strongly self-absorbing, then $\mathfrak{D}$ is $\mathcal{Z}$-stable [25]. Therefore, by Proposition 5.4, $ACov(\mathfrak{D}, m) \leq 2$ and $UACov(\mathfrak{D}, m) \leq 2$. □

Corollary 5.15. *If $\mathfrak{A}$ is $\mathfrak{D}$-stable, then $ACov(\mathfrak{A}, m) \leq 2$ and $UACov(\mathfrak{A}, m) \leq 2$.*

Proof. The proof follows from Corollary 5.14. □

Remark 5.16. If $\mathfrak{A}$ is weakly purely infinite and $UACov(\mathfrak{A}, m) = 1$ for all $m \geq 2$. Then, $\mathfrak{A}$ is purely infinite.

6. Tracial covering numbers

In this section, we define three types of tracial covering numbers: the tracial covering number, the tracial approximate covering number, and the tracial unit approximate covering number. We demonstrate that if Λ represents a collection of unital nuclear C^*-algebras with tracial covering number not exceeding m, then any simple unital C^*-algebra $\mathfrak{A}$ that belongs to the associated class of weakly tracially approximated C^*-algebras will also have tracial covering number of at most m. We provide analogous results for the tracial approximate covering number and the tracial unit approximate covering number. Finally, we establish the connections among these three tracial covering numbers.

In this section, $\mathfrak{p}$ always denotes a projection, and we use "c.c.p" as an abbreviation for "contractive completely positive linear map".

Definition 6.1. A unital C^*-algebra $\mathfrak{A}$ is said to have tracial covering number at most m, denoted as $TrCov(\mathfrak{A}) \leq m$, if for every $\epsilon > 0$, any finite subset $F \subset \mathfrak{A}$, and any non-zero element $a \in \mathfrak{A}_+$, there exists a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ with $Cov(\mathfrak{B}) \leq m$, along with c.c.p maps $\Gamma : \mathfrak{A} \to \mathfrak{A}$ and $\Delta : \mathfrak{A} \to \mathfrak{B}$ such that...

(1) $\Gamma(1) \overset{\mathfrak{A}}{\precsim} a$
(2) $\|w - \Gamma(w) - \Delta(w)\| < \epsilon$, for all $w \in F$

Proposition 6.2. *Let Λ denote a collection of unital nuclear C^*-algebras that have tracial covering number not exceeding m. Then, any simple unital C^*-algebra $\mathfrak{A}$ that falls within the corresponding collection of weakly tracially approximated C^*-algebras will also possess tracial covering number of at most m.*

Proof. We prove the Proposition using the method employed in the proof of Theorem 3.7 in [8]. Let $\mathfrak{A}$ be a simple unital C^*-algebra in $WTA\Lambda$. Take $\epsilon > 0$, a finite subset $F = \{x_1, x_2, \ldots, x_n\}$ of $\mathfrak{A}$, and a non-zero element $a \in \mathfrak{A}_+$. By [[19], Lemma 2.4], there exist positive elements a_1, a_2 such that $\|a_1\| = 1$, $\|a_2\| = 1$, $a_1 a_2 = 0$, $a_1 \overset{\mathfrak{A}}{\sim} a_2$, and $a_1 + a_2 \overset{\mathfrak{A}}{\precsim} a$. Let $\delta > 0$ and set $G = F \cup \{a_2\}$. Since $\mathfrak{A} \in WTA\Lambda$, there exist a $\mathfrak{p} \in \mathfrak{A}$, $h \in \mathfrak{A}$ with $0 \leq h \leq 1$, and a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $h \in \mathfrak{B}$, $1_{\mathfrak{B}} = \mathfrak{p}$, and $\mathfrak{B} \in \Lambda$, and such that:

(1) $(\mathfrak{p} - h)w \in_\delta \mathfrak{B}$, $w(\mathfrak{p} - h) \in_\delta \mathfrak{B}$, for $w \in G$,
(2) $\|(\mathfrak{p} - h)w - w(\mathfrak{p} - h)\| < \delta$, for $w \in G$,
(3) $1 - (\mathfrak{p} - h) \overset{\mathfrak{A}}{\precsim} a_1 \overset{\mathfrak{A}}{\sim} a_2$, and
(4) $\|(\mathfrak{p} - h)a_2(\mathfrak{p} - h)\| > 1 - \delta$.

Now, take $X = [0, 1]$ and $f(t) = t^{\frac{1}{2}}$ in [[17], Lemma 2.5.11] with δ sufficiently small. By [[17], Lemma 2.5.11] and (2), we have:

(5) $\|(\mathfrak{p} - h)^{\frac{1}{2}} w - w(\mathfrak{p} - h)^{\frac{1}{2}}\| < \epsilon$ for all $w \in G$,
(6) $\|(1 - (\mathfrak{p} - h))^{\frac{1}{2}} w - w(1 - (\mathfrak{p} - h))^{\frac{1}{2}}\| < \epsilon$ for all $w \in G$.

From (1) and (5), there exist elements $y_1, y_2, \ldots, y_n \in \mathfrak{B}$ and $y \in \mathfrak{B}_+$ such that $\|(p-h)^{\frac{1}{2}} x_i (\mathfrak{p}-h)^{\frac{1}{2}} - y_i\| < \epsilon$ for $i = 1, 2, \ldots, n$ and $\|(\mathfrak{p} - h)^{\frac{1}{2}} a_2 (\mathfrak{p} - h)^{\frac{1}{2}} - y\| < \epsilon$. Using this inequality, (5), and (6), for $1 \leq i \leq n$,

$$\begin{aligned}
&\|x_i - y_i - (1 - (\mathfrak{p} - h))^{\frac{1}{2}} x_i (1 - (\mathfrak{p} - h))^{\frac{1}{2}}\| \\
&\leq \|x_i - (\mathfrak{p} - h)x_i - (1 - (\mathfrak{p} - h))x_i\| \\
&+ \|(\mathfrak{p} - h)x_i - (\mathfrak{p} - h)^{\frac{1}{2}} x_i (\mathfrak{p} - h)^{\frac{1}{2}}\| \\
&+ \|(1 - (\mathfrak{p} - h))x_i - (1 - (\mathfrak{p} - h))^{\frac{1}{2}} x_i (1 - (\mathfrak{p} - h))^{\frac{1}{2}}\| \\
&+ \|(\mathfrak{p} - h)^{\frac{1}{2}} x_i (\mathfrak{p} - h)^{\frac{1}{2}} - y_i\| \\
&< 3\epsilon
\end{aligned} \tag{6.1}$$

Since $\|(\mathfrak{p} - h)^{\frac{1}{2}} a_2 (\mathfrak{p} - h)^{\frac{1}{2}} - y\| < \epsilon$, by [[17], Corollary 1.6], we have:

(7) $(y - 3\epsilon)_+ \overset{\mathfrak{A}}{\precsim} ((\mathfrak{p} - h)^{\frac{1}{2}} a_2 (\mathfrak{p} - h)^{\frac{1}{2}} - 2\epsilon)_+$.

From (4), with $\delta \leq \epsilon < \frac{1}{5}$, $\|(\mathfrak{p} - h)^{\frac{1}{2}} a_2 (\mathfrak{p} - h)^{\frac{1}{2}}\| \geq \|(\mathfrak{p} - h)a_2(\mathfrak{p} - h)\| \geq 1 - \epsilon$. Therefore, by choosing y, $\|(y - 3\epsilon)_+\| \geq \|(\mathfrak{p} - h)^{\frac{1}{2}} a_2 (\mathfrak{p} - h)^{\frac{1}{2}}\| - 4\epsilon \geq 1 - 5\epsilon$, so $(y - 3\epsilon)_+ \neq 0$.

Define a c.c.p $\Theta : \mathfrak{A} \to \mathfrak{A}$ by $\Theta(b) = (1 - (\mathfrak{p} - h))^{\frac{1}{2}} b (1 - (\mathfrak{p} - h))^{\frac{1}{2}}$. Since $\mathfrak{B}$ is a nuclear C^*-algebra, there exists a c.c.p $\rho : \mathfrak{A} \to \mathfrak{B}$ such that $\|\rho(\mathfrak{p}-h) - (\mathfrak{p}-h)\| < \epsilon$ and $\|\rho(y_i) - y_i\| < \epsilon$ for all $1 \leq i \leq n$. Since $TrCov(\mathfrak{B}) \leq m$, there exist a c.c.p $\theta : \mathfrak{B} \to \mathfrak{B}$ and a c.c.p $\tau : \mathfrak{B} \to \mathfrak{D}$ with $Cov(\mathfrak{D}) \leq m$ such that:

1*. $\theta(\mathfrak{p}) \overset{\mathfrak{B}}{\precsim} (y-3\epsilon)_+$,
2*. $\|(\mathfrak{p}-h)-\theta(\mathfrak{p}-h)-\tau(\mathfrak{p}-h)\| < \epsilon$, and $\|y_i-\theta(y_i)-\tau(y_i)\| < \epsilon$ for all $1 \le i \le n$.

Now, define $\gamma : \mathfrak{A} \to \mathfrak{A}$ by $\gamma(b) = \Theta(b)+\theta(\rho((\mathfrak{p}-h)^{\frac{1}{2}}b(\mathfrak{p}-h)^{\frac{1}{2}}))$, and let $\Gamma = \frac{1}{1+2\epsilon}\gamma$, and $\Delta : \mathfrak{A} \to \mathfrak{D}$ by $\Delta(b) = \tau(\rho((\mathfrak{p}-h)^{\frac{1}{2}}b(\mathfrak{p}-h)^{\frac{1}{2}}))$. Then:

$$\begin{aligned}
\|\Gamma(1)\| &= \frac{1}{2\epsilon+1}\|\gamma(1)\| \\
&= \frac{1}{2\epsilon+1}\|\Theta(1)+\theta(\rho(\mathfrak{p}-h))\| \\
&= \frac{1}{2\epsilon+1}\|1-(\mathfrak{p}-h)+\theta(\rho(\mathfrak{p}-h))\| \\
&= \frac{1}{2\epsilon+1}\Big\|1-(\mathfrak{p}-h)+\theta(\mathfrak{p}-h)+\tau(\mathfrak{p}-h) \\
&\quad +\theta(\rho(\mathfrak{p}-h))-\theta(\mathfrak{p}-h)-\tau(\mathfrak{p}-h)\Big\| \\
&\le \frac{1}{2\epsilon+1}\Big(\|(\mathfrak{p}-h)-\theta(\mathfrak{p}-h)-\tau(\mathfrak{p}-h)\| \\
&\quad +\|\theta(\rho(\mathfrak{p}-h))-\theta(\mathfrak{p}-h)\|+\|1-\tau(\mathfrak{p}-h)\|\Big) \\
&\le \frac{1}{2\epsilon+1}(2\epsilon+1) = 1.
\end{aligned}$$

Similarly, we can demonstrate that Δ is a c.c.p. Using (3), (7), and (1*), we have:

$$\begin{aligned}
\Gamma(1) &= \frac{1}{2\epsilon+1}\left(\Theta(1)+\theta\left(\rho\left((\mathfrak{p}-h)^{\frac{1}{2}}1(\mathfrak{p}-h)^{\frac{1}{2}}\right)\right)\right) \\
&= \frac{1}{2\epsilon+1}\left(\Theta(1)+\theta(\rho(\mathfrak{p}-h))\right) \\
&\overset{\mathfrak{A}}{\sim} \Theta(1)+\theta(\rho(\mathfrak{p}-h)) \\
&\overset{\mathfrak{A}}{\precsim} 1-(\mathfrak{p}-h)\oplus\theta(\mathfrak{p}) \\
&\overset{\mathfrak{A}}{\precsim} a_1 \oplus (y-3\epsilon)_+ \\
&\overset{\mathfrak{A}}{\precsim} a_1 \oplus \left((\mathfrak{p}-h)^{\frac{1}{2}}a_2(\mathfrak{p}-h)^{\frac{1}{2}}-2\epsilon\right)_+ \\
&\overset{\mathfrak{A}}{\precsim} a_1 \oplus a_2 \overset{\mathfrak{A}}{\sim} a_1+a_2 \overset{\mathfrak{A}}{\precsim} a.
\end{aligned}$$

Therefore, $\Gamma(1) \overset{\mathfrak{A}}{\precsim} a$.
By (6.1) and (*2**), we have:

$$\|x_i - \Gamma(x_i) - \Delta(x_i)\|$$

$$= \left\| x_i - \frac{1}{2\epsilon+1}\left(\Theta(x_i) + \theta\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right)\right) - \tau\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right)\right\|$$

$$\leq \left\| x_i - \frac{1}{2\epsilon+1}\left((1-(\mathfrak{p}-h))^{1/2}x_i(1-(\mathfrak{p}-h))^{1/2}\right) - y_i \right\|$$
$$+ \left\| y_i - \frac{1}{2\epsilon+1}\left(\theta\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right)\right) - \left(\tau\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right)\right)\right\|$$

$$\leq \left\| x_i - (1-(\mathfrak{p}-h))^{1/2}x_i(1-(\mathfrak{p}-h))^{1/2} - y_i \right\|$$
$$+ \left\| (1-(\mathfrak{p}-h))^{1/2}x_i(1-(\mathfrak{p}-h))^{1/2} - \frac{1}{2\epsilon+1}\left((1-(\mathfrak{p}-h))^{1/2}x_i(1-(\mathfrak{p}-h))^{1/2}\right) \right\|$$
$$+ \|y_i - \theta(y_i) - \tau(y_i)\| + \|\theta(y_i) - \theta(\rho(y_i))\|$$
$$+ \left\| \theta(\rho(y_i)) - \theta\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right) \right\|$$
$$+ \left\| \theta\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right) - \frac{1}{2\epsilon+1}\theta\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right) \right\|$$
$$+ \|\tau(y_i) - \tau(\rho(y_i))\| + \left\| \tau(\rho(y_i)) - \tau\left(\rho\left((\mathfrak{p}-h)^{1/2}x_i(\mathfrak{p}-h)^{1/2}\right)\right) \right\|$$

$$\leq 3\epsilon + \frac{2\epsilon}{2\epsilon+1} + 3\epsilon + \frac{2\epsilon}{2\epsilon+1} + 2\epsilon$$

$$= 8\epsilon + \frac{4\epsilon}{2\epsilon+1} < 12\epsilon.$$

Therefore, we conclude that: $\|x_i - \Gamma(x_i) - \Delta(x_i)\| < 12\epsilon$.

Thus, for any $\epsilon > 0$, any finite subset $F \subset \mathfrak{A}$, and any non-zero element $a \in \mathfrak{A}_+$, there exist a unital sub-C^*-algebra $\mathfrak{D}$ of $\mathfrak{A}$ with $Cov(\mathfrak{D}) \leq m$, a c.c.p $\Gamma : \mathfrak{A} \to \mathfrak{A}$, and a c.c.p $\Delta : \mathfrak{A} \to \mathfrak{D}$ such that:

(1) $\Gamma(1) \overset{\mathfrak{A}}{\precsim} a$,
(2) $\|w - \Gamma(w) - \Delta(w)\| < \epsilon$ for all $w \in F$.

□

Corollary 6.3. *Let Λ denote a collection of unital nuclear C^*-algebras that have tracial covering number not exceeding m. Then, any simple unital C^*-algebra $\mathfrak{A}$ belonging to the class of tracially approximated C^*-algebras will also have tracial covering number of at most m.*

Proof. Since $TA\Lambda \subset WTA\Lambda$, the proof follows from Proposition 6.2.

□

Definition 6.4. A unital C^*-algebra $\mathfrak{A}$ is said to have tracial approximate covering number at most m, denoted $TrACov(\mathfrak{A}) \leq m$, if for every $\epsilon > 0$, every finite subset $F \subset \mathfrak{A}$, and every non-zero element $a \in \mathfrak{A}_+$, there exists a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ with $ACov(\mathfrak{B}) \leq m$, along with c.c.p maps $\Gamma : \mathfrak{A} \to \mathfrak{A}$ and $\Delta : \mathfrak{A} \to \mathfrak{B}$ satisfying the following conditions:

(1) $\Gamma(1) \overset{\mathfrak{A}}{\precsim} a$
(2) $\|w - \Gamma(w) - \Delta(w)\| < \epsilon$ for all $w \in F$.

Proposition 6.5. *Let Λ be a collection of unital nuclear C^*-algebras with tracial approximate covering number not exceeding m. Then any simple unital C^*-algebra $\mathfrak{A}$ in the corresponding class of weakly tracially approximated C^*-algebras also has tracial approximate covering number at most m.*

Proof. The justification is based on a framework similar to Proposition 6.2. □

Corollary 6.6. *Let Λ be a collection of unital nuclear C^*-algebras with tracial approximate covering number at most m. Then any simple unital C^*-algebra $\mathfrak{A}$ in the corresponding class of tracially approximated C^*-algebras also has tracial approximate covering number at most m.*

Proof. Since $TA\Lambda \subset WTA\Lambda$, the proof follows from Proposition 6.5. □

Definition 6.7. A unital C^*-algebra $\mathfrak{A}$ is said to have tracial unit approximate covering number at most m, denoted $TrUACov(\mathfrak{A}) \leq m$, if for every $\epsilon > 0$, every finite subset $F \subset \mathfrak{A}$, and every non-zero element $a \in \mathfrak{A}_+$, there exists a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ with $UACov(\mathfrak{B}) \leq m$, along with c.c.p maps $\Gamma : \mathfrak{A} \to \mathfrak{A}$ and $\Delta : \mathfrak{A} \to \mathfrak{B}$ satisfying the following conditions:

(1) $\Gamma(1) \overset{\mathfrak{A}}{\precsim} a$
(2) $\|w - \Gamma(w) - \Delta(w)\| < \epsilon$ for all $w \in F$.

Proposition 6.8. *Let Λ be a collection of unital nuclear C^*-algebras with tracial unit approximate covering number at most m. Then any simple unital C^*-algebra $\mathfrak{A}$ in the corresponding class of weakly tracially approximated C^*-algebras also has tracial unit approximate covering number at most m.*

Proof. The justification is based on a framework similar to Proposition 6.2. □

Corollary 6.9. *Let Λ be a collection of unital nuclear C^*-algebras with tracial unit approximate covering number at most m. Then any simple unital C^*-algebra $\mathfrak{A}$ in the corresponding class of tracially approximated C^*-algebras also has tracial unit approximate covering number at most m.*

Proof. Since $TA\Lambda \subset WTA\Lambda$, the proof follows from Proposition 6.8. □

Next, we prove relation between $TrACov(\mathfrak{A})$, $TrUACov(\mathfrak{A})$ and $TrCov(\mathfrak{A})$

Proposition 6.10. *Let be $\mathfrak{A}$ is unital C^*-algebra.*

(1) If $TrCov(\mathfrak{A}) \leq m$, then $TrACov(\mathfrak{A}) \leq m$ and $TrUACov(\mathfrak{A}) \leq m$.

(2) If $TrACov(\mathfrak{A}) \leq m$, then $TrUACov(\mathfrak{A}) \leq m$.

Proof. (1) Assume that $TrCov(\mathfrak{A}) \leq m$. By definition, for every $\epsilon > 0$, any finite set $F \subset \mathfrak{A}$, and any non-zero positive element $a \in \mathfrak{A}_+$, there exists a unital sub-C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ with $Cov(\mathfrak{B}) \leq m$, along with c.c.p maps $\Gamma : \mathfrak{A} \to \mathfrak{A}$ and $\Delta : \mathfrak{A} \to \mathfrak{B}$ such that:

(1*) $\Gamma(1) \underset{\mathfrak{A}}{\precsim} a$

(2*) $\|w - \Gamma(w) - \Delta(w)\| < \epsilon$ for all $w \in F$.

Since $UACov(\mathfrak{B}) \leq ACov(\mathfrak{B}) \leq Cov(\mathfrak{B}) \leq m$, it follows that $TrACov(\mathfrak{A}) \leq m$ and $TrUACov(\mathfrak{A}) \leq m$.

(2) Suppose that $TrACov(\mathfrak{A}) \leq m$. By definition and Proposition 4.8, we have $TrUACov(\mathfrak{A}) \leq m$.

□

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[1], Research Scholar, Department of Mathematics, Payyanur College, Payyanur, India, Department of Mathematics, Sir Syed College, Taliparamba, India
Email address: sirajmp@sirsyedcollege.ac.in

[2] Department of Mathematics, Payyanur College, Payyanur, India
Email address: drktravindran@gmail.com