

DECOMPOSITION OF OPERATORS INTO SPAN OF UNITARY OPERATORS

BADAL SAHOO¹ and PABITRA KUMAR JENA^{2*}

ABSTRACT. The span of unitary operators is crucial in the study of Bergman space Toeplitz operators. This article delves mainly into the span of unitary Toeplitz operators and unitary combination of certain transforms of Toeplitz operators such as the Aluthge transform, the exponential transform, the Cayley transform and the mean transform of operators on the Bergman space.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the Banach algebra of linear bounded operators (or transformations) on $\mathcal{H}$, endowed with the standard operator norm $\|\cdot\|$. A transformation $T \in B(\mathcal{H})$ is unitary if it satisfies the condition $T^*T = TT^* = I$. In the early years, unitary operators received greater attention to study. It is a classical result that the unit sphere of any C^* -algebra $\mathfrak{M}$ can be uniformly approximated by convex combinations of unitary operators. If $T \in \mathfrak{M}$ and $\|T\| \leq \frac{n-2}{n}$ for $n > 2$ and $n \in \mathbb{N}$, then by [11] T can be written as mean of the number of unit operators n . Furthermore, a new approach was proposed to prove this result [8]. The index of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as $\text{ind}(T) = \dim \ker(T) - \dim \ker(T^*)$, where $\ker(T)$ denotes the kernel (null space) of T . In any von Neumann algebra $\mathfrak{N}$, a partial isometry T can be transformed into a unitary operator U provided that $\text{ind}(T) = 0$ (see [16]). Suppose $\mathfrak{N}^1$ denotes the closed unit ball in $\mathfrak{N}$ and $T \in \mathfrak{N}^1$. This paper also reveals that then T will be the average of two unitary operators iff $\text{ind}(T) = 0$. Further, $\text{ind}(T) = 0$ is the sufficient condition for converting the partial isometry to a unitary operator of a polar decomposition of T . PK. Jena [9, 10] derived certain sufficient conditions for converting non-unitary operators to unitary operators on the Bergman space. More general theory is also available in the literature [20] that any bounded linear operator on a Hilbert space can be decomposed into a linear combination of four unitary operators.

Now, we provide some definitions of certain transforms of bounded linear operators which are instrumental for development of main results.

Every operator $T \in \mathcal{B}(\mathcal{H})$ has a polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry. The Aluthge transform $\mathfrak{A}(T)$ and Duggal

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* Corresponding author.

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transform $T^{\mathfrak{D}}$ of T are defined as $\mathfrak{A}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T^{\mathfrak{D}} = |T|U$, respectively. A rigorous discussion of the Aluthge transform and Duggal transform for binormal operators and centered operators has been taken place in [15]. Also this paper [15] establishes certain conditions under which the Duggal transform of T yields the polar decomposition of T . The mean transform of T is denoted by $\mu(T)$ and defined by $\mu(T) = \frac{U\|T\| + \|T\|U}{2} = \frac{T+T^{\mathfrak{D}}}{2}$, which is the average of polar decomposition and Duggal transform. A comparison between the behavioral change of mean transform and Aluthge transform is presented by S. H. Lee, W. Y. Lee and J. Yoon [12]. For $T \in \mathcal{B}(\mathcal{H})$, the Cayley transform $Cl(T)$ of T is $(I+T)^{-1}(I-T)$, provided that $(I+T)$ is invertible. Cayley transform is considered as an extension of the conformal mapping $f(z) = \frac{1-z}{1+z}$ from the complex plane to itself, later extended to the matrix classes.

An operator T is normaloid if its norm equals the supremum of the absolute values of its numerical radii and T is quasinormal if it satisfies the equation $(TT^*)T = (T^*T)T$. $T \in \mathcal{B}(\mathcal{H})$ is hyponormal if $[T^*, T] = T^*T - TT^* \geq 0$. The operator T is paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all x . Further, T is binormal if its absolute values commute, i.e., $[|T|, |T^*|] = 0$.

2. PRELIMINARIES

Definition 2.1. The Bergman space $A^2(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ with inner product $\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$. For $\psi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ψ on $A^2(\mathbb{D})$ is defined by $T_\psi f = P(\psi f)$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

For each $a \in \mathbb{D}$, there exists a unique automorphism $\psi_a \in \text{Aut}(\mathbb{D})$ satisfying:

1. ψ_a is an involution (i.e., $\psi_a \circ \psi_a = id_{\mathbb{D}}$.)
2. ψ_a swaps 0 and a (i.e., $\psi_a(0) = a$ and $\psi_a(a) = 0$.)
3. The disk automorphism ψ_a fixes exactly one point in the unit disk $\mathbb{D}$.

The explicit form of the automorphism ψ_a on $\mathbb{D}$ is $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$, with derivative equals $-k_a(z)$ at z . For $a \in \mathbb{D}$, the operator U_a acts on the space of measurable functions f in $\mathbb{D}$ via $U_a f(t) = k_a(t)f(\psi_a(t))$. The U_a operator is a continuous linear map on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$, preserving the structure of the Hilbert space, and it has the following properties: $U_a^2 = I$, $U_a^* = U_a$, U_a preserves the subspaces $A^2(\mathbb{D})$ and $(A^2(\mathbb{D}))^\perp$. The projection operator P commutes with U_a , that is, $PU_a = U_aP$.

We now unfold this paper as follows. This research article consists of seven sections, beginning with an introduction in Section 1, followed by a comprehensive review of relevant notation and definitions in section 2. In Section-3, we consider the span of unitary Toeplitz operators. Aluthge transform of Toeplitz operators is studied in Section 4. Section 5,6 and 7 includes the exponential transform, Cayley transform and mean transform of Toeplitz Operators on the Bergman space, respectively.

3. MAIN RESULTS

3.1. Span of Unitary Toeplitz operators : This section provides sufficient conditions for the decomposability of Toeplitz operators on the Bergman space into a linear combination of unitary operators.

To guarantee the finite-rank property of T_ψ , we now impose conditions on ψ , enabling its application in the forthcoming theorem.

Theorem 3.1. *Suppose ψ be the complex regular Borel measure on $\mathbb{D}$, supported on finitely many point masses, with $\|\psi\|_\infty \leq 1$. Then, the Toeplitz operator T_ψ on $A^2(\mathbb{D})$ admits a representation as a span of eight unitary transformations.*

Proof. Let $T_\psi = \mathfrak{U}Q$ be the polar decomposition of T_ψ . If ψ is a finite linear combination of point masses then by [13], T_ψ is of finite rank. Thus, the partial isometry $\mathfrak{U}$ can be converted into a unitary operator. Again, as $\|\psi\|_\infty \leq 1$ then $\|T_\psi\| \leq 1$. That implies $\|Q\| \leq 1$, $I - Q^2 \geq 0$ and $\|I - Q^2\| \leq 1$. Define, $\mathfrak{U}_1 = iQ + \sqrt{I - Q^2}$, $\mathfrak{U}_2 = iQ - \sqrt{I - Q^2}$, $\mathfrak{U}_3 = Q + i\sqrt{I - Q^2}$, $\mathfrak{U}_4 = Q - i\sqrt{I - Q^2}$, $\mathfrak{U}_5 = -iQ + \sqrt{I - Q^2}$, $\mathfrak{U}_6 = -iQ - \sqrt{I - Q^2}$, $\mathfrak{U}_7 = -Q + i\sqrt{I - Q^2}$, $\mathfrak{U}_8 = -Q - i\sqrt{I - Q^2}$. Now, $\sum_{j=1}^4 \mathfrak{U}_j - \sum_{j=5}^8 \mathfrak{U}_j = 4(1+i)Q$.

Therefore, $Q = \frac{1}{4(1+i)} \left\{ \sum_{j=1}^4 \mathfrak{U}_j - \sum_{j=5}^8 \mathfrak{U}_j \right\}$. Thus, $T_\psi = \mathfrak{U}Q = \frac{1}{4(1+i)} \mathfrak{U} \left\{ \sum_{j=1}^4 \mathfrak{U}_j - \sum_{j=5}^8 \mathfrak{U}_j \right\}$, where $\mathfrak{U}\mathfrak{U}_j$ are unitary operators for $j = 1, 2, \dots, 8$. Hence proven. $\square$

In the following theorem, we are providing the sufficient condition $\|\psi\|_\infty \leq 1$ ensures that the Toeplitz operator T_ψ is a contraction mapping on the Bergman space $A^2(\mathbb{D})$.

Theorem 3.2. *Let ψ be an element of $L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then, T_ψ on $A^2(\mathbb{D})$ can be represented as a linear combination of four unitary operators.*

Proof. Consider the Cartesian decomposition $T_\psi = T_{\psi_1} + iT_{\psi_2}$, where $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Since $\|\psi\|_\infty \leq 1$, then $\|T_\psi\| \leq 1$. Hence T_ψ is a contraction operator. Thus the real and imaginary part of T_ψ are contractions. Thus, $\|T_{\psi_1}\| \leq 1$ and $\|T_{\psi_2}\| \leq 1$. So $I - T_{\psi_1}^2$ and $I - T_{\psi_2}^2$ are positive and $\|I - T_{\psi_1}^2\| \leq 1$ and $\|I - T_{\psi_2}^2\| \leq 1$. Let us define the operators $M_1 = T_{\psi_1} + i\sqrt{I - T_{\psi_1}^2}$, $M_2 = T_{\psi_1} - i\sqrt{I - T_{\psi_1}^2}$, $M_3 = iT_{\psi_2} + \sqrt{I - T_{\psi_2}^2}$, $M_4 = iT_{\psi_2} - \sqrt{I - T_{\psi_2}^2}$. It is easy to verify that $M_j, j = 1, \dots, 4$ are unitary operators. Thus, $T_\psi = T_{\psi_1} + iT_{\psi_2} = \frac{1}{2} \sum_{j=1}^4 M_j$ which is a linear combination of four unitary operators. $\square$

Corollary 3.3. Suppose $\psi \in L^\infty(\mathbb{D})$ and T_ψ satisfies the norm inequality $\|T_\psi f\| \leq \|f\|$ for all $f \in A^2(\mathbb{D})$. Then, there exist unitary operators M_i and scalars c_i where

$$i = 1, \dots, 8 \text{ such that } T_\psi = \sum_{i=1}^8 c_i M_i.$$

Proof. Since $\|T_\psi f\| \leq \|f\|$ for all $f \in A^2(\mathbb{D})$, then $\|T_\psi\| \leq 1$. Therefore, by Theorem (3.2), real and imaginary parts of T_ψ are contractions. Define the operators $M_1 = T_{\psi_1} + i\sqrt{I - T_{\psi_1}^2}$, $M_2 = T_{\psi_1} - i\sqrt{I - T_{\psi_1}^2}$, $M_3 = -T_{\psi_1} + i\sqrt{I - T_{\psi_1}^2}$, $M_4 = -T_{\psi_1} - i\sqrt{I - T_{\psi_1}^2}$, $M_5 = iT_{\psi_2} + \sqrt{I - T_{\psi_2}^2}$, $M_6 = iT_{\psi_2} - \sqrt{I - T_{\psi_2}^2}$, $M_7 = -iT_{\psi_2} + \sqrt{I - T_{\psi_2}^2}$, $M_8 = -iT_{\psi_2} - \sqrt{I - T_{\psi_2}^2}$. Direct computation shows $\{M_i : i = 1, 2, \dots, 8\}$ are unitary operators. Hence $T_\psi = T_{\psi_1} + iT_{\psi_2} = \text{span}\{M_i : i = 1, 2, \dots, 8\}$. Thus proof is accomplished. $\square$

Remark 3.4. Let $\psi \in L^\infty(\mathbb{D})$ with $\|T_\psi\| \leq 1$, then T_ψ is the linear combination of sixteen unitary operators.

Proof. By the similar arguments of corollary (3.3), we may extend the idea to define another eight unitary operators as $M_9 = iT_{\psi_1} + \sqrt{I - T_{\psi_1}^2}$, $M_{10} = iT_{\psi_1} - \sqrt{I - T_{\psi_1}^2}$, $M_{11} = -iT_{\psi_1} + \sqrt{I - T_{\psi_1}^2}$, $M_{12} = -iT_{\psi_1} - \sqrt{I - T_{\psi_1}^2}$, $M_{13} = T_{\psi_2} + i\sqrt{I - T_{\psi_2}^2}$, $M_{14} = T_{\psi_2} - i\sqrt{I - T_{\psi_2}^2}$, $M_{15} = -T_{\psi_2} + i\sqrt{I - T_{\psi_2}^2}$, $M_{16} = -T_{\psi_2} - i\sqrt{I - T_{\psi_2}^2}$. Consequently, this leads to the result. $\square$

Theorem 3.5. Consider $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then T_ψ on $A^2(\mathbb{D})$ can be written as sum of five unitary operators.

Proof. As $\|\psi\|_\infty \leq 1$, then $\|T_\psi\| \leq \|\psi\|_\infty \leq 1$. Suppose $T_\psi = T_{\psi_1} + iT_{\psi_2}$ be the cartesian decomposition of T_ψ for some $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Notice that T_{ψ_1} and T_{ψ_2} are contractions. As $I - T_{\psi_2}^2$ is contraction, so we define $C = \frac{1}{2(1+i)}(T_{\psi_1} - \sqrt{I - T_{\psi_2}^2})$. Now, $\|C\| \leq \frac{1}{2\sqrt{2}}(\|T_{\psi_1}\| + \|I - T_{\psi_2}^2\|) \leq 1$. This implies C is a contraction operator. Let's now construct $V_1 = C + i\sqrt{I - C^2}$, $V_2 = C - i\sqrt{I - C^2}$, $V_3 = iC + \sqrt{I - C^2}$, $V_4 = iC - \sqrt{I - C^2}$ and $V_5 = iT_{\psi_2} + \sqrt{I - T_{\psi_2}^2}$. One can easily verify that $V_i, \forall i = 1, \dots, 5$ are unitary operators. Therefore, $T_\psi = \sum_{i=1}^5 V_i = T_{\psi_1} + iT_{\psi_2}$. This yields the desired result. $\square$

Corollary 3.6. Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. The operator T_ψ will be the sum of nine unitary operators.

Proof. Since $\|\psi\|_\infty \leq 1$, then $\|T_\psi\| \leq 1$. Let $T_\psi = T_{\psi_1} + iT_{\psi_2}$ be the cartesian decomposition of T_ψ with $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Define $C = \frac{1}{4(1+i)}(T_{\psi_1} - \sqrt{I - T_{\psi_2}^2})$. Here C is a contraction operator. Suppose $V_6 = -C + i\sqrt{I - C^2}$, $V_7 = -C - i\sqrt{I - C^2}$, $V_8 = -iC + \sqrt{I - C^2}$, $V_9 = -iC - \sqrt{I - C^2}$. Here V_6, V_7, V_8 and V_9 are

unitary operators. Thus from Theorem- (3.5), $\sum_{i=1}^5 V_i - \sum_{j=6}^9 V_j = T_{\psi_1} + iT_{\psi_2}$. Hence the result proven. $\square$

Corollary 3.7. *If $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$, then T_ψ has the representation of addition of thirty six unitary operators.*

Proof. Since $\|\psi\|_\infty \leq 1$, then $\|T_\psi\| \leq 1$. Let $T_\psi = T_{\psi_1} + iT_{\psi_2}$ be the cartesian decomposition of T_ψ with $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Define $C_1 = \frac{1}{4(1+i)}(T_{\psi_1} + \sqrt{I - T_{\psi_2}^2})$, $C_2 = \frac{1}{4(1+i)}(T_{\psi_1} - \sqrt{I - T_{\psi_2}^2})$, $C_3 = \frac{i}{4(1+i)}(T_{\psi_2} - \sqrt{I - T_{\psi_1}^2})$, $C_4 = \frac{i}{4(1+i)}(T_{\psi_2} + \sqrt{I - T_{\psi_1}^2})$. Here C_j are contraction operators. Let's define the operators

$W_j^{(1)} = \pm C_j \mp i\sqrt{I - C_j^2}$, $W_j^{(2)} = \pm iC_j \mp \sqrt{I - C_j^2}$, $j = 1, \dots, 4$. Notice that $W_j^{(1)}, W_j^{(2)}$ are unitary operators $\forall j$. Now, $W^{\diamond 1} = iT_{\psi_2} + \sqrt{I - (T_{\psi_2})^2}$, $W^{\diamond 2} = iT_{\psi_2} - \sqrt{I - (T_{\psi_2})^2}$, $W^{\diamond 3} = T_{\psi_1} + i\sqrt{I - (T_{\psi_1})^2}$, $W^{\diamond 4} = T_{\psi_1} - i\sqrt{I - (T_{\psi_1})^2}$. Thus, $W^{\diamond 1}, W^{\diamond 2}, W^{\diamond 3}, W^{\diamond 4}$ are unitary operators. Hence, $T_\psi = \sum_{j=1}^4 W_j^{(1)} + W_j^{(2)} + W^{\diamond j}$

which is the span of thirty six unitary operators. We thus obtained the result. $\square$

Theorem 3.8. *Let $\psi_i \in L^\infty(\mathbb{D})$ with $\|\psi_i\|_\infty \leq 1$ for $i = 1, 2$. If $\psi = \psi_1 + \psi_2$, then T_ψ can be written as sum of two unitary operators.*

Proof. Let $\mathfrak{C} = \frac{1}{2}(T_{\psi_1} + T_{\psi_2})$, where $\|\mathfrak{C}\| \leq 1$ as $\|\psi_i\|_\infty \leq 1$ for $i = 1, 2$. Define $\mathfrak{V}_1 = \mathfrak{C} + i\sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_2 = \mathfrak{C} - i\sqrt{I - \mathfrak{C}^2}$, where $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are unitary operators. Here $\mathfrak{V}_1 + \mathfrak{V}_2 = T_{\psi_1} + T_{\psi_2} = T_\psi$. This yields the desired result. $\square$

Corollary 3.9. *Let $\psi_i \in L^\infty(\mathbb{D})$ with $\|\psi_i\|_\infty \leq 1$ for $i = 1, 2$. If $\psi = \psi_1 + \psi_2$, then T_ψ can be written as linear combination of eight unitary operators.*

Proof. Let $\mathfrak{C} = \frac{1}{4(1+i)}(T_{\psi_1} + T_{\psi_2})$, where $\|\mathfrak{C}\| \leq 1$ as $\|\psi_i\|_\infty \leq 1$ for $i = 1, 2$. Define $\mathfrak{V}_1 = \mathfrak{C} + i\sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_2 = \mathfrak{C} - i\sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_3 = -\mathfrak{C} + i\sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_4 = -\mathfrak{C} - i\sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_5 = i\mathfrak{C} + \sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_6 = i\mathfrak{C} - \sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_7 = -i\mathfrak{C} + \sqrt{I - \mathfrak{C}^2}$, $\mathfrak{V}_8 = -i\mathfrak{C} - \sqrt{I - \mathfrak{C}^2}$ where $\mathfrak{V}_i, i = 1, 2, \dots, 8$ are unitary operators. Here $\mathfrak{V}_1 + \mathfrak{V}_2 - \mathfrak{V}_3 - \mathfrak{V}_4 + \mathfrak{V}_5 + \mathfrak{V}_6 - \mathfrak{V}_7 - \mathfrak{V}_8 = 4(1+i)\mathfrak{C} = T_{\psi_1} + T_{\psi_2} = T_\psi$. This yields the desired outcome. $\square$

4. ALUTHGE TRANSFORM OF TOEPLITZ OPERATORS

In this section, we investigate the Aluthge transform $\mathfrak{A}(T_\psi)$ through its representation as a linear combination of unitary operators.

Theorem 4.1. *If $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$, then the Aluthge transform $\mathfrak{A}(T_\psi)$ of T_ψ can be expressed as the sum of five unitary transformations.*

Proof. If $T_\psi = T_{\psi_1} + iT_{\psi_2}$ is the cartesian decomposition of T_ψ with $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$, then, $\mathfrak{A}(T_\psi) = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2})$. Since $\|\psi\|_\infty \leq 1$, then $\|\mathfrak{A}(T_\psi)\|_\infty \leq 1$. So, $\|\mathfrak{A}(T_{\psi_1})\| \leq 1$ and $\|\mathfrak{A}(T_{\psi_2})\| \leq 1$. Thus, $1 - \mathfrak{A}(T_\psi)^2 \geq 0$. Let's construct

$C = \frac{1}{2(1+i)}(\mathfrak{A}(T_{\psi_1} + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2})$. Now, $\|C\| \leq 1$. Let's define the unitary operators $W_1 = C + i\sqrt{I - C^2}$, $W_2 = C - i\sqrt{I - C^2}$, $W_3 = iC + \sqrt{I - C^2}$, $W_4 = iC - \sqrt{I - C^2}$, $W^\diamond = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$. Hence, $\mathfrak{A}(T_\psi) = \sum_{i=1}^4 W_i + W^\diamond = 2(1+i)C + i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2} = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2}) = \mathfrak{A}(T_\psi)$. $\square$

Corollary 4.2. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then the Aluthge transform $\mathfrak{A}(T_\psi)$ of T_ψ can be expressed as the span of ten unitary operators.*

Proof. It follows from the Theorem (4.1) that $\mathfrak{A}(T_\psi) = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2})$ and $\mathfrak{A}(T_\psi), \mathfrak{A}(T_{\psi_1}), \mathfrak{A}(T_{\psi_2})$ are contractions.

Let's define $C_1 = \frac{1}{2(1+i)}(\mathfrak{A}(T_{\psi_1}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2})$, $C_2 = \frac{1}{2(1+i)}(\mathfrak{A}(T_{\psi_1}) + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2})$. Here $\|C_1\| \leq 1$ and $\|C_2\| \leq 1$. Introduce the operators $W_1 = C_2 + i\sqrt{I - C_2^2}$, $W_2 = C_2 - i\sqrt{I - C_2^2}$, $W_3 = iC_2 + \sqrt{I - C_2^2}$, $W_4 = iC_2 - \sqrt{I - C_2^2}$, $W_5 = C_1 + i\sqrt{I - C_1^2}$, $W_6 = C_1 - i\sqrt{I - C_1^2}$, $W_7 = iC_1 + \sqrt{I - C_1^2}$, $W_8 = iC_1 - \sqrt{I - C_1^2}$, $W^{\diamond 1} = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $W^{\diamond 2} = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$. Notice that $W^{\diamond 1}, W^{\diamond 2}$ are unitary operators. Hence, $\mathfrak{A}(T_\psi) = \frac{1}{2}(\sum_{j=1}^8 W_j + W^{\diamond 1} + W^{\diamond 2})$. Thus the result is obtained. $\square$

Corollary 4.3. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then $\mathfrak{A}(T_\psi)$ can be expressed as span of thirty six unitary operators.*

Proof. Let us define the operators

$$\begin{aligned} C_1 &= \frac{1}{4(1+i)}(\mathfrak{A}(T_{\psi_1}) + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}) \\ C_2 &= \frac{1}{4(1+i)}(\mathfrak{A}(T_{\psi_1}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}) \\ C_3 &= \frac{i}{4(1+i)}(\mathfrak{A}(T_{\psi_2}) + \sqrt{I - \mathfrak{A}(T_{\psi_1})^2}) \\ C_4 &= \frac{i}{4(1+i)}(\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_1})^2}) \end{aligned}$$

Further, define

$$W_j^{(1)} = \pm C_j \mp i\sqrt{I - C_j^2}, j = 1, 2, 3, 4. \quad W_j^{(2)} = \pm iC_j \mp \sqrt{I - C_j^2}, j = 1, 2, 3, 4.$$

Here $W_j^{(1)}, W_j^{(2)}$ are unitary operators $\forall j$. Now,

$$\begin{aligned} W^{\diamond 1} &= i\mathfrak{A}(T_{\psi_2}) + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2} \\ W^{\diamond 2} &= i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2} \\ W^{\diamond 3} &= \mathfrak{A}(T_{\psi_1}) + i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2} \\ W^{\diamond 4} &= \mathfrak{A}(T_{\psi_1}) - i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2} \end{aligned}$$

Thus, $W^{\diamond_1}$, $W^{\diamond_2}$, $W^{\diamond_3}$, $W^{\diamond_4}$ are unitary operators.

Therefore, $\mathfrak{A}(T_\psi) = \sum_{j=1}^4 W_j^{(1)} + W_j^{(2)} + W^{\diamond_j}$ equals to span of thirty six unitary operators. □

Theorem 4.4. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then, Aluthge transform $\mathfrak{A}(T_\psi)$ of T_ψ can be decomposed into the span of four unitaries.*

Proof. Since $\|T_\psi\| \leq \|\psi\|_\infty \leq 1$, then by [3], $\|\mathfrak{A}(T_\psi)\| \leq \|T_\psi\| \leq 1$. So, $\mathfrak{A}(T_\psi)$ is a contraction transformation on $A^2(\mathbb{D})$. Let $\mathfrak{A}(T_\psi) = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2})$ be the cartesian decomposition of $\mathfrak{A}(T_\psi)$ with $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Again, since $\mathfrak{A}(T_\psi)$ be a contraction operator, then its real and imaginary parts are contractions. So, $\|\mathfrak{A}(T_{\psi_1})\| \leq 1$ and $\|\mathfrak{A}(T_{\psi_2})\| \leq 1$. Thus, $1 - \mathfrak{A}(T_\psi)^2 \geq 0$. Now, define $P_1 = \mathfrak{A}(T_{\psi_1}) + i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$, $P_2 = \mathfrak{A}(T_{\psi_1}) - i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$, $Q_1 = i\mathfrak{A}(T_{\psi_2}) + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $Q_2 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$.

Hence $\mathfrak{A}(T_\psi) = \frac{1}{2}(P_1 + P_2 + Q_1 + Q_2)$. □

Corollary 4.5. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then, the Aluthge transform $\mathfrak{A}(T_\psi)$ of T_ψ can be expressed additive combination of sixteen unitary operators.*

Proof. Since $\|\psi\|_\infty \leq 1$, then from Theorem (4.4), we get $\mathfrak{A}(T_\psi)$ is a contraction transformation on $A^2(\mathbb{D})$. Let $\mathfrak{A}(T_\psi) = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2})$ represents the cartesian decomposition of $\mathfrak{A}(T_\psi)$ with $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$. Thus, $\|\mathfrak{A}(T_{\psi_1})\| \leq 1$ and $\|\mathfrak{A}(T_{\psi_2})\| \leq 1$. Then, $1 - \mathfrak{A}(T_\psi)^2 \geq 0$. Let us define $P_1 = \mathfrak{A}(T_{\psi_1}) + i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$, $P_2 = \mathfrak{A}(T_{\psi_1}) - i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$, $P_3 = i\mathfrak{A}(T_{\psi_2}) + \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $P_4 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $P_5 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $P_6 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $P_7 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$, $P_8 = i\mathfrak{A}(T_{\psi_2}) - \sqrt{I - \mathfrak{A}(T_{\psi_2})^2}$. Hence $\mathfrak{A}(T_\psi) = \frac{1}{2}(P_1 + P_2 + Q_1 + Q_2)$. □

Theorem 4.6. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. Then, the Aluthge transform of the Toeplitz operator T_ψ , denoted by $\mathfrak{A}(T_\psi)$, can be represented as the sum of two unitary operators.*

Proof. The cartesian decomposition of T_ψ is $T_{\psi_1} + iT_{\psi_2}$. So by [3], $\mathfrak{A}(T_\psi) = \mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2})$. Again, since $\|\psi\|_\infty \leq 1$, then by [3], $\|\mathfrak{A}(T_\psi)\| \leq \|T_\psi\| \leq 1$. So, $\mathfrak{A}(T_\psi)$ is a contraction operator, then its real and imaginary parts are contractions i.e $\|\mathfrak{A}(T_{\psi_1})\| \leq 1$ and $\|\mathfrak{A}(T_{\psi_2})\| \leq 1$. Construct $C = \frac{1}{2}(\mathfrak{A}(T_{\psi_1}) + i\mathfrak{A}(T_{\psi_2}))$. Here $\|C\| \leq 1$. Let us define $V_1 = C + i\sqrt{I - C^2}$, $V_2 = C - i\sqrt{I - C^2}$. Now, $V_1 + V_2 = 2C = \mathfrak{A}(T_\psi)$. Hence proven. □

5. EXPONENTIAL TRANSFORM OF TOEPLITZ OPERATORS

The representation of the exponential transform of Toeplitz operators on $A^2(\mathbb{D})$ as a linear combination of unitary operators is explored in this section.

Theorem 5.1. *Let $\psi \in L^\infty(\mathbb{D})$ with $\psi \geq 0$. Then $\cos(T_\psi), \sin(T_\psi)$ can be expressed as span of two unitary operators.*

Proof. Since $\psi \geq 0$, then $T_\psi \geq 0$. Thus, we can define two operators $e^{iT_\psi}, e^{-iT_\psi}$, which are unitary operators. Hence $\cos(T_\psi) = \frac{e^{iT_\psi} + e^{-iT_\psi}}{2}$, which is the average of two unitary operators. The proof of $\sin(T_\psi)$ parallels that of $\cos(T_\psi)$. $\square$

Theorem 5.2. *Let $\psi \in L^\infty(\mathbb{D})$ and $T_\psi = V - V^*$ for any $V \in \mathcal{L}(A^2(\mathbb{D}))$, then e^{T_ψ} is unitary.*

Proof. Since $T_\psi = V - V^*$, let us define the operator $U = e^{T_\psi} = \sum_{n=0}^{\infty} \frac{1}{n} T_\psi^n$. Now, $T_\psi^* = -T_\psi$. Thus, $U^* = e^{T_\psi^*} = e^{-T_\psi} = U^{-1}$. Hence U is unitary. $\square$

Corollary 5.3. *Let $\psi \in L^\infty(\mathbb{D})$ with $\psi \geq 0$. Then e^{T_ψ} is unitary.*

Proof. Since $\psi \geq 0, T_\psi \geq 0$. Hence, T_ψ is self-adjoint. As we know that every self-adjoint can be expressed as difference of two positive operators, let's say (P, P^*) so $T_\psi = P - P^*$ where P and P^* are two positive operators. Hence, the proof follows from Theorem 5.2. $\square$

The Schatten p -class $C_p(H)$ or C_p , $1 \leq p < \infty$, comprises compact operators T on Hilbert space H whose singular values satisfy $(\lambda_n) \in l^p$. For more details see [6].

Theorem 5.4. *Let $\psi \in L^p(\mathbb{D}, dA)$, $1 \leq p < \infty$ and $\psi_1, \psi_2 \in L^\infty(\mathbb{D})$ with $\psi_2 - \bar{\psi}_2 \geq 0$. If $(T_{\psi_1 - \bar{\psi}_1})(T_{\psi_2 - \bar{\psi}_2}) = (T_{\psi_2 - \bar{\psi}_2})(T_{\psi_1 - \bar{\psi}_1})$, then e^{T_ψ} is unitary on $A^2(\mathbb{D})$.*

Proof. Since $\psi \in L^p(\mathbb{D}, dA)$, then by ([20], page-125), $T_\psi \in C_p$. Thus, there exists four positive operators ([20], p-125), $T_{\psi_1}, T_{\psi_1}^*, T_{\psi_2}, T_{\psi_2}^*$ in C_p such that $T_\psi = (T_{\psi_1} - T_{\psi_1}^*) + i(T_{\psi_2} - T_{\psi_2}^*)$. Again, since $(T_{\psi_1 - \bar{\psi}_1})(T_{\psi_2 - \bar{\psi}_2}) = (T_{\psi_2 - \bar{\psi}_2})(T_{\psi_1 - \bar{\psi}_1})$, so $e^{T_\psi} = e^{(T_{\psi_1} - T_{\psi_1}^*) + i(T_{\psi_2} - T_{\psi_2}^*)} = e^{(T_{\psi_1} - T_{\psi_1}^*)} e^{i(T_{\psi_2} - T_{\psi_2}^*)}$. Thus, by Theorem (5.2), $e^{(T_{\psi_1} - T_{\psi_1}^*)}$ is unitary. As $\psi_2 - \bar{\psi}_2 \geq 0$, then $T_{\psi_2} - T_{\psi_2}^* = T_{\psi_2 - \bar{\psi}_2} \geq 0$. Therefore, $e^{i(T_{\psi_2} - T_{\psi_2}^*)}$ is unitary on $A^2(\mathbb{D})$. Hence e^{T_ψ} is unitary. $\square$

Corollary 5.5. *Let $T_\psi \in \mathcal{L}(H)$ with $\|T_\psi f\| \leq \|f\|$ for all $f \in A^2(\mathbb{D})$. If $T_\psi = UP$ be the polar decomposition of T_ψ with $U_1 = iP + \sqrt{I - P^2}, U_2 = iP - \sqrt{I - P^2}, U_3 = -iP + i\sqrt{I - P^2}, U_4 = -iP - i\sqrt{I - P^2}$ then $e^{U_1+U_2}, e^{U_1+U_2-U_3-U_4}$ are unitary operators.*

Proof. From the corollary (5.5), $U_1 + U_2 = 2iP$. Thus, $e^{U_1+U_2}$ is unitary. Other cases can be verified in the similar manner. $\square$

Corollary 5.6. *Let $U = \pm P \pm i\sqrt{I - P^2}$ or $W = \pm iP \pm \sqrt{I - P^2}$, then e^{U-U^*} and e^{W-W^*} are unitary operators.*

Corollary 5.7. *Let $U = \pm \mathfrak{A}(T_{\psi_1}) \pm i\sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$ and $V = \pm i\mathfrak{A}(T_{\psi_1}) \pm \sqrt{I - \mathfrak{A}(T_{\psi_1})^2}$, then e^{U-U^*} and e^{V-V^*} are unitary operators.*

Corollary 5.8. *Let $U = \pm Cl(T_\psi) \pm i\sqrt{I - Cl(T_\psi)^2}$ and $V = \pm iCl(T_\psi) \pm \sqrt{I - Cl(T_\psi)^2}$, then e^{U-U^*} and e^{V-V^*} are unitary operators.*

Corollary 5.9. *Let $U = \pm \mu(T_\psi) \pm i\sqrt{I - \mu(T_\psi)^2}$ and $V = \pm i\mu(T_\psi) \pm \sqrt{I - \mu(T_\psi)^2}$, then e^{U-U^*} and e^{V-V^*} are unitary operators.*

6. CAYLEY TRANSFORM OF TOEPLITZ OPERATORS

The Cayley transform of T_ψ is analyzed here, showcasing its expression as a linear combination of unitary operators.

Theorem 6.1. *Let $\psi_1, \psi_2, \psi \in L^\infty(\mathbb{D})$ with $\psi_1 \geq 0$ and $\psi_2 \geq 0$. If T_ψ is a quasi unitary operator, then $Cl(T_\psi)$, Cayley transform of T_ψ is the span of four unitary operators.*

Proof. Since ψ_1 and ψ_2 are positive, then $T_{\psi_1} \geq 0$ and $T_{\psi_2} \geq 0$. Let $Cl(T_\psi) = T_{\psi_1} + iT_{\psi_2}$ be any cartesian decomposition of $Cl(T_\psi)$. Again, Since T_ψ is quasiunitary, then $T_\psi - I$ is unitary. Thus by [18], $Cl(T_\psi)$ is a contraction operator. Hence, the real and imaginary part of $Cl(T_\psi)$ are contractions i.e. $\|T_{\psi_1}\| \leq 1$ and $\|T_{\psi_2}\| \leq 1$. Thus, $I - T_{\psi_1}^2$ is positive and contraction operator. Similarly, $\|I - T_{\psi_2}^2\| \leq 1$ and $I - T_{\psi_2}^2$ is positive. Let's define $N_1 = T_{\psi_1} + i\sqrt{I - T_{\psi_1}^2}$, $N_2 = T_{\psi_1} - i\sqrt{I - T_{\psi_1}^2}$, $N_3 = iT_{\psi_2} + \sqrt{I - T_{\psi_2}^2}$ and $N_4 = iT_{\psi_2} - \sqrt{I - T_{\psi_2}^2}$. Here N_1, N_2, N_3 and N_4 are unitary operators. Therefore, $Cl(T_\psi) = T_{\psi_1} + iT_{\psi_2} = \frac{1}{2}(N_1 + N_2 + N_3 + N_4)$, which is the span of four unitary operators. $\square$

7. MEAN TRANSFORM OF TOEPLITZ OPERATORS

In this section, we study the mean transform of T_ψ , uncovering its unitary decomposition.

Theorem 7.1. *Let $\psi \in L^\infty(\mathbb{D})$ with $\|\psi\|_\infty \leq 1$. If T_ψ is invertible, then the mean transform of T_ψ i.e $\hat{T}_\psi$ can be expressed as the span of sixteen unitary operators.*

Proof. Let $T_\psi = U|T_\psi|$ be the polar decomposition of T_ψ with U as partial isometry. Since T_ψ is invertible, then U can be converted into unitary operator. Since $\|T_\psi\| \leq 1$, then $\||T_\psi|\| \leq 1$. Thus $I - |T_\psi|^2 \geq 0$ and $\|I - |T_\psi|^2\| \leq 1$. Let us define $W_1 = |T_\psi| + i\sqrt{I - |T_\psi|^2}$, $W_2 = |T_\psi| - i\sqrt{I - |T_\psi|^2}$, $W_3 = i|T_\psi| + \sqrt{I - |T_\psi|^2}$, $W_4 = i|T_\psi| - \sqrt{I - |T_\psi|^2}$, $W_5 = -|T_\psi| + i\sqrt{I - |T_\psi|^2}$, $W_6 = -|T_\psi| - i\sqrt{I - |T_\psi|^2}$, $W_7 = -i|T_\psi| + \sqrt{I - |T_\psi|^2}$, $W_8 = -i|T_\psi| - \sqrt{I - |T_\psi|^2}$. Now, $|T_\psi| = \frac{1}{4(1+i)}\left(\sum_{j=1}^4 W_j - \sum_{j=5}^8 W_j\right)$.

Hence $T_\psi = U|T_\psi| = \frac{U}{4(1+i)}\left(\sum_{j=1}^4 W_j - \sum_{j=5}^8 W_j\right) = \frac{U}{4(1+i)}\left(\sum_{j=1}^4 U_j - \sum_{j=5}^8 U_j\right)$, where $U_i = UW_i$ are unitary operators for all $i = 1, 2, \dots, 8$. Now the Duggal transform of T_ψ is $T^\mathfrak{D} = |T|U = |T_\psi|U = \frac{\left(\sum_{j=1}^4 W_j - \sum_{j=5}^8 W_j\right)U}{4(1+i)} = \frac{\sum_{j=1}^4 V_j - \sum_{j=5}^8 V_j}{4(1+i)}$, where

$V_i = W_i U$ are unitary operators for $i = 1, 2 \dots 8$.

$$\begin{aligned}\mu(T_\psi) &= \frac{U|T_\psi| + |T_\psi|U}{2} \\ &= \frac{\sum_{j=1}^4 U_j - \sum_{j=5}^8 U_j + \sum_{j=1}^4 V_j - \sum_{j=5}^8 V_j}{8(1+i)} \\ &= \text{span}\{U_1, U_2, \dots, U_8, V_1, V_2, \dots, V_8\}.\end{aligned}$$

Hence Proven. $\square$

Theorem 7.2. Let $\mu(T_\psi) = \frac{T_\psi + T_\psi^*}{2}$. If $\mu(T) = 0$, then the mean of Cayley transform of T_ψ i.e $\mu(Cl(T_\psi))$ can be represented as average of two unitary operators.

Proof. Notice that T_ψ is skew-hermitian implies $Cl(T_\psi) = (I + iT_\psi)(I - iT_\psi)^{-1}$ is unitary. Hence $\mu(Cl(T_\psi)) = \frac{Cl(T_\psi) + Cl(T_\psi)^*}{2}$. Hence proven. $\square$

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¹ DEPARTMENT OF MATHEMATICS, GOVERNMENT SCIENCE COLLEGE, CHATRAPUR, PO BOX-761020, INDIA.

Email address: badalsahoo206@gmail.com

² DEPARTMENT OF MATHEMATICS, BERHAMPUR UNIVERSITY, BHANJA BIHAR, PO BOX-760007, INDIA

Email address: pabitrmath@gmail.com