

ON BLOCK-RELATED DERIVED GRAPHS

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ABSTRACT. This paper introduces and analyses the block-degree of a vertex and the cut-degree of a block. The block-degree of a vertex v is the number of blocks containing v . The cut-degree of a block b is the number of cut vertices of G contained in b . The block-degree sequence of cut vertices of the graph and the cut-degree sequence of the graph are defined. A few characterizations of the block-degree and cut-degree sequence of the graph are established. Given a graph, its block graph ($\mathfrak{B}(G)$) is a graph where each vertex represents a block, and two vertices are connected if their blocks intersect. The number of cut vertices of $\mathfrak{B}(G)$ is determined. Further, an investigation is carried out on the traversability of $\mathfrak{B}(G)$. A block cutpoint graph ($BC(G)$) of a graph represents a graph where each vertex corresponds to either a block or a cut vertex, and two vertices are connected if one represents a block and the other represents a cut vertex contained within that block. The properties of $BC(G)$ and its iterations are studied. The graph G for which $BC(G)$ is a perfect m -ary tree is characterized.

1. INTRODUCTION

Graph theory is an emerging branch of Mathematics known for its significant role in various fields by providing solutions for problems concerning relationships and interactions. One of the concepts within the graph theory is that of cut vertices whose removal increases the number of connected components in the graph. The study of these vertices is pivotal for the structural integrity and resilience of networks. On the other hand, blocks which are the maximal connected subgraphs with no cut vertices are essential for understanding the structural properties, connectivity, and robustness of the graph. The block which contains exactly one cut vertex of G is known as a pendant block and a block which contains more than one cut vertex of G is known as a non-pendant block. Any connected graph with at least three vertices has at least two pendant blocks. Let $B(G) = \{b_l | 1 \leq l \leq k\}$ and $C(G) = \{c_j | 1 \leq j \leq t\}$ be the set of all blocks and cut vertices, respectively in G . We say any two blocks are adjacent if they have a common cut vertex. We consider a simple connected graph G of order n , k blocks and t cut vertices, where $n \geq 2, k \geq 1, t \geq 0$. Note that, $C(G)$ is empty when the graph is a block.

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The relationship between cut vertices and blocks reveals insights into the connectivity and vulnerability of complex networks. Though an extensive study on these parameters has been carried out in the literature, the study on the inter-relation between cut vertices and blocks was limited. The purpose of the study is to bridge the gap by defining and examining the concepts of block-degree of a vertex and cut-degree of a block. Several properties of block-degree of a cut vertex and the cut-degree of a block are examined. The findings on block-degree and cut-degree facilitate in exploring some block and cut vertex related derived graphs namely, the block graph of a graph ($\mathfrak{B}(G)$) and the block cut point graph of a graph ($BC(G)$).

The block graph $\mathfrak{B}(G)$ of a graph G is a graph with a vertex set as $B(G)$ and two vertices are adjacent if the corresponding blocks have a common cut vertex. This concept was introduced in [6], where some of its properties were studied and it was proved a graph is a block graph if and only if all its blocks are complete. The chordal graphs are graphs in which every cycle of length at least 4 has a chord. Thus, block graphs are a subclass of chordal graphs. Various characterization of block graphs were presented based on some metric conditions, refer [8, 2, 4]. A characterization based on a forbidden subgraph was presented in [1]. Mudler established an interestingly simple characterization in [9]. Based on the blocks and cut vertices of the graph, a derived graph called block cut point graph of a graph, denoted by $BC(G)$ was defined in [7]. It is a graph with vertex set $B(G) \cup C(G)$ and any two vertices are adjacent if one is a cut vertex and the other is a block containing it. In [7], they proved every block cut point graph of a graph is a tree. They characterized graphs which are block cutpoint trees or BC -trees. Barefoot in [3] proved that a tree has a BC -tree partition if and only if it lacks a perfect matching. Several concepts and algorithms about BC -tree partitions have also been introduced. For terminology and notations, we refer to [11, 5]. A rooted tree is classified as m -ary if its internal vertices have a maximum of m children. It's considered full m -ary if all internal vertices have exactly m children. The following results are necessary for further studies in this paper.

Theorem 1.1. [10] *A full m -ary tree with*

- (1) *n vertices has $I = \frac{(n-1)}{m}$ internal vertices and $L = \frac{((m-1)n)+1}{m}$ leaves.*
- (2) *I internal vertices has $n = mI + 1$ vertices and $L = (m-1)I + 1$ leaves.*
- (3) *L leaves has $n = \frac{(mL-1)}{(m-1)}$ vertices and $I = \frac{(L-1)}{(m-1)}$ internal vertices.*

Theorem 1.2. [5] *The vertex v of a connected graph G is a cut vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every $(u - w)$ path of G .*

Theorem 1.3. [5] *A graph of order at least 3 is a block if and only if every two vertices lie on a common cycle.*

Theorem 1.4. [11] *Every vertex of a graph has an even degree if and only if every block is Eulerian.*

Theorem 1.5. [5] *A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degrees.*

2. ON BLOCK-DEGREE OF THE VERTEX AND THE CUT-DEGREE OF THE BLOCK

The number of cut vertices helps to determine the number of blocks in G . Now, we define the block-degree of a vertex in the graph.

Definition 2.1. The block-degree of a v_i in a graph G , written $\deg_b(v_i)$, is the number of blocks containing v_i , where $1 \leq i \leq n$. Let us denote $\deg_b(v_i) = \beta_i$.

The block-degree of a vertex in G which is not a cut vertex is 1, as it is contained in exactly one block. Throughout this paper, we study the block-degree of cut vertices c_j of G , where $1 \leq j \leq t$.

Example 2.2. In Figure 1, $C(G) = \{v_2, v_4, v_6\}$. The vertices in $V(G) \setminus C(G)$ are non-cut vertices with block degree 1. The block-degree of the vertices in $C(G)$ is 3, 4 and 2 respectively.

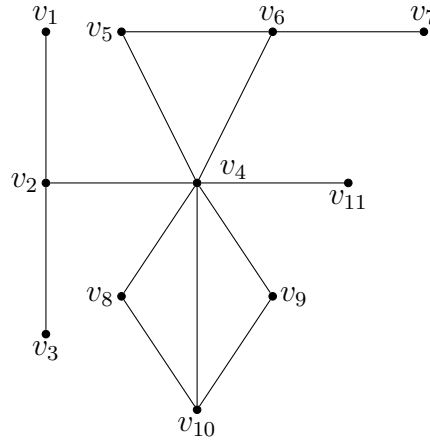


FIGURE 1. A graph G to illustrate block-degree of vertices

Proposition 2.3. For any graph G with $|B(G)| = k$, the following statements are true:

- (1) For any vertex $c_j \in C(G)$, $2 \leq \deg_b(c_j) \leq k$, where $1 \leq j \leq t$.
- (2) For any $c_j \in C(G)$, $\deg_b(c_j) \leq \deg(c_j)$.
- (3) The $\deg_b(c_j) = k$ if and only if G has exactly one cut vertex.

Proof. Since any cut vertex of G belongs to at least two blocks and at most all blocks of G , the block-degree of cut vertices in G lies between 2 and k , hence the result (1).

Since for any non-trivial graph, K_2 is a block of least order and the degree of a vertex is the number of edges incident to it, the degree of a cut vertex is at least its block-degree. Hence, the result (2).

Suppose G has exactly one cut vertex, say c_1 . Then, all the blocks of G contain c_1 . Thus, the block-degree of c_1 is k . The converse is easy to see. $\square$

Now, we characterize the graphs for which the block-degree of a cut vertex equals its degree.

Theorem 2.4. *In a graph G of order n , $n \geq 3$, $\deg(c_j) = \deg_b(c_j)$, where $c_j \in C(G)$ and $1 \leq j \leq t$ if and only if G is a tree.*

Proof. Consider G to be a tree of order n , $n \geq 3$. As every edge of G is a block, the number of edges incident to a cut vertex is its block degree. Thus, the degree of the cut vertex equals its block-degree.

Conversely, suppose G is cyclic. Then, there exists at least one cut vertex which belongs to a block which has cycles. Let that cut vertex be c_x . As c_x belongs to some block which has cycles, say b_l , at least two edges incident with c_x lie in b_l . Thus, $\deg(c_x) \geq \beta_x + 1$, where $\deg_b(c_x) = \beta_x$. Therefore, $\deg_b(c_x) \neq \deg(c_x)$. Hence the result. $\square$

Motivated by the definition of the degree sequence of a graph G , we now, define the block-degree sequence of cut vertices of a graph.

Definition 2.5. The block-degree sequence of cut vertices of a graph G is a list of block-degrees of cut vertices $\beta_1, \beta_2, \dots, \beta_t$ in the non-increasing order. If multiple cut vertices share the same block-degree, for example, if r cut vertices have a block-degree β (where $r \leq t$), we denote this by $(\beta)_r$. This notation indicates that there are r cut vertices each with a block-degree β .

Remark 2.6. The following are the observations from the block-degree sequence of cut vertices of a graph.

- (1) Since the block-degree of any cut vertex is at least 2, in the block-degree sequence of cut vertices of a graph, $\beta_t \geq 2$.
- (2) Given any sequence of non-increasing positive integers $S : \beta_1, \dots, \beta_t$ with $\beta_t \geq 2$, there exists a graph G with S being its block-degree sequence of cut vertices of G .

Given the block-degree sequence of cut vertices of the graph G , we give the bounds on the order, size of G and the number of blocks of G .

Theorem 2.7. *Let $\beta_1, \beta_2, \dots, \beta_t$ be the block-degree sequence of cut vertices of the graph G , where $|C(G)| = t$. The following statements are true.*

- (1) $|V(G)| \geq 2 + \sum_{j=1}^t (\beta_j - 1)$.
- (2) $|E(G)| \geq 1 + \sum_{j=1}^t (\beta_j - 1)$.
- (3) $|B(G)| = 1 + \sum_{j=1}^t (\beta_j - 1)$.

Proof. Let β_j be the block-degree of c_j , where $1 \leq j \leq t$ and $|C(G)| = t$. As the minimum possible order of any block is 2 and these β_j blocks share a common vertex c_j , these β_j blocks contribute at least $\beta_j + 1$ and β_j to the order and size of the graph, respectively. Let c_1 and c_2 share a block. Then, one block of $\beta_1 + \beta_2$ blocks contains both c_1 and c_2 . Thus, these $\beta_1 + \beta_2$ blocks contribute at least $(\beta_1 + 1) + (\beta_2 - 1)$ and $\beta_1 + (\beta_2 - 1)$ to the order and size of the graph, respectively. In addition, the number of distinct blocks containing c_1 and c_2 is $\beta_1 + (\beta_2 - 1)$. As any two cut vertices have at most one block in common, the minimum order and size of G are $(\beta_1 + 1) + (\beta_2 - 1) + \dots + (\beta_t - 1)$ and $\beta_1 + (\beta_2 - 1) + \dots + (\beta_t - 1)$, respectively. Further, the number of blocks in G is $\beta_1 + (\beta_2 - 1) + \dots + (\beta_t - 1)$. $\square$

The characterization of graphs which attain the lower bound on the order and size of the graph with a given block-degree sequence of cut vertices of G is given below.

Corollary 2.8. *Let $\beta_1, \beta_2, \dots, \beta_t$ be the block-degree sequence of cut vertices of the graph G and $|C(G)| = t$. Then, G is a tree if and only if $|V(G)| = 2 + \sum_{j=1}^t (\beta_j - 1)$ and $|E(G)| = 1 + \sum_{j=1}^t (\beta_j - 1)$.*

Proof. Let G be a tree. As every edge in G is a block, the order of each block is 2. By Theorem 2.7, the order and size of G are $2 + \sum_{j=1}^t (\beta_j - 1)$ and $1 + \sum_{j=1}^t (\beta_j - 1)$, respectively. Suppose $|V(G)| = 2 + \sum_{j=1}^t (\beta_j - 1)$ and $|E(G)| = 1 + \sum_{j=1}^t (\beta_j - 1)$. Trees are unique among graphs whose size is always one less than their order. Hence the result. $\square$

Corollary 2.9. *If $(\beta)_t$ is the block-degree sequence of cut vertices of the graph G , then $|V(G)| \geq t(\beta - 1) + 2$, $|E(G)| \geq t(\beta - 1) + 1$ and $|B(G)| = t(\beta - 1) + 1$.*

Proof. Suppose the block-degree of all cut vertices in the graph G are the same. Then, in Theorem 2.7, $\beta_1 = \beta_2 = \dots = \beta_t = \beta$. Hence the result follows. $\square$

The bounds on the number of pendant blocks are given below.

Theorem 2.10. *For any graph G with $|B(G)| = k$ and $k \geq 2$, we have, $2 \leq \text{number of pendant blocks} \leq k$.*

Proof. Let G be a graph with $|B(G)| = k$, $k \geq 2$. The lower bound is obvious as any non-trivial graph which is not a block has at least two pendant blocks. Suppose G has exactly one cut vertex. Then, all blocks of G are pendant blocks and hence the upper bound is attained. Suppose G has more than one cut vertex. Then, there exist at least two cut vertices of G that share a common block, which is a non-pendant block. Thus, when $|C(G)| \geq 2$, the number of pendant blocks is at most $k - 1$. Hence the result. $\square$

As the lower bound on the number of pendant blocks gives the upper bound on the number of non-pendant blocks and vice versa, we give the bounds on the number of non-pendant blocks.

Corollary 2.11. *For any graph with $|B(G)| = k$ and $k \geq 2$, we have, $0 \leq \text{number of non-pendant blocks} \leq k - 2$.*

Now, we characterize graphs whose block-degree sequence of cut vertices is $(2)_t$.

Theorem 2.12. *A graph G has t cut vertices and $t + 1$ blocks if and only if the block-degree sequence of cut vertices of G is $(2)_t$.*

Proof. Assume $C(G) = \{c_1, \dots, c_t\}$ and $B(G) = \{b_1, \dots, b_{t+1}\}$. Now, we construct G . As any cut vertex belongs to at least two blocks, assume that c_1 belongs to x blocks, where $x \geq 3$. There are remaining $t - 1$ cut vertices and $t + 1 - x$ blocks. We know that the number of blocks in G is at least one more than the number of cut vertices in G . As $x \geq 3$, $t + 1 - x \leq t - 2$, it leads to a contradiction. Thus, all the vertices of $C(G)$ belong to exactly two blocks. Therefore, the block-degree sequence of the cut vertices of G is $(2)_t$. The converse is easy to see. $\square$

Corollary 2.13. *If the graph G has exactly two pendant blocks, then the block-degree sequence of cut vertices of the graph G is $(2)_t$, where $|C(G)| = t$.*

Proof. Let G be a graph with t cut vertices and exactly two pendant blocks, say, b_1, b_2 . Thus, $t \geq 1$. By Theorem 2.12, it suffices to prove $|B(G)| = t + 1$. We prove the statement “P(t): The graph G with t cut vertices and two pendant blocks has $t + 1$ blocks” by induction. Suppose $t = 1$. Then, all blocks of G are pendant blocks. Thus, by hypothesis, $|B(G)| = 2 = t + 1$. Hence, $P(1)$ is true. Suppose $P(t)$ is true when $t = x$, where $x > 1$. Then, G has $C(G) = \{c_1, \dots, c_x\}$ and $B(G) = \{b_1, \dots, b_{x+1}\}$. Suppose $t = x + 1$. Then, G has $x + 1$ cut vertices. By hypothesis, G with x cut vertices has $x + 1$ blocks. Now, add the vertex c_{x+1} to $C(G)$. The cut vertex c_{x+1} has to belong to any one of the pendant blocks of G say, b_1 , otherwise, there exist at least three pendant blocks, which is a contradiction. As each cut vertex belongs to at least two blocks and G has exactly two pendant blocks, c_{x+1} belongs to exactly two blocks, say, b_1 and b_{x+2} . Therefore, G has $x + 2$ blocks with exactly two pendant blocks namely, b_2 and b_{x+2} . Hence, $P(t)$ is true for any t , $t \geq 1$. Therefore, by Theorem 2.12, the block-degree sequence of cut vertices of G is $(2)_t$. $\square$

The previous theorem gives insight to generalize the existence of a graph with all cut vertices of the same block degree.

Theorem 2.14. *For any positive integers t, p, k , where $t \geq 1$, $p \geq 1$ and $k = pt + 1$, there exists a graph G with t cut vertices, k blocks and $(p + 1)_t$ as its block degree sequence of cut vertices of the graph.*

Proof. Let t, p, k be the positive integers with $t \geq 1, p \geq 1$ and $k = pt + 1$. Let us construct the graph G . Without loss of generality, assume t and k as the number of cut vertices and blocks of the graph G , respectively. We know that the number of blocks in a graph is at least one more than the number of cut vertices in the graph. Thus, when G has t cut vertices and $t + 1$ blocks, by Theorem 2.12, the block-degree sequence of cut vertices of G is $(2)_t$. This implies that each cut vertex of G belongs to two blocks of G . Since the total number of blocks is $pt + 1$ out of which $t + 1$ blocks are present in G , it is left to add $t(p - 1)$ more blocks to G . Add $(p - 1)$ blocks to each cut vertex of G . As each cut vertex already belongs to two blocks and the addition of $(p - 1)$ blocks to a cut vertex makes its block-degree $p + 1$. Therefore, the block-degree sequence of cut vertices of G is $(p + 1)_t$. Hence the result. $\square$

With a given number of cut vertices and blocks of the graph, some properties on the block-degree of the cut vertex and block-degree sequence of cut vertices of the graph are discussed below.

Theorem 2.15. *For any given $|C(G)| = A$ and $|B(G)| = B$, where $B > A \geq 1$, the following hold true.*

- (1) *There exists a graph G with block-degree sequence of cut vertices as $(\beta)_A$ if and only if $\beta = \frac{B+A-1}{A}$.*
- (2) *When $B \leq A + 2$, there exists a unique block-degree sequence.*

- (3) For any graph G , $2 \leq \deg_b(c_j) \leq B - (A - 1)$, where $c_j \in C(G)$ and $1 \leq j \leq t$.
- (4) For any graph G of order n , $n \geq 3$, $\deg_b(c_j) \leq \deg(c_j) \leq n - A$, where $1 \leq j \leq t$.

Proof. (1) Suppose the block-degree sequence of the graph G is $(\beta)_A$. Then, by Theorem 2.7,

$$\begin{aligned} B &= (\beta + \beta + \dots (A \text{ times}) + \beta) - (A - 1) \\ \beta(A) &= B + (A - 1) \\ \beta &= \frac{B + A - 1}{A} \end{aligned}$$

The converse is easy to see.

- (2) Suppose $B \leq A + 2$.
As $B \geq A + 1$, by Theorem 2.12, when $B = A + 1$, the unique block-degree sequence of cut vertices of G is $(2)_A$. Let $B = A + 2$. As G with $A + 1$ blocks has the unique block-degree sequence of cut vertices and the minimum block-degree of any cut vertex is 2, the addition of one block to $A + 1$ blocks gives a unique block-degree sequence of cut vertices as $(3)_1, (2)_{(A-1)}$.
- (3) By Proposition 2.3, the lower bound follows. Given A cut vertices, there exists at least $A + 1$ blocks in G . By Theorem 2.12, with A cut vertices and $A + 1$ blocks of G , every cut vertex of G belongs to two blocks. The remaining $B - (A + 1)$ blocks can contain any cut vertex of G . Thus, a cut vertex of G can belong to at most $2 + B - (A + 1)$. Therefore, for any $c_j \in C(G)$, $\deg_b(c_j) \leq B - (A - 1)$.
- (4) By Proposition 2.3, the lower bound follows. Since for any given B the order of G is at least $B + 1$, the maximum degree of any cut vertex c_j given by,

$$\begin{aligned} \deg(c_j) &\leq \text{maximum}\{\deg_b(c_j) | 1 \leq j \leq t\} + n - (B + 1) \\ &\leq B - (A - 1) + n - (B + 1) \quad (\text{By (3) of Theorem 2.15}) \\ &\leq n - A. \end{aligned}$$

Hence the proof. □

We now define the cut-degree of a block in the graph.

Definition 2.16. The cut-degree of a block in a graph G , written $\deg_c(b_l)$ is the number of cut vertices of G contained in b_l , where $1 \leq l \leq k$. Let us denote $\deg_c(b_l) = \beta'_l$.

Proposition 2.17. For any graph G with $|C(G)| = t$, $b_l \in B(G)$ and $1 \leq l \leq k$, the following statements are true.

- (1) $0 \leq \deg_c(b_l) \leq t$.
- (2) $\deg_c(b_l) = |V(b_l)|$ if and only if b_l is a non-pendant block and all its vertices are cut vertices of G .

Proof. The lower bound attains when the graph G itself is a block, as it has no cut vertex. A block can contain at most all the cut vertices of G in it. Thus, the cut-degree of a block in G is at most k . Hence the result. $\square$

Definition 2.18. The cut-degree sequence of the graph G is a list of cut-degrees of blocks $\beta'_1, \beta'_2, \dots, \beta'_k$ in the non-increasing order. If multiple blocks share the same cut-degree, for example, if s blocks have a cut-degree β' (where $s \leq t$), we denote this by $(\beta')_s$. This notation indicates that there are s blocks each with a cut-degree β' .

Remark 2.19. The following are the observations from the definition of the cut-degree of the graph.

- (1) Since the graph which is not a block has at least two pendant blocks, the cut-degree sequence of the graph has at least two ones, i.e., $\beta'_{k-1} = \beta'_k = 1$.
- (2) Not all non-increasing positive integer sequences are a cut-degree sequence of some graph G .

Example 2.20. In Figure 2, $B(G) = \{B_1, B_2, B_3, B_4, B_5\}$. The cut-degree of the vertices in $B(G)$ are 1, 3, 1, 2 and 1 respectively.

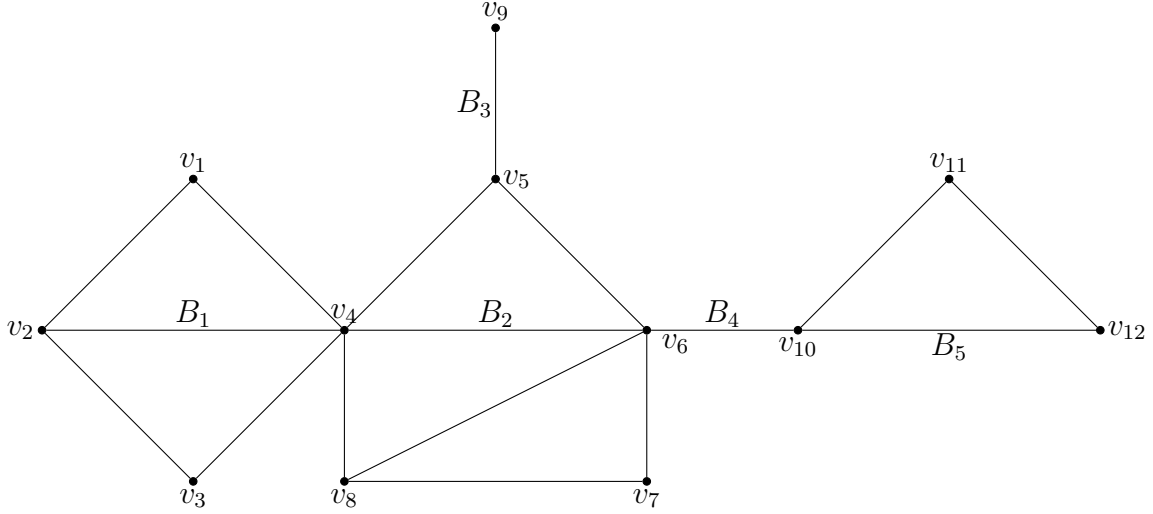


FIGURE 2. A graph G to illustrate cut-degree of blocks

From Remark 2.19 (2), we characterize certain sequences of positive integers which are cut-degree sequences of some graph G .

Proposition 2.21. *The cut-degree sequence of the graph G is $(1)_k$, where $|B(G)| = k$ if and only if G has exactly one cut vertex.*

Proof. Suppose G has exactly one cut vertex. Then, all blocks contain the cut vertex, which implies the cut-degree of all blocks in G is one. Hence the cut-degree sequence of G is $(1)_k$. Conversely, suppose the cut-degree sequence of the graph is $(1)_k$. Then, every block contains exactly one cut vertex. Thus, all blocks are pendant blocks. Therefore, G has exactly one cut vertex. Hence the result. $\square$

Given the cut-degree sequence of a graph, the lower bound on the number of pendant blocks of the graph is determined in the following theorem.

Theorem 2.22. *Let $S' : (\beta')_r, (1)_s$ be a sequence of non-increasing positive integers, where $\beta' \geq 2, r \geq 1, s \geq 2$. If S' is the cut degree sequence of some graph G , then $s \geq r(\beta' - 2) + 2$.*

Proof. Consider $S' : (\beta')_r, (1)_s$ to be a non-increasing sequence of positive integers. Let S' be the cut-degree sequence of the graph G . Then G has $r + s$ blocks in which r are non-pendant blocks each of cut-degree β' and s are pendant blocks each of cut-degree 1. We prove $s \geq r(\beta' - 2) + 2$ by induction on r . Let the statement be “P(r): If the graph G has $S' : (\beta')_r, (1)_s$ as its cut-degree sequence, then $s \geq r(\beta' - 2) + 2$ ”.

Suppose $r = 1$. Then, $S' : (\beta')_1, (1)_s$. This implies that the graph has exactly one non-pendant block containing β' cut vertices of G and s pendant blocks. Since the non-pendant block contains β' cut vertices, the non-pendant block shares a common vertex with at least β' pendant blocks. Therefore, $s \geq \beta'$. Hence, $P(1)$ is true.

Assume $P(r = x)$ is true, where $x > 1$. This implies that $S' : (\beta')_x, (1)_s$ is the cut-degree sequence of the graph G such that $s \geq x(\beta' - 2) + 2$.

Suppose $r = x + 1$. Then, cut-degree sequence of the graph is $S' : (\beta')_{x+1}, (1)_s$. This implies that the graph has $x + 1$ non-pendant blocks with each of its cut-degree being β' and let the graph be G_1 . Now, we construct G_1 from G . By hypothesis, the graph G with x non-pendant blocks has at least $x(\beta' - 2) + 2$ pendant blocks. Adding blocks to a non-pendant block of G does not increase the number of non-pendant blocks; hence, G_1 is not formed. Thus, adding blocks to a pendant block of G gives G_1 . By hypothesis, every non-pendant block of G_1 is of cut-degree β' . Further, any pendant block of G contains exactly one cut vertex of G . Thus, adding at least $\beta' - 1$ non-adjacent blocks sharing a common vertex with a pendant block of G gives G_1 . Now, we consider the following possibilities based on the order of pendant blocks in G .

Case 1. Suppose G has a pendant block of order at least β' . Then, the pendant block, say b_y , has at least β' vertices out of which one is a cut vertex of G . Add at least one block to the remaining $\beta' - 1$ vertices of b_y which makes b_y a non-pendant block with cut-degree β' and hence the graph G_1 is formed.

Case 2. Suppose G has no pendant block of order at least β' . Then, all pendant blocks of G are of the order less than β' . To make a pendant block as a non-pendant block with cut-degree β' , there should be at least β' vertices in the pendant block. Thus, consider a pendant block in G , say b_z , which has maximum order among all the pendant blocks of G , say $\alpha < \beta'$. Now, perform the edge subdivision operation on an (the) edge (edges) of b_z until the block b_z has β' vertices. This implies that $\beta' - \alpha$ times the edge subdivision operation is performed on an (the) edge (edges) of the block b_z to make its order β' . Now, the pendant block b_z is of order β' . Similar to Case 1, adding at least one block to $\beta' - 1$ vertices of the block b_z makes b_z a non-pendant block with cut-degree β' and hence G_1 is formed.

Concluding both cases, the number of pendant blocks in G_1 is the sum of the number of newly added blocks and one less than the number of pendant blocks in G as one of the pendant blocks of G has become the non-pendant block in G_1 . Therefore in G_1 , the number of pendant blocks is,

$$\begin{aligned} s &\geq (\beta' - 1) + x(\beta' - 2) + 1 \\ &\geq (\beta' - 2) + x(\beta' - 2) + 2 \\ s &\geq (x + 1)(\beta' - 2) + 2 \end{aligned}$$

Hence, $S' : (\beta')_{x+1}, (1)_s$ is the cut-degree sequence of the graph such that $s \geq (x + 1)(\beta' - 2) + 2$. Therefore, $P(r)$ is true when $r = x + 1$. Hence the statement $P(r)$ is true for any $r \geq 1$. Hence the result. $\square$

For the given cut-degree sequence of the graph G , the bounds on the order, size and the number of cut vertices of G are discussed below.

Theorem 2.23. *Let $\beta'_1, \beta'_2, \dots, \beta'_k$ be the cut-degree sequence of the graph G , where $|B(G)| = k$. The following statements are true.*

- (1) $|V(G)| \geq 1 + m + \sum_{l=1}^{k-m} (\beta'_l - 1)$, where m is the number of pendant blocks.
- (2) $|E(G)| \geq p + \sum_{l=1}^{k-p} \beta'_l$, where p is the number of blocks with cut-degree at most 2.
- (3) $|C(G)| = 1 - k + \sum_{l=1}^k \beta'_l$.

Proof. Let β'_l be the cut-degree of the block b_l , where $1 \leq l \leq k$.

- (1) Each b_l block has at least β'_l vertices. Let m be the number of pendant blocks in G . Clearly, $m \geq 2$. Thus, there are m blocks with cut-degree 1. Suppose $m = k$, then G has exactly one cut vertex. By Proposition 2.21, the cut-degree sequence of G is $(1)_k$. Thus, $|V(G)| \geq 1 + m$. Suppose $m < k$. Then, there are $k - m$ non-pendant blocks in G . As the minimum order of any block is 2 and the cut-degree of a non-pendant block is at least 2, the order of each non-pendant block is at least its cut-degree and the minimum order of a pendant block is 2. As any two blocks share at most one cut vertex, these $k - m$ blocks contribute at least $\beta'_1 + (\beta'_2 - 1) + \dots + (\beta'_{k-m} - 1)$ to the order of the graph. Further, m pendant blocks share a common vertex with some $k - m$ non-pendant blocks. As pendant blocks contain exactly one cut vertex and the minimum order of a pendant block is 2, m pendant blocks contribute m to the order of G . Thus, the minimum possible order of G is $\beta'_1 + (\beta'_2 - 1) + \dots + (\beta'_{k-m} - 1) + m$. In both cases, $|V(G)| \geq 1 + m + \sum_{l=1}^{k-m} (\beta'_l - 1)$.
- (2) The minimum size of any block is 1 as the smallest order block is K_2 . Let p be the number of blocks with cut-degree at most 2. Then there are $k - p$ blocks with cut-degree at least 3. Let those blocks be $b_1, \dots, b_{k-p}$. Then each block b_q has at least three vertices, where $1 \leq q \leq k - p$. By Theorem 1.3, in any block b_q , every two vertices lie in a common cycle. Thus, the size of any b_q block is at least its cut-degree. Thus, these $k - p$ blocks contribute at least $\beta'_1 + \beta'_2 + \dots + \beta'_{k-p}$ to the size of the graph. Further, p blocks with cut-degree at most 2 contribute at least p to the size of the

graph. As every block of G is edge-disjoint, the minimum possible size of G is $\beta'_1 + \beta'_2 + \dots + \beta'_{k-p} + p$. Therefore, $|E(G)| \geq p + \sum_{l=1}^{k-p} \beta'_l$.

- (3) As any two blocks share at most one cut vertex, the number of cut vertices in G is $\beta'_1 + (\beta'_2 - 1) + \dots + (\beta'_k - 1)$. Therefore, $|C(G)| = 1 - k + \sum_{l=1}^k \beta'_l$. $\square$

3. BLOCK GRAPH OF A GRAPH

Some basic properties of block graph of a graph are presented here. As every block in $\mathfrak{B}(G)$ is complete, we have the following results.

Proposition 3.1. *For any graph G with $|C(G)| \geq 1$, the following statements are true.*

- (1) $\mathfrak{B}(G)$ is complete if and only if G has exactly one cut vertex.
- (2) $\mathfrak{B}(G)$ is cyclic if and only if G has at least one cut vertex with block-degree at least 3.
- (3) $\mathfrak{B}(G)$ is unicyclic if and only if exactly one vertex of $C(G)$ has block-degree 3 and the remaining vertices of $C(G)$ with block-degree less than 3.
- (4) $\mathfrak{B}(G)$ is acyclic if and only if all the vertices of $C(G)$ has block-degree less than 3.
- (5) The number of blocks in $\mathfrak{B}(G)$ is the number of cut vertices in G and the order of each block is a block-degree of a cut vertex in G .

Proof. The results (1) – (4) follow by the adjacency of vertices in the block graph of a graph G , that is, two vertices in $\mathfrak{B}(G)$ are adjacent only if the corresponding blocks in G share a cut vertex. All vertices in a block in $\mathfrak{B}(G)$ are blocks in G which share a cut vertex. Thus, every block in $\mathfrak{B}(G)$ represents a cut vertex in the corresponding graph G . The number of vertices in each block in $\mathfrak{B}(G)$ is the number of blocks sharing a cut vertex in G . Thus, the order of each block in $\mathfrak{B}(G)$ is the block-degree of a cut vertex in G . Hence the result. $\square$

The following theorem gives the number of cut vertices in $\mathfrak{B}(G)$ in terms of the blocks of G .

Theorem 3.2. *The number of cut vertices in $\mathfrak{B}(G)$ is the number of non-pendant blocks in G .*

Proof. Let $b_z \in B(G)$ be a non-pendant block in G . Consider two non-adjacent blocks, say b_x and b_y , which belong to different components when $G - V(b_z)$. Then, for $u \in b_x, v \in b_y$, any $(u - v)$ path contains vertices of the block b_z . In $\mathfrak{B}(G)$, the vertex b_z lies in $(b_x - b_y)$ path, where b_x and b_y are in different components of $G - \{b_z\}$. Therefore, by Theorem 1.2, b_z is a cut vertex in $\mathfrak{B}(G)$. Thus, in general, we conclude that all non-pendant blocks of G are the cut vertices in $\mathfrak{B}(G)$. It suffices to establish that pendant blocks in G do not act as cut vertices in $\mathfrak{B}(G)$. On the contrary, consider a pendant block in G say b_p , which is a cut vertex in $\mathfrak{B}(G)$. Since b_p is a pendant block in G , there exists exactly one cut vertex contained in b_p . As b_p is a cut vertex in $\mathfrak{B}(G)$, by Theorem 1.2, in $\mathfrak{B}(G)$, every $(b_e - b_f)$ path has the vertex b_p , where $b_e, b_f \in V(\mathfrak{B}(G))$ and they belong

to different components in $\mathfrak{B}(G) - \{b_p\}$. Thus, the blocks b_e, b_f are non-adjacent blocks which belong to different components when $G - V(b_p)$. In addition every $(r - s)$ path in G contains the vertices of the block b_p , where $r \in b_e, s \in b_f$, which is a contradiction because when b_e, b_f are not adjacent blocks of b_p . Therefore, no pendant blocks of G are cut vertices in $\mathfrak{B}(G)$. Hence, the number of cut vertices in $\mathfrak{B}(G)$ is the number of non-pendant blocks in G . $\square$

The bounds on the number of cut vertices of the block graph of a graph are given below.

Corollary 3.3. *For any graph G , $0 \leq |C(\mathfrak{B}(G))| \leq |C(G)| - 1$.*

Proof. By Theorem 3.2, $|C(\mathfrak{B}(G))|$ is the number of non-pendant blocks in G . By corollary 2.11, $0 \leq |C(\mathfrak{B}(G))| \leq k - 2$, where $|B(G)| = k$. Thus, when $|C(\mathfrak{B}(G))| = k - 2$, G has exactly two pendant blocks. By Theorem 2.12 and Corollary 2.13, $|C(G)| = k - 1$. Thus, $|C(\mathfrak{B}(G))| \leq |C(G)| - 1$. $\square$

The graphs for which its $\mathfrak{B}(G)$ is a tree are characterized.

Theorem 3.4. *For any graph G with $|C(G)| = t$ and $t \geq 1$, $\mathfrak{B}(G)$ is a tree if and only if the block-degree sequence of cut vertices of G is $(2)_t$.*

Proof. Let G be a graph with t cut vertices. Suppose $\mathfrak{B}(G)$ is a tree. By Proposition 3.1, the order of each block in $\mathfrak{B}(G)$ is the block-degree of a cut vertex in G . As every block in $\mathfrak{B}(G)$ is an edge, the block-degree of every cut vertex in G is 2. Hence the block-degree sequence of cut vertices of G is $(2)_t$. Conversely, suppose the block-degree of all vertices in G is 2. By Theorem 2.12, G has t cut vertices and $t + 1$ blocks. By Proposition 3.1, $\mathfrak{B}(G)$ has t blocks. Thus, the graph $\mathfrak{B}(G)$ with order $t + 1$ and t blocks is a tree. Hence the result. $\square$

The term traversability predominantly refers to the Eulerian and Hamiltonian properties of the graph. We discuss the traversability of $\mathfrak{B}(G)$.

Proposition 3.5. *For any graph G with $|C(G)| \geq 1$, $\mathfrak{B}(G)$ is Hamiltonian if and only if G has exactly one cut vertex.*

Proof. Suppose G has exactly one cut vertex. Then, all blocks of G share the same common vertex. Thus, in $\mathfrak{B}(G)$, every vertex is connected to every other vertex, forming a complete graph. Therefore, $\mathfrak{B}(G)$ is Hamiltonian. Conversely, suppose $\mathfrak{B}(G)$ is Hamiltonian. We know that any graph which has a cut vertex is not Hamiltonian which implies, $\mathfrak{B}(G)$ is a block. Thus, by Proposition 3.1, $\mathfrak{B}(G)$ is a block if and only if G has exactly one cut vertex. Hence the result. $\square$

Theorem 3.6. *For any graph G with $|C(G)| \geq 1$, $\mathfrak{B}(G)$ has a Hamiltonian path if G satisfies any one of the following:*

- (1) G has exactly one cut vertex.
- (2) All non pendant blocks of G are of cut-degree 2.

Proof. Suppose (1) is true. Then, by Proposition 3.5, $\mathfrak{B}(G)$ is Hamiltonian. Therefore, $\mathfrak{B}(G)$ has Hamiltonian path.

Suppose (2) is true. Then, by Theorem 3.2, as all non-pendant blocks of G are cut vertices in $\mathfrak{B}(G)$, all cut vertices of $\mathfrak{B}(G)$ belongs to exactly two blocks. As each block in $\mathfrak{B}(G)$ is complete and the block-degree of all cut vertices being 2, $\mathfrak{B}(G)$ has a Hamiltonian path. $\square$

Now, we classify graphs based on whether their block graphs are Eulerian.

Theorem 3.7. *$\mathfrak{B}(G)$ is Eulerian if and only if the block-degrees of all cut vertices of G are odd.*

Proof. Let G be a graph with t cut vertices. Suppose $\mathfrak{B}(G)$ is Eulerian. Then, each block in $\mathfrak{B}(G)$ is Eulerian. By Proposition 3.1 (5), the number of blocks in $\mathfrak{B}(G)$ is the number of cut vertices in G . Thus, B_j ($1 \leq j \leq t$) are the blocks in $\mathfrak{B}(G)$. Thus, for given j and any l , $\deg_{B_j}(b_l)$ is even. As each block is complete with all vertices of each block are of even degree with respect to that block, the order of each block is odd. Each block of odd order in $\mathfrak{B}(G)$ implies that every cut vertex belongs to an odd number of blocks in G . Thus, the block-degree of all cut vertices in G is odd. As the block-degree of any cut vertex is at least 2, the block degree of all cut vertices in G is $2a + 1$, for any $a \geq 1$. The converse is easy to see. $\square$

Now, we give a necessary condition for graphs whose $\mathfrak{B}(G)$ has no Eulerian trail.

Theorem 3.8. *If the block-degree of cut vertices of G are even, then $\mathfrak{B}(G)$ has no eulerian trail.*

Proof. Consider a graph G with t cut vertices such that their block-degrees are even. By Proposition 3.1, $B(\mathfrak{B}(G)) = \{B_j | 1 \leq j \leq t\}$. Let G have a cut vertex with block-degree $2a$, $a \geq 2$. In $\mathfrak{B}(G)$, there exists a block, say, B_x with $2a$ vertices. Let $b_1, \dots, b_{2a}$ be the vertices of B_x in $\mathfrak{B}(G)$. As each block is complete in $\mathfrak{B}(G)$, $\deg_{B_x}(b_q)$ is odd, where $1 \leq q \leq 2a$. Assume $\mathfrak{B}(G)$ has an Eulerian trail. By Theorem 1.5, $\mathfrak{B}(G)$ has exactly two vertices of odd degree. We consider the following cases:

Case 1. Suppose all vertices of B_x are not cut vertices of $\mathfrak{B}(G)$.

Then, for all q , $\deg_{\mathfrak{B}(G)}(b_i)$ is odd. As the order of B_x is $2a$, $\mathfrak{B}(G)$ has at least four vertices of odd degree, which is a contradiction.

Case 2. Suppose all vertices of B_x are cut vertices of $\mathfrak{B}(G)$.

Then, all vertices of B_x belong to at least one block other than B_x . We consider the following possibility on the degree of vertices of B_x .

Suppose exactly two vertices of B_x are of odd degree, say, b_1 and b_2 . Then, $b_3, \dots, b_{2a}$ are of even degree. Thus, these vertices belong to an even number of blocks with even order and any number of blocks with odd order. Consider $b_3 \in B_y$, where $|V(B_y)|$ is even. Thus, the degree of all the vertices of B_y with respect to B_y is odd. All the vertices of $V(B_y)$ are cut vertices, otherwise, their degree will be odd which leads to a contradiction. Similar to even degree vertices of the block B_x , all vertices of $V(B_y)$ belong to an even number of blocks with even order. Continuing this way, there exists at least $2a$ vertices of odd degree,

which is a contradiction.

Suppose no vertex of $V(B_x)$ is of odd degree. Then, the number of even order blocks containing b_1 or b_2 is even which is similar to the number of even order blocks containing any of $b_3, \dots, b_{2a}$ in the previous case. Thus, it leads to a contradiction.

The above cases conclude that $\mathfrak{B}(G)$ has no Eulerian trail. Hence the result. $\square$

4. BLOCK CUT POINT GRAPH OF A GRAPH

The properties of the block cut point graph of a graph are studied. Note that, every $BC(G)$ is acyclic.

Proposition 4.1. *For any graph G with $|C(G)| \geq 1$, $BC(G) \not\cong K_2$.*

Proof. As a cut vertex of G belongs to at least two blocks, $BC(G)$ has at least three vertices. Since any $BC(G)$ is acyclic, the result follows. $\square$

The characterization of the graphs isomorphic to $BC(G)$ is discussed below.

Theorem 4.2. *Let G be a graph of order n , $n \geq 3$. Then, $G \cong BC(G)$ if and only if $G \cong K_{1,n-1}$.*

Proof. Suppose $G \cong BC(G)$ with order n , $n \geq 3$. As every $BC(G)$ is acyclic, G is acyclic. If G has t ($t \geq 1$) cut vertices, then G has at least $t + 1$ blocks. As any connected graph with at least three vertices has at least two non-cut vertices, the order of the graph is at least $t + 2$. Thus, the order of $BC(G)$ is at least $2t + 1$. The equality $2t + 1 = t + 2$ is attained only when $t = 1$. The only acyclic graph G of order n with one cut vertex is $K_{1,n-1}$. Therefore, $G \cong K_{1,n-1}$.

Conversely, suppose $G \cong K_{1,n-1}$, $n \geq 3$. Then, G has $n - 1$ blocks and one cut vertex, say v . In $BC(G)$, $n - 1$ block vertices are adjacent to v . Clearly, $BC(G) \cong K_{1,n-1}$. Thus, $G \cong BC(G)$.

Hence the result. $\square$

Motivated by the definition of the iterated line graphs, we define iterated block cut point graphs as follows.

Definition 4.3. For any positive integer p , the iterated block cut point graph $BC^p(G)$ is defined recursively by $BC^p(G) \cong BC(BC^{p-1}(G))$.

Theorem 4.4. *For $p \geq 1$, $BC^p(G) \cong BC^{p+1}(G)$ if and only if G has exactly one cut vertex.*

Proof. Suppose G has exactly one cut vertex say, c . Then in $BC(G)$, c is adjacent to all the block vertices forming $K_{1,n}$, $n \geq 2$. Let $BC(G)$ be some graph, say G_1 . As G_1 has exactly one cut vertex, $BC(G_1) \cong K_{1,n}$. Thus, we have, $BC(G) \cong BC^2(G) \cong K_{1,n}$. Therefore, in general, for any p , $p \geq 1$, $BC^p(G) \cong BC^{p+1}(G)$. The converse is easy to see. Hence the result. $\square$

The properties on graphs whose $BC(G)$ is not a full m -ary tree are given below.

Theorem 4.5. *Let G be a graph with t cut vertices and k blocks. $BC(G)$ is not a full m -ary tree if any one of the following conditions is satisfied:*

- (1) *$BC(G)$ is a rooted tree of odd height with the root belonging to $B(G)$.*
- (2) *$BC(G)$ is a rooted tree of even height with the root belonging to $C(G)$.*

Proof. Suppose (1) is true. Note that any path in $BC(G)$ has an alternating sequence of vertices $C(G)$ and $B(G)$. Thus, the vertices of $B(G)$ lie in the levels $2p$, $p \geq 0$. As all the leaves in $BC(G)$ belong to $B(G)$, the height of $BC(G)$ has to be even, which is a contradiction. Hence, $BC(G)$ is not a full m -ary tree.

The proof of (2) is similar to the proof of (1). Hence the result. $\square$

A perfect m -ary tree is a full m -ary tree in which all the leaves are at the same height. we establish a characterization on graphs whose $BC(G)$ is a perfect m -ary tree.

Theorem 4.6. *$BC(G)$ is a perfect m -ary tree of height h if and only if it satisfies one of the following.*

- (1) (a) $|B(G)| = \sum_{p=0}^{\frac{h}{2}} m^{2p}$ in which one block has cut-degree m , m^h blocks have cut-degree 1 and the remaining $\sum_{p=1}^{\frac{h-2}{2}} m^{2p}$ blocks have cut-degree $m+1$.
 (b) *The root of $BC(G)$ belongs to $B(G)$.*
- (2) (a) $|C(G)| = \sum_{p=0}^{\frac{h-1}{2}} m^{2p}$ in which one vertex has block-degree m , the remaining $\sum_{p=1}^{\frac{h-1}{2}} m^{2p-1}$ cut vertices have block-degree $m+1$ and
 $|B(G)| = \sum_{p=1}^{\frac{h+1}{2}} m^{2p-1}$ in which m^h blocks have cut -degree 1.
 (b) *The root of $BC(G)$ belongs to $C(G)$.*

Proof. Assume that statement (1) is true. Let the block with a cut-degree m be represented by b_1 , which is also the root of $BC(G)$. The vertices of $B(G)$ belong to levels $2p$ in $BC(G)$, where $0 \leq p \leq \frac{h}{2}$. There are m^h blocks in G with cut-degree 1, which are the leaves in $BC(G)$. The remaining blocks in G with cut-degree $m+1$ are internal vertices of $BC(G)$ with degree $m+1$. In $BC(G)$, the vertices of $C(G)$ are adjacent only to the vertices of $B(G)$ and none of them are leaves. Hence, in $BC(G)$, the vertices of $C(G)$ lie in the levels $2p-1$, $1 \leq p \leq \frac{h}{2}$ which are also internal vertices with degree $m+1$. Therefore, $BC(G)$ has a root with degree m , $\sum_{p=1}^{\frac{h-1}{2}} m^p$ internal vertices with degree $m+1$ and m^h leaves. Consequently, $BC(G)$ is a perfect m -ary tree. In addition, as the root and leaves of $BC(G)$ belong to $B(G)$, the height of $BC(G)$ is even.

Assume that statement (2) is true. Without loss of generality, let $c_x \in C(G)$ be the cut vertex with block-degree m and the root of $BC(G)$. The remaining cut vertices of G lie in the levels $2p$ in $BC(G)$, where $1 \leq p \leq \frac{h-1}{2}$ with degree $m+1$ which are internal vertices. The m^h blocks in G with cut-degree 1 are leaves in $BC(G)$. As the cut vertices of G are adjacent only to the block vertices of G , in $BC(G)$, the remaining $\sum_{p=1}^{\frac{h-1}{2}} m^{2p-1}$ blocks of G are internal vertices of $BC(G)$ with degree $m+1$. Therefore, $BC(G)$ is a rooted tree with degree of its root being m , $\sum_{p=1}^{\frac{h-1}{2}} m^p$ internal vertices of degree $m+1$ and m^h leaves. Consequently,

$BC(G)$ is a perfect m -ary tree. Moreover, in $BC(G)$, as the root belongs to $C(G)$ and leaves belong to $B(G)$, its height is odd.

Conversely, suppose $BC(G)$ is a perfect m -ary tree with height h . Then, the degree of the root, internal vertices and leaves of $BC(G)$ are $m, m+1$, and m , respectively. As $BC(G)$ has m^h leaves and all leaves belong to the vertices of $B(G)$, m^h vertices of $B(G)$ are of degree 1. Since any path in $BC(G)$ has an alternating sequence of vertices of $C(G)$ and $B(G)$, we consider the following cases based on the height of $BC(G)$.

Case 1. Suppose h is even.

Then, in $BC(G)$ the vertices of $B(G)$ lie in the levels $2p$, where $0 \leq p \leq \frac{h}{2}$ and the cut vertices of G lie in the level $2p-1$, where $1 \leq p \leq \frac{h}{2}$. The number of vertices of $B(G)$ and $C(G)$ in $BC(G)$ are $\sum_{p=0}^{\frac{h}{2}} m^{2p}$ and $\sum_{p=1}^{\frac{h}{2}} m^{2p-1}$, respectively. Thus, the root of $BC(G)$ belongs to $B(G)$. The number of vertices of $B(G)$ other than the root and leaves of $BC(G)$ is $\sum_{p=1}^{\frac{h-2}{2}} m^{2p}$ which are the internal vertices of $BC(G)$. Consequently, all the cut vertices of G are internal vertices of $BC(G)$. As the degrees of vertices in $BC(G)$ are the cut-degrees (block-degrees) of the blocks (cut vertices) in G , respectively, condition (1) follows.

Case 2. Suppose h is odd.

Then, in $BC(G)$ the vertices of $B(G)$ lie in the level $2p-1$, where $1 \leq p \leq \frac{h+1}{2}$ and the cut vertices of G lie in the level $2p$, where $0 \leq p \leq \frac{h-1}{2}$. The number of vertices of $B(G)$ and $C(G)$ in $BC(G)$ are $\sum_{p=1}^{\frac{h+1}{2}} m^{2p-1}$ and $\sum_{p=0}^{\frac{h-1}{2}} m^{2p}$, respectively. Thus, the root of $BC(G)$ belongs to $C(G)$. As m^h vertices of $B(G)$ are leaves of $BC(G)$, the remaining $\sum_{p=1}^{\frac{h-1}{2}} m^{2p-1}$ vertices of $B(G)$ and $\sum_{p=1}^{\frac{h-1}{2}} m^{2p}$ vertices of $C(G)$ are internal vertices of $BC(G)$. As the cut-degrees (block-degrees) of blocks (cut vertices) in G are the degree of vertices in $BC(G)$, the condition (2) follows.

Hence the result. □

5. CONCLUSION

This study introduces two terminologies based on the block and cut vertex of the graph namely, the block-degree of a vertex and the cut-degree of a block. The properties of the block-degree of a cut vertex in the graph are explored and a few characterizations are established. Given the number of cut vertices and blocks of G , properties of the block-degree and block-degree sequence of cut vertices of the graph are analyzed. The study on the properties of cut-degree of a block and cut-degree sequence of G is carried out. The study on these parameters provides further insights in facilitating the study of two block-related derived graphs namely, the block graph of a graph and the block cut point of a graph. The characterization of graphs whose $\mathfrak{B}(G)$ is a tree is established. The Hamiltonicity and Eulerian properties of $\mathfrak{B}(G)$ are investigated. The graphs whose $BC(G)$ is perfect m -ary tree are characterized. Further, the structural properties of G whose $BC(G)$ is not a full m -ary tree are investigated.

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