

MERTENS'S FORMULA, k -FULL NUMBERS AND SUM OF TWO SQUARES

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ABSTRACT. In this article we prove generalizations of the well-known Mertens's formula. We also define some arithmetic functions related to the exponents of the primes in the prime factorization of the positive integers and study its average. Finally, we study square-free numbers, s -full numbers and perfect powers not exceeding x which can be represented as sums of two squares.

1. GENERALIZATIONS OF MERTENS'S FORMULA

The Mertens's formula [8] is stated as follow:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{e^{a\sqrt{\log x}}}\right)\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x}\right)\right), \quad (1.1)$$

where a is a positive constant, m is an arbitrary positive integer and p denotes a positive prime. Williams [9] generalized this formula for arithmetic progressions.

As usual, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, denotes the Riemann zeta function.

We begin with the following generalizations.

Theorem 1.1. *Let $k \geq 2$ a positive integer. The following asymptotic formula holds*

$$\prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{k-1}}\right) = \frac{e^{\gamma}}{\zeta(k)} \log x \left(1 + O\left(\frac{1}{\log^m x}\right)\right). \quad (1.2)$$

If $k \geq 2$ is even then the following asymptotic formula holds

$$\begin{aligned} & \prod_{p \leq x} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + (-1)^{k-1} \frac{1}{p^{k-1}}\right) \\ &= \frac{\zeta(2)}{\zeta(k)} \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x}\right)\right). \end{aligned} \quad (1.3)$$

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If $k \geq 3$ is odd then the following asymptotic formula holds

$$\begin{aligned} & \prod_{p \leq x} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + (-1)^{k-1} \frac{1}{p^{k-1}} \right) \\ &= \frac{\zeta(2)\zeta(k)}{\zeta(2k)} \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x}\right) \right). \end{aligned} \quad (1.4)$$

Proof. We have (sum of a geometric progression)

$$\begin{aligned} & \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{k-1}} \right) = \prod_{p \leq x} \left(\frac{1 - \frac{1}{p^k}}{1 - \frac{1}{p}} \right) = \frac{\prod_{p \leq x} \left(1 - \frac{1}{p^k} \right)}{\prod_{p \leq x} \left(1 - \frac{1}{p} \right)} \\ &= \frac{\prod_p \left(1 - \frac{1}{p^k} \right)}{\prod_{p > x} \left(1 - \frac{1}{p^k} \right) \prod_{p \leq x} \left(1 - \frac{1}{p} \right)} = \frac{e^\gamma}{\zeta(k)} \log x \left(1 + O\left(\frac{1}{\log^m x}\right) \right). \end{aligned}$$

Since (see (1.1))

$$\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} = \frac{1}{\frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x}\right) \right)} = e^\gamma \log x \left(1 + O\left(\frac{1}{\log^m x}\right) \right)$$

and $((-\log(1-x) \sim x) \text{ as } x \rightarrow 0)$

$$\sum_{p > x} \left(-\log \left(1 - \frac{1}{p^k} \right) \right) \leq 2 \sum_{p > x} \frac{1}{p^k} \leq 2 \sum_{p > x} \frac{1}{p^2} \leq \frac{2}{x},$$

that is,

$$\sum_{p > x} \left(\log \left(1 - \frac{1}{p^k} \right) \right) = O\left(\frac{1}{x}\right),$$

that is,

$$\prod_{p > x} \left(1 - \frac{1}{p^k} \right) = e^{O(\frac{1}{x})} = 1 + O\left(\frac{1}{x}\right).$$

Therefore equation (1.2) is proved.

Analogously if k is even we have (sum of a geometric progression)

$$\begin{aligned} & \prod_{p \leq x} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + (-1)^{k-1} \frac{1}{p^{k-1}} \right) = \prod_{p \leq x} \left(\frac{1 - \frac{1}{p^k}}{1 + \frac{1}{p}} \right) = \frac{\prod_{p \leq x} \left(1 - \frac{1}{p^k} \right)}{\prod_{p \leq x} \left(1 + \frac{1}{p} \right)} \\ &= \frac{\zeta(2)}{\zeta(k)} \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x}\right) \right) \end{aligned}$$

and if k is odd we have

$$\begin{aligned} \prod_{p \leq x} \left(1 - \frac{1}{p} + \frac{1}{p^2} - \cdots + (-1)^{k-1} \frac{1}{p^{k-1}} \right) &= \prod_{p \leq x} \left(\frac{1 + \frac{1}{p^k}}{1 + \frac{1}{p}} \right) = \frac{\prod_{p \leq x} \left(1 + \frac{1}{p^k} \right)}{\prod_{p \leq x} \left(1 + \frac{1}{p} \right)} \\ &= \frac{\prod_{p \leq x} \left(1 - \frac{1}{p^{2k}} \right)}{\prod_{p \leq x} \left(1 - \frac{1}{p^k} \right) \prod_{p \leq x} \left(1 + \frac{1}{p} \right)} = \frac{\zeta(2)\zeta(k)}{\zeta(2k)} \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log^m x} \right) \right). \end{aligned}$$

Therefore equations (1.3) and (1.4) are proved. The theorem is proved. $\square$

Now, we generalize Theorem 1.1 We have the following theorem.

Theorem 1.2. *Let k be an arbitrary fixed positive integer. Let a_i ($i = 1, 2, \dots, k$) be real numbers and $a_1 \neq 0$. Then the following asymptotic formula holds*

$$\prod_{p \leq x} \left(1 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) = C(\log x)^{a_1} \left(1 + O\left(\frac{1}{\log^m x} \right) \right), \quad (1.5)$$

where C is a positive constant and its value is

$$\begin{aligned} C &= \exp \left(a_1 M + \sum_p \left(\frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \right. \\ &\quad \left. + \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_p \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right)^i \right) \\ &= \exp \left(a_1 M + \sum_p \left(\frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \right. \\ &\quad \left. + \sum_p \left(\log \left(1 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) - \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \right) \right), \quad (1.6) \end{aligned}$$

where $M = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$ is the Mertens's constant [2] and γ is Euler's constant [2].

Proof. We have the well-known logarithmic power series

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad (|x| < 1).$$

Note that there exists a prime $p' \geq 2$ such that if $p \geq p'$ then

$$\left| \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right| < 1,$$

since $\left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \sim \frac{a_1}{p}$ as $p \rightarrow \infty$. Without loss generality and for sake of simplicity we suppose in the theorem and in this proof that $p' = 2$. Therefore

we have

$$\begin{aligned} \sum_{p \leq x} \log \left(1 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) &= \sum_{p \leq x} \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \\ &+ \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_{p \leq x} \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right)^i. \end{aligned} \quad (1.7)$$

On the other hand, we have the well-known equation [8]

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + O \left(\frac{1}{e^{a\sqrt{\log x}}} \right) = \log \log x + M + O \left(\frac{1}{\log^m x} \right).$$

where a is a positive constant, $M = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$ is Mertens's constant and m is an arbitrary positive integer.

Now, we put

$$\begin{aligned} \log C &= a_1 M + \sum_p \left(\frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right) \\ &+ \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_p \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right)^i \end{aligned} \quad (1.8)$$

This definition of $\log C$ is correct because the series in the right hand converges since it is absolutely convergent as it is showed by the following inequalities

$$\begin{aligned} &\sum_{i=2}^{\infty} \frac{1}{i} \sum_p \left| \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right)^i \right| \leq \sum_{i=2}^{\infty} \sum_p \left(\left| \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right| \right)^i \\ &\leq \sum_{i=2}^{\infty} \sum_p \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^i = \sum_p \sum_{i=2}^{\infty} \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^i \\ &= \sum_p \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^2 \frac{1}{1 - \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)}, \end{aligned} \quad (1.9)$$

where clearly the last series converges and consequently the series that defines $\log C$ is absolutely convergent.

Now, the absolute value of the error between (1.7) and (1.8) is $O\left(\frac{1}{x}\right)$, since (compare with (1.9))

$$\begin{aligned} & \sum_{i=2}^{\infty} \frac{1}{i} \sum_{p>x} \left| \left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right)^i \right| \leq \sum_{i=2}^{\infty} \sum_{p>x} \left(\left| \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} \right| \right)^i \\ & \leq \sum_{i=2}^{\infty} \sum_{p>x} \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^i = \sum_{p>x} \sum_{i=2}^{\infty} \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^i \\ & = \sum_{p>x} \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)^2 \frac{1}{1 - \left(\frac{\sum_{j=1}^k |a_j|}{p} \right)} \leq c \frac{1}{x}, \end{aligned}$$

where c is a positive constant. The theorem is proved. $\square$

A more general theorem than Theorem 1.2 can be established, namely

Theorem 1.3. *Let k be an arbitrary fixed positive integer. Let a_i ($i = 1, 2, \dots, k$) be real numbers and $a_1 \neq 0$. Let β_i ($i = 2, \dots, k$) be positive real numbers such that $1 < \beta_2 < \beta_3 < \cdots < \beta_k$. Then the following asymptotic formula holds*

$$\prod_{p \leq x} \left(1 + \frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \cdots + \frac{a_k}{p^{\beta_k}} \right) = C(\log x)^{a_1} \left(1 + O\left(\frac{1}{\log^m x} \right) \right), \quad (1.10)$$

where C is a positive constant and its value is

$$\begin{aligned} C = & \exp \left(a_1 M + \sum_p \left(\frac{a_2}{p^{\beta_2}} + \cdots + \frac{a_k}{p^{\beta_k}} \right) \right. \\ & \left. + \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_p \left(\frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \cdots + \frac{a_k}{p^{\beta_k}} \right)^i \right) \end{aligned} \quad (1.11)$$

where M is Mertens's constant.

Proof. The proof is identical to the proof of Theorem 1.2. The theorem is proved. $\square$

Theorems 1.1, 1.2 and 1.3 for primes are particular cases of a more general theorem valid for many integer sequences, namely

Theorem 1.4. *Let c_n be a strictly increasing sequence of positive integers. We shall denote a generic element of the sequence c . Suppose that $\sum_c \frac{1}{c}$ diverges. Let a_i ($i = 1, 2, \dots, k$) be real numbers and $a_1 \neq 0$. The following asymptotic formula holds*

$$\prod_{c \leq x} \left(1 + \frac{a_1}{c} + \frac{a_2}{c^2} + \cdots + \frac{a_k}{c^k} \right) \sim D \exp \left(a_1 \sum_{c \leq x} \frac{1}{c} \right), \quad (1.12)$$

where D is a positive constant and its value is

$$D = \exp \left(\sum_c \left(\frac{a_2}{c^2} + \cdots + \frac{a_k}{c^k} \right) + \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_c \left(\frac{a_1}{c} + \frac{a_2}{c^2} + \cdots + \frac{a_k}{c^k} \right)^i \right). \quad (1.13)$$

Proof. The proof is the same as the proof of Theorem 1.2. The theorem is proved. $\square$

Example 1.5. Suppose that c_n is the sequence of all positive integers, that is $c_n = n$. If we put $c = n$ then, as it is well-known [4],

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right). \quad (1.14)$$

As application we shall obtain an Euler's formula for γ in terms of the Riemann zeta function. Consider the trivial product

$$\prod_{n \leq x} \left(1 + \frac{1}{n}\right) = \prod_{n \leq x} \left(\frac{n+1}{n}\right) = ([x] + 1) \sim x. \quad (1.15)$$

On the other hand equations (1.12), (1.13) and (1.14) give

$$\prod_{n \leq x} \left(1 + \frac{1}{n}\right) \sim \exp(\log x + \gamma + o(1)) \exp\left(-\frac{1}{2}\zeta(2) + \frac{1}{3}\zeta(3) - \cdots\right). \quad (1.16)$$

Therefore equations (1.15) and (1.16) give the Euler's formula [2]

$$\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \cdots$$

Let us consider the sequence $c = 2, 3, 4, 5, \dots$ (all positive integers except 1). Consider the trivial product

$$\prod_{c \leq x} \left(1 - \frac{1}{c}\right) = \prod_{c \leq x} \left(\frac{c-1}{c}\right) = \frac{1}{[x]} \sim \frac{1}{x}.$$

On the other hand, we have

$$\sum_{c \leq x} \frac{1}{c} = \log x + \gamma - 1 + O\left(\frac{1}{x}\right).$$

Therefore equations (1.12) and (1.13) give

$$\prod_{c \leq x} \left(1 - \frac{1}{c}\right) \sim \exp(-\log x - \gamma + 1 + o(1)) \exp\left(-\frac{1}{2}(\zeta(2) - 1) - \frac{1}{3}(\zeta(3) - 1) - \frac{1}{4}(\zeta(4) - 1) - \cdots\right).$$

Therefore we obtain another Euler's formula for γ in terms of the zeta function, namely

$$\gamma = 1 - \frac{1}{2}(\zeta(2) - 1) - \frac{1}{3}(\zeta(3) - 1) - \frac{1}{4}(\zeta(4) - 1) - \dots \quad (1.17)$$

Formula (1.17) for γ in terms of the zeta function appear in [2].

Formula (1.17) can be generalized. We recall the generalized harmonic number ($k \geq 1$) ($i \geq 1$)

$$H_i(k) = 1 + \frac{1}{2^i} + \frac{1}{3^i} + \dots + \frac{1}{k^i}$$

If $i = 1$ then we obtain the harmonic number

$$H_1(k) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

The constant γ is defined by the limit related with the harmonic number and the logarithm

$$\lim_{k \rightarrow \infty} (H_1(k) - \log k) = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \log k \right) = \gamma$$

We shall prove a formula for $H_1(k) - \log k$ that generalize (1.17) and where the zeta function and generalized harmonic numbers are related. We consider the sequence $c = k + 1, k + 2, k + 3, \dots$ where k is an arbitrary fixed positive integer. Now consider the trivial product

$$\prod_{c \leq x} \left(1 - \frac{1}{c} \right) = \prod_{c \leq x} \left(\frac{c-1}{c} \right) = \frac{k}{[x]} \sim \frac{k}{x} = \frac{e^{\log k}}{x}.$$

we have

$$\sum_{c \leq x} \frac{1}{c} = \log x + \gamma - H_1(k) + O\left(\frac{1}{x}\right)$$

Therefore by use of (1.12) and (1.13) as in the proof of (1.17) we obtain the wished formula generalization of (1.17), namely

$$\gamma = H_1(k) - \log k - \sum_{i=2}^{\infty} \frac{1}{i} (\zeta(i) - H_i(k)) \quad (1.18)$$

If $k = 1$ then (1.18) becomes (1.17).

If c_n is the sequence of primes in the arithmetic progression $m n + r$, where m and r are relatively prime, then, as it is well-known [8],

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv r \pmod{m}}} \frac{1}{p} &= \frac{1}{\varphi(m)} \log \log x + M_{m,r} + O\left(\frac{1}{e^{a\sqrt{\log x}}}\right) \\ &= \frac{1}{\varphi(m)} \log \log x + M_{m,r} + O\left(\frac{1}{\log^s x}\right), \end{aligned} \quad (1.19)$$

where $M_{m,r}$ is a constant depending of the modulus m and the residue r relatively prime to m , a is a positive constant, $\varphi(m)$ is the Euler's function and s is an arbitrary positive integer.

Suppose that the modulus m is constant. The constant $M_{m,r}$, depending of r , can be written in the form

$$M_{m,r} = A_{m,r} + \frac{1}{\varphi(m)}\gamma + \sum_{p \equiv r} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right), \quad (1.20)$$

where $A_{m,r}$ is a certain constant depending of r .

Now, we have

$$\begin{aligned} \sum_{p \leq x} \frac{1}{p} &= \sum_r \frac{1}{\varphi(m)} \log \log x + \sum_{p|m} \frac{1}{p} + \sum_r M_{m,r} + o(1) \\ &= \log \log x + \sum_{p|m} \frac{1}{p} + \sum_r A_{m,r} + \sum_r \frac{1}{\varphi(m)}\gamma \\ &+ \sum_r \sum_{p \equiv r} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) + o(1) = \log \log x + \sum_{p|m} \frac{1}{p} \\ &+ \sum_r A_{m,r} + M - \sum_{p|m} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) + o(1), \end{aligned}$$

where the sums runs over all residue r relatively prime to m . Therefore we find that

$$\sum_r A_{m,r} = \sum_{p|m} \log \left(1 - \frac{1}{p} \right),$$

that is,

$$\exp\left(\sum_r A_{m,r}\right) = \prod_{p|m} \left(1 - \frac{1}{p} \right) = \frac{\varphi(m)}{m}.$$

Now, equation (1.13) becomes

$$\begin{aligned} D_{m,r} &= \exp \left(\sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_{p \equiv r} \left(\frac{-1}{p} \right)^i \right) \\ &= \exp \left(\sum_{p \equiv r} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \right). \end{aligned} \quad (1.21)$$

Therefore equation (1.12) becomes (see equations (1.19), (1.20) and (1.21))

$$\begin{aligned}
& \prod_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \left(1 - \frac{1}{p}\right) = \exp \left(-\frac{1}{\varphi(m)} \log \log x - A_{m,r} - \frac{1}{\varphi(m)} \gamma \right. \\
& \quad \left. - \sum_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) \right) \exp \left(\sum_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) \right) \\
& \quad \left(1 + O\left(\frac{1}{\log^s x}\right)\right) \\
& = \frac{1}{(\log x)^{\frac{1}{\varphi(m)}}} \exp \left(-A_{m,r} - \frac{1}{\varphi(m)} \gamma \right) \left(1 + O\left(\frac{1}{\log^s x}\right)\right) \tag{1.22}
\end{aligned}$$

for all positive integer s , that is, a generalization of the Mertens's formula (1.1) if $m = 1$, since in this case $A_{1,0} = 0$ and $\varphi(1) = 1$.

Note that equation (1.22) gives

$$A_{m,r} = -\frac{\gamma}{\varphi(m)} + \lim_{x \rightarrow \infty} \left(\sum_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \left(-\log \left(1 - \frac{1}{p}\right) \right) - \frac{1}{\phi(m)} \log \log x \right)$$

Now, we establish two very general theorems.

Theorem 1.6. *Let a_i ($i \geq 1$) be an infinite sequence of real numbers such that $a_1 \neq 0$. Let β_i ($i \geq 2$) be an infinite sequence of real numbers such that $1 < \beta_2 < \beta_3 < \dots$.*

Suppose that

$$\left| \frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \frac{a_3}{p^{\beta_3}} + \dots \right| \leq \frac{|a_1|}{p} + \frac{|a_2|}{p^{\beta_2}} + \frac{|a_3|}{p^{\beta_3}} + \dots \leq \frac{c_1}{p}$$

and

$$\left| \frac{a_2}{p^{\beta_2}} + \frac{a_3}{p^{\beta_3}} + \dots \right| \leq \frac{|a_2|}{p^{\beta_2}} + \frac{|a_3|}{p^{\beta_3}} + \dots \leq \frac{c_2}{p^\alpha}$$

where c_1 and c_2 are positive constant and $\alpha > 1$.

Then the following asymptotic formula holds

$$\prod_{p \leq x} \left(1 + \frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \frac{a_3}{p^{\beta_3}} + \dots \right) = C(\log x)^{a_1} \left(1 + O\left(\frac{1}{\log^m x}\right) \right),$$

where C is a positive constant and its value is

$$\begin{aligned}
 C &= \exp \left(a_1 M + \sum_p \left(\frac{a_2}{p^{\beta_2}} + \frac{a_3}{p^{\beta_3}} + \cdots \right) \right. \\
 &+ \left. \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_p \left(\frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \cdots \right)^i \right) \\
 &= \exp \left(a_1 M + \sum_p \left(\frac{a_2}{p^{\beta_2}} + \frac{a_3}{p^{\beta_3}} + \cdots \right) \right. \\
 &+ \left. \sum_p \left(\log \left(1 + \frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \cdots \right) - \left(\frac{a_1}{p} + \frac{a_2}{p^{\beta_2}} + \cdots \right) \right) \right),
 \end{aligned}$$

where $M = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$ is the Mertens's constant and γ is Euler's constant.

Proof. The proof is the same as the proof of Theorem 1.2. The theorem is proved. $\square$

Theorem 1.7. Let c_n be a strictly increasing sequence of positive integers. We shall denote a generic element of the sequence c . Suppose that $\sum_c \frac{1}{c}$ diverges. Let a_i ($i \geq 1$) be an infinite sequence of real numbers such that $a_1 \neq 0$. Let β_i ($i \geq 2$) be an infinite sequence of real numbers such that $1 < \beta_2 < \beta_3 < \cdots$.

Suppose that

$$\left| \frac{a_1}{c} + \frac{a_2}{c^{\beta_2}} + \frac{a_3}{c^{\beta_3}} + \cdots \right| \leq \frac{|a_1|}{c} + \frac{|a_2|}{c^{\beta_2}} + \frac{|a_3|}{c^{\beta_3}} + \cdots \leq \frac{c_1}{c}$$

and

$$\left| \frac{a_2}{c^{\beta_2}} + \frac{a_3}{c^{\beta_3}} + \cdots \right| \leq \frac{|a_2|}{c^{\beta_2}} + \frac{|a_3|}{c^{\beta_3}} + \cdots \leq \frac{c_2}{c^\alpha}$$

where c_1 and c_2 are positive constant and $\alpha > 1$.

Then the following asymptotic formula holds

$$\prod_{c \leq x} \left(1 + \frac{a_1}{c} + \frac{a_2}{c^{\beta_2}} + \frac{a_3}{c^{\beta_3}} + \cdots \right) \sim D \exp \left(a_1 \sum_{c \leq x} \frac{1}{c} \right),$$

where D is a positive constant and its value is

$$\begin{aligned}
 D &= \exp \left(\sum_c \left(\frac{a_2}{c^{\beta_2}} + \frac{a_3}{c^{\beta_3}} + \cdots \right) \right. \\
 &+ \left. \sum_{i=2}^{\infty} (-1)^{i+1} \frac{1}{i} \sum_c \left(\frac{a_1}{c} + \frac{a_2}{c^{\beta_2}} + \cdots \right)^i \right) \\
 &= \exp \left(\sum_c \left(\frac{a_2}{c^{\beta_2}} + \frac{a_3}{c^{\beta_3}} + \cdots \right) \right. \\
 &+ \left. \sum_c \left(\log \left(1 + \frac{a_1}{c} + \frac{a_2}{c^{\beta_2}} + \cdots \right) - \left(\frac{a_1}{c} + \frac{a_2}{c^{\beta_2}} + \cdots \right) \right) \right),
 \end{aligned}$$

Proof. The proof is the same as the proof of Theorem 1.2. The theorem is proved. $\square$

2. SOME NEWS ARITHMETIC FUNCTIONS

Let us consider the prime factorization of a positive integer $n = q_1^{s_1} \cdots q_r^{s_r}$, where the q_i ($i = 1, \dots, r$) are the distinct primes and the s_i ($i = 1, \dots, r$) are the multiplicity or exponents. The number of distinct primes in the prime factorization of n is denoted as usual $\omega(n)$, that is, $\omega(n) = r$. The total number of primes in the prime factorization of n is denoted as usual $\Omega(n)$, that is, $\Omega(n) = s_1 + \cdots + s_r$ [4]. We put $\omega(1) = 0$ and $\Omega(1) = 1$. Let $s = \sigma + it$ be a complex number. The prime zeta function is $P(s) = \sum_p \frac{1}{p^s}$ defined for $\sigma > 1$. This function is related with the prime counting function $\pi(x)$, the arithmetical function $\omega(n)$ and the zeta function $\zeta(s)$ since we have [1]

$$P(s) = \sum_p \frac{1}{p^s} = s \int_1^{\infty} \frac{\pi(x)}{x^{s+1}} dx = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s} \quad (\sigma > 1)$$

Let us consider newly the prime factorization of $n = q_1^{s_1} \cdots q_r^{s_r}$. We define the arithmetical function $a(n) = \frac{1}{s_1! s_2! \cdots s_r!}$. Clearly it is a multiplicative arithmetical function. Its generating Dirichlet series is related to the prime zeta function since we have for $\sigma > 1$

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{2! p^{2s}} + \cdots \right) = \prod_p e^{\frac{1}{p^s}} = e^{\sum_p \frac{1}{p^s}} = e^{P(s)}$$

If $s = 1$ then we have

$$\begin{aligned}
 \sum_n \frac{a(n)}{n} &= \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{2! p^2} + \cdots \right) = \prod_{p \leq x} e^{\frac{1}{p}} = e^{\sum_{p \leq x} \frac{1}{p}} \\
 &= e^M \log x \left(1 + O \left(\frac{1}{\log^m x} \right) \right)
 \end{aligned} \tag{2.1}$$

Note that in the first sum n run over all positive integer such that all its prime factors does not exceed x . This include all $n \leq x$.

We shall study the average of $a(n)$. Before, we need some definitions and lemmas.

A number is square-free or quadratfrei [4] if either it is the product of distinct primes or 1, that is, its prime factorization is of the form $q_1 \cdots q_r$ where the q_i ($i = 1, \dots, r$) ($r \geq 1$) are the distinct primes. We denote a generic square-free number q .

Lemma 2.1. *Let $Q_{q_1 \dots q_r}(x)$ the number of square-free not exceeding x relatively primes to the square-free $q_1 \cdots q_r$. The following formula holds.*

$$Q_{q_1 \dots q_r}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x + o(x).$$

Proof. See [5]. The lemma is proved. $\square$

A number is square-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to 2, that is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is square-full if $s_i \geq 2$ ($i = 1, \dots, r$). We denote a generic square-full number Q . The sum of reciprocal of square-full numbers $\sum_Q \frac{1}{Q}$ is a convergent series [3]

Now, we can study the average of $a(n)$ in the following theorem.

Theorem 2.2. *The following asymptotic formulas hold.*

$$\sum_{n \leq x} a(n) = e^{M-\gamma} x + o(x) \quad (2.2)$$

$$\sum_{n \leq x} \frac{a(n)}{n} = e^{M-\gamma} \log x + o(\log x) \quad (2.3)$$

Compare with (2.1).

Remember that $M - \gamma = \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$.

Proof. Equation (2.3) is an immediate consequence of equation (2.2) and Abel summation. Therefore we shall prove equation (2.2).

The number of square-free numbers not exceeding x is [4] $\sum_{q \leq x} 1 = \frac{6}{\pi^2} x + o(x)$. On the other hand $a(q) = \frac{1}{1!1! \cdots 1!} = 1$. Therefore we have

$$\sum_{q \leq x} a(q) = \sum_{q \leq x} 1 = \frac{6}{\pi^2} x + o(x) \quad (2.4)$$

The rest of the numbers is the product of a square-free and a square-full, that is, of the form qQ , where q is the product of the primes with exponent 1 in the prime factorization and Q is the product of the rest of the prime powers in the prime factorization. Note that then $\gcd(q, Q) = 1$ and $a(qQ) = a(q)a(Q) = 1a(Q) = a(Q)$.

Let us consider the prime factorization of a square-full number Q , that is, $Q = q_1^{r_1} \cdots q_t^{r_t}$. Let us consider the numbers of the form qQ , where Q is fixed,

that is, the numbers with the same $Q = q_1^{r_1} \cdots q_t^{r_t}$. The number of these numbers not exceeding x , that is, $qQ \leq x$ will be (see Lemma 2.1)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_t}{(q_1 + 1) \cdots (q_t + 1)} \frac{x}{Q} + o(x) = \frac{6}{\pi^2} b(Q) \frac{x}{Q} + o(x)$$

and the contribution of these numbers to the sum $\sum_{n \leq x} a(n)$ will be

$$\frac{6}{\pi^2} b(Q) a(Q) \frac{x}{Q} + o(x) \quad (2.5)$$

where by sake of simplicity we put $b(Q) = \frac{q_1 \cdots q_t}{(q_1 + 1) \cdots (q_t + 1)}$.

Let $\epsilon > 0$. We choose B such that (remember that the series of the reciprocal of square-full numbers converges)

$$\sum_{Q > B} \frac{1}{Q} < \epsilon \quad (2.6)$$

Therefore we have (see (2.4), (2.5) and (2.6))

$$\begin{aligned} \sum_{n \leq x} a(n) &= \frac{6}{\pi^2} \left(1 + \sum_{Q \leq B} \frac{b(Q) a(Q)}{Q} \right) x + o(x) + F(x) \\ &= Cx - \frac{6}{\pi^2} \left(\sum_{Q > B} \frac{b(Q) a(Q)}{Q} \right) x + o(x) + F(x) \end{aligned} \quad (2.7)$$

where $F(x)$ is the contribution of Q such that $B < Q \leq x$ and consequently (see (2.6))

$$0 \leq F(x) \leq \sum_{Q > B} a(Q) \left\lfloor \frac{x}{Q} \right\rfloor \leq \sum_{Q > B} \frac{x}{Q} \leq \epsilon x \quad (2.8)$$

since if Q is fixed, the number of numbers $qQ \leq x$ does not exceed the number of positive integers n that satisfies the inequality $n \leq \frac{x}{Q}$ and this number is $\left\lfloor \frac{x}{Q} \right\rfloor$.

On the other hand C is the sum of the series

$$C = \frac{6}{\pi^2} \left(1 + \sum_Q \frac{b(Q) a(Q)}{Q} \right) \quad (2.9)$$

where the sum run over all square-full Q .

Consequently (see (2.7), (2.6) and (2.8))

$$\left| \frac{\sum_{n \leq x} a(n)}{x} - C \right| \leq \epsilon + \epsilon + \epsilon = 3\epsilon \quad (x \geq x_\epsilon)$$

That is, since $\epsilon > 0$ can be arbitrarily small,

$$\sum_{n \leq x} a(n) = Cx + o(x) \quad (2.10)$$

Now, we have prove that $C = e^{M-\gamma}$. We have (see (2.9))

$$\begin{aligned}
 C &= \frac{6}{\pi^2} \left(1 + \sum_Q \frac{b(Q)a(Q)}{Q} \right) \\
 &= \frac{6}{\pi^2} \prod_p \left(1 + \frac{p}{p+1} \left(\frac{1}{2!p^2} + \frac{1}{3!p^3} + \cdots \right) \right) \\
 &= \lim_{x \rightarrow \infty} \frac{6}{\pi^2} \prod_{p \leq x} \left(1 + \frac{p}{p+1} \left(\frac{1}{2!p^2} + \frac{1}{3!p^3} + \cdots \right) \right) \\
 &= \lim_{x \rightarrow \infty} \prod_{p \leq x} \left(1 - \frac{1}{p^2} \right) \prod_{p \leq x} \frac{p}{p+1} e^{\frac{1}{p}} = \lim_{x \rightarrow \infty} \prod_{p \leq x} \left(1 - \frac{1}{p} \right) e^{\frac{1}{p}} \\
 &= \lim_{x \rightarrow \infty} \left(\prod_{p \leq x} \left(1 - \frac{1}{p} \right) \right) e^{\sum_{p \leq x} \frac{1}{p}} = e^{M-\gamma}
 \end{aligned} \tag{2.11}$$

Equations (2.10) and (2.11) give (2.2). The theorem is proved. $\square$

We define the arithmetical functions $g(n) = s_1 s_2 \cdots s_k$, that is, the product of the exponents in the prime factorization of n , and the arithmetical function $\frac{2^{\omega(n)}}{2^{\Omega(n)}}$. In the following two theorems we study their average.

Theorem 2.3. *The following asymptotic formula holds*

$$\sum_{n \leq x} g(n) = Cx + o(x), \tag{2.12}$$

where the positive constant C is $C = \prod_p \left(1 + \frac{1}{p(p-1)} \right)$

Proof. The proof is the same as the proof of Theorem 2.2. The theorem is proved. $\square$

Theorem 2.4. *The following asymptotic formula holds*

$$\sum_{n \leq x} \frac{2^{\omega(n)}}{2^{\Omega(n)}} = Cx + o(x), \tag{2.13}$$

where the positive constant C is $C = \prod_p \left(1 - \frac{1}{p(2p-1)} \right)$

Proof. Note that if $n = q$ then $\omega(q) = \Omega(q)$ and consequently $\frac{2^{\omega(n)}}{2^{\Omega(n)}} = 1$. On the other hand, if $n = qQ$ then $\frac{2^{\omega(qQ)}}{2^{\Omega(qQ)}} = \frac{1}{2^{\Omega(Q)-\omega(Q)}}$. Now, the proof is the same as the proof of Theorem 2.2. The theorem is proved. $\square$

Let us consider newly the prime factorization of $n = q_1^{s_1} \cdots q_r^{s_r}$. We define the arithmetical function $\frac{(s_1 + \cdots + s_r)!}{s_1! s_2! \cdots s_r!} = \frac{\Omega(n)!}{s_1! s_2! \cdots s_r!}$. As it is well known of combinatorial calculus this arithmetical function give us the number of permutations of all primes in the prime factorization of n . Its generating Dirichlet series is also

related to the prime zeta function since we have (we consider s real).

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Omega(n)!}{s_1! s_2! \cdots s_r!} \frac{1}{n^s} &= 1 + \left(\sum_p \frac{1}{p^s} \right) + \left(\sum_p \frac{1}{p^s} \right)^2 + \left(\sum_p \frac{1}{p^s} \right)^3 + \cdots \\ &= \frac{1}{1 - \left(\sum_p \frac{1}{p^s} \right)} = \frac{1}{1 - P(s)} \end{aligned} \quad (2.14)$$

Note that there exist a real number s_0 such that $P(s_0) = 1$. Therefore the Dirichlet series converges for $P(s) < 1$, that is, $s > s_0$ and diverges for $P(s) > 1$, that is, $1 < s \leq s_0$. Now,

$$P(2) = \sum_p \frac{1}{p^2} < \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - 1 < 1$$

and consequently $1 < s_0 < 2$. By application of Abel summation we obtain some information on the average of this arithmetical function, namely

$$\sum_{n \leq x} \frac{\Omega(n)!}{s_1! s_2! \cdots s_r!} = o(x^{s_0 + \alpha}) \quad (2.15)$$

for all $\alpha > 0$.

The normal order of $\Omega(n)$ is $\log \log n$ [4], that is, for all $\epsilon > 0$ we have

$$(1 + \epsilon) \log \log n < \Omega(n) < (1 + \epsilon) \log \log n \quad (2.16)$$

where $n \in A$ and A is a set of 1 density, that is, $\sum_{n \leq x, n \in A} 1 \sim x$ and consequently the complementary set B where equation (2.16) does not hold has zero density, that is, $\sum_{n \leq x, n \in B} 1 = o(x)$. The sets A and B depend of ϵ . Note that $A \cup B$ is the set of all positive integers. We shall prove that the series (2.14) diverges for $1 < s \leq s_0$ as consequence of the zero density set B of abnormal numbers. We have the well-known Stirling's formula $n! \sim \sqrt{2\pi} \frac{n^n \sqrt{n}}{e^n}$. Therefore there exists a positive constant c such that

$$n! < cn^{n+\frac{1}{2}} \quad (2.17)$$

from a certain value of n . Now, for the elements of the set A we have (see (2.16))

$$\Omega(n) < (1 + \epsilon) \log \log n < \lfloor (1 + \epsilon) \log \log n \rfloor + 1 \quad (2.18)$$

and consequently (see (2.17) and (2.18))

$$\frac{\Omega(n)!}{s_1! \cdots s_k!} \leq \Omega(n)! < (\lfloor (1 + \epsilon) \log \log n \rfloor + 1)! < n^{o(1)} \quad (2.19)$$

Therefore the series (2.14) over the numbers of the set A of 1 density converges, since (see (2.19))

$$\sum_{n \in A} \frac{\Omega(n)!}{s_1! \cdots s_k!} \frac{1}{n^s} < \sum_{n \in A} \frac{n^{o(1)}}{n^{1+\frac{s-1}{2}} n^{\frac{s-1}{2}}} < c' \sum_{n \in A} \frac{1}{n^{1+\frac{s-1}{2}}} \quad (1 < s \leq s_0)$$

where c' is a positive constant. Hence the series (2.14) diverges for $1 < s \leq s_0$ as consequence of the zero density set B of rare abnormal numbers, as desired.

3. SQUARE-FREE AND s -FULL NUMBERS NOT EXCEEDING x REPRESENTABLE AS THE SUM OF TWO SQUARES

Let $Q(x)$ be the number of square-free not exceeding x . It is well-known (see [4]) these numbers have positive density $\frac{6}{\pi^2}$. That is,

$$Q(x) = \frac{6}{\pi^2}x + o(x) \quad (3.1)$$

We recall that a number is representable as the sum of two squares if and only if all prime factor of the form $4n - 1$ in its prime factorization (if there is some) has even exponent. In Example 1.4 of the reference [6] we observe that if $B(x)$ is the number of positive integers not exceeding x representable as the sum of two nonnegative squares then we have

$$B(x) \sim B \frac{x}{\sqrt{\log x}} \quad (3.2)$$

where B is the Landau-Ramanujan constant.

We are interested in the number of square-free not exceeding x representable as the sum of two squares. We have the following theorem.

Theorem 3.1. *Let $N(x)$ be the number of square-free not exceeding x representable as the sum of two squares. The following formula holds*

$$N(x) \sim \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \quad (3.3)$$

Proof. We shall give two proofs. Now, we give a proof where we use former theorems and formulas. In the end of the section we give a proof that use only equation (3.2) and equation (1.7) of the reference [6]. Equation (3.2) was proved by Landau with a contour integral (see [7]).

First proof. Note that the square-free sum of two nonnegative squares are a product of primes of the form $4n + 1$ and or 2.

We have the following formula (see Example 1.4 of the reference [6] and equation 1.19)

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv 1, \text{mod}(4)}} \frac{1}{p} &= \frac{1}{2} \log \log x \\ &+ \left(\ln \left(\frac{\sqrt{\pi}}{4B} \right) + \frac{\gamma}{2} + \sum_{p \equiv 1, \text{mod}(4)} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \right) + o(1) \end{aligned}$$

Therefore (see equation 1.19 in Example 1.5) we have

$$M_{4,1} = \ln \left(\frac{\sqrt{\pi}}{4B} \right) + \frac{\gamma}{2} + \sum_{p \equiv 1, \text{mod}(4)} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$$

and since (see equation 1.20 in Example 1.5)

$$M_{4,1} = A_{4,1} + \frac{\gamma}{2} + \sum_{p \equiv 1, \text{mod}(4)} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right)$$

we obtain

$$A_{4,1} = \log \left(\frac{\sqrt{\pi}}{4B} \right)$$

Therefore by Wirsing's Theorem 1.7 in the reference [6] we obtain that the number $A_{4,1}(x)$ of numbers not exceeding x whose prime factors are of the form $4n + 1$ is

$$A_{4,1}(x) \sim \frac{e^{A_{4,1}}}{\Gamma(1/2)} \frac{x}{\sqrt{\log x}} = \frac{1}{4B} \frac{x}{\sqrt{\log x}}$$

since $\Gamma(1/2) = \sqrt{\pi}$.

Therefore, by Corollary 1.11 in the reference [6], the number of square-free such that their prime factors are of the form $4n + 1$ will be

$$S_{4,1,2}(x) \sim \frac{1}{4B} \prod_{p \equiv 1, \text{mod}(4)} \left(1 - \frac{1}{p^2} \right) \frac{x}{\sqrt{\log x}}$$

and the number of square-free multiple of 2 such that the rest of their prime factors are of the form $4n + 1$ will be

$$S_{4,1,2} \left(\frac{x}{2} \right) \sim \frac{1}{2} \frac{1}{4B} \prod_{p \equiv 1, \text{mod}(4)} \left(1 - \frac{1}{p^2} \right) \frac{x}{\sqrt{\log x}}$$

Therefore if $S(x)$ is the number of square-free not exceeding x representable as the sum of two squares, then

$$S(x) \sim \left(1 + \frac{1}{2} \right) \frac{1}{4B} \prod_{p \equiv 1, \text{mod}(4)} \left(1 - \frac{1}{p^2} \right) \frac{x}{\sqrt{\log x}} = \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}}$$

The theorem is proved. $\square$

The s -th powers not exceeding x and sum of two squares can be studied easily. If s is even then the s -th power is a square and consequently a sum of two squares. Therefore all s -th power such that s is even is sum of two squares and then there are $\left\lfloor x^{\frac{1}{s}} \right\rfloor = x^{\frac{1}{s}} + o \left(x^{\frac{1}{s}} \right)$ not exceeding x . If s is odd then a^s is sum of two squares if and only if a is sum of two squares. Therefore their number not exceeding x will be

$$B \left(x^{\frac{1}{s}} \right) \sim \sqrt{s} B \frac{x^{\frac{1}{s}}}{\sqrt{\log x}}$$

Hence, if s is odd, the proportion of s -th powers not exceeding x representable as the sum of two squares is negligible compared with the number of s -th powers not exceeding x , namely $\left\lfloor x^{\frac{1}{s}} \right\rfloor = x^{\frac{1}{s}} + o \left(x^{\frac{1}{s}} \right)$.

Theorem 3.2. *Let $N_{q_1, q_2, \dots, q_s}(x)$ be the number of square-free multiple of the primes $q_1, q_2, \dots, q_s$ ($s \geq 1$) not exceeding x and representable as the sum of two squares. The following formula holds.*

$$N_{q_1, q_2, \dots, q_s}(x) \sim \frac{1}{(q_1 + 1)(q_2 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \quad (3.4)$$

Proof. For sake of simplicity we put (see equations (3.3) and (3.4))

$$C_0(x) = N(x) = \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) = A_0 \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) \quad (3.5)$$

$$\begin{aligned} C_{s-1}(x) = N_{q_1, q_2, \dots, q_{s-1}}(x) &= \frac{1}{(q_1 + 1)(q_2 + 1) \cdots (q_{s-1} + 1)} \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \\ &+ o\left(\frac{x}{\sqrt{\log x}}\right) = A_{s-1} \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) \quad (s = 2, 3, \dots) \end{aligned} \quad (3.6)$$

Note that

$$C_{s-1}(x) < C_{s-1} \frac{x}{\sqrt{\log x}} \quad (x \geq 2) \quad (s = 1, 2, 3, \dots) \quad (3.7)$$

Suppose that the theorem is true for $s-1$ ($s \geq 1$), we shall prove that the theorem is also true for s . If we put $\frac{x}{q_s^r} = q_1 \cdots q_{s-1} q_s$ ($s \geq 2$) or $\frac{x}{q_s^r} = q_s$ ($s = 1$) then we have

$$\begin{aligned} C_s(x) = N_{q_1, \dots, q_{s-1} q_s}(x) &= \left(\sum_{n=1}^{\lfloor r \rfloor} (-1)^{n+1} C_{s-1} \left(\frac{x}{q_s^n} \right) \right) \\ &+ (-1)^{\lfloor r \rfloor + 2} C_{s-1} \left(\frac{x}{q_s^{\lfloor r \rfloor + 1}} \right) \end{aligned} \quad (3.8)$$

Note that

$$C_{s-1} \left(\frac{x}{q_s^{\lfloor r \rfloor + 1}} \right) = O(1) = o\left(\frac{x}{\sqrt{\log x}}\right) \quad (3.9)$$

Let $\epsilon > 0$. We choose M depending of ϵ such that

$$\sum_{n=M+1}^{\infty} \frac{1}{q_s^n} < \epsilon \quad (3.10)$$

Now

$$\begin{aligned}
\sum_{n=1}^M (-1)^{n+1} C_{s-1} \left(\frac{x}{q_s^n} \right) &= \sum_{n=1}^M (-1)^{n+1} \left(A_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log \left(\frac{x}{q_s^n} \right)}} + o \left(\frac{x}{\sqrt{\log x}} \right) \right) \\
&= \left(\sum_{n=1}^M (-1)^{n+1} A_{s-1} \frac{1}{q_s^n} \frac{x}{\sqrt{\log x}} \right) + o \left(\frac{x}{\sqrt{\log x}} \right) = A_s \frac{x}{\sqrt{\log x}} \\
&- \left(\sum_{n=M+1}^{\infty} (-1)^{n+1} \frac{1}{q_s^n} \right) A_{s-1} \frac{x}{\sqrt{\log x}} + o \left(\frac{x}{\sqrt{\log x}} \right)
\end{aligned} \tag{3.11}$$

Besides

$$\left| \sum_{n=M+1}^{\lfloor r \rfloor} (-1)^{n+1} C_{s-1} \left(\frac{x}{q_s^n} \right) \right| \leq \sum_{n=M+1}^{\lfloor r \rfloor} C_{s-1} \left(\frac{x}{q_s^n} \right) \tag{3.12}$$

Now, we have (see equations (3.7) and (3.10))

$$\begin{aligned}
\sum_{q_s^{M+1} \leq q_s^n \leq \sqrt{x}} C_{s-1} \left(\frac{x}{q_s^n} \right) &\leq \sum_{q_s^{M+1} \leq q_s^n \leq \sqrt{x}} C_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log \frac{x}{q_s^n}}} \\
&= \sum_{q_s^{M+1} \leq q_s^n \leq \sqrt{x}} C_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log q_s^n}{\log x}}} \leq \sum_{q_s^{M+1} \leq q_s^n \leq \sqrt{x}} C_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log x}} \sqrt{2} \\
&\leq \epsilon C_{s-1} \sqrt{2} \frac{x}{\sqrt{\log x}}
\end{aligned} \tag{3.13}$$

since the inequality $q_s^n \leq \sqrt{x}$ implies the inequality $\frac{1}{\sqrt{1 - \frac{\log q_s^n}{\log x}}} \leq \sqrt{2}$.

On the other hand, note that the inequality $\sqrt{x} < q_s^n$ implies the inequality $\frac{x}{q_s^n} < \sqrt{x}$. Besides, the equation $\frac{x}{q_s^r} = q_1 \cdots q_{s-1} q_s$ (see above) implies the inequality $\frac{x}{q_s^n} \geq \frac{x}{q_s^{\lfloor r \rfloor}} \geq q_1 \cdots q_{s-1} q_s$, that is, the inequality $q_s^n \leq \frac{x}{q_1 \cdots q_{s-1} q_s}$ ($n \leq r$) and this last inequality implies the inequality $\frac{1}{\sqrt{1 - \frac{\log q_s^n}{\log x}}} \leq \frac{\sqrt{\log x}}{\sqrt{\log(q_1 \cdots q_{s-1} q_s)}}$. Therefore we have

$$\begin{aligned}
\sum_{\sqrt{x} < q_s^n \leq q_s^{\lfloor r \rfloor}} C_{s-1} \left(\frac{x}{q_s^n} \right) &\leq \sum_{\sqrt{x} < q_s^n \leq q_s^{\lfloor r \rfloor}} C_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log \frac{x}{q_s^n}}} \\
&= \sum_{\sqrt{x} < q_s^n \leq q_s^{\lfloor r \rfloor}} C_{s-1} \frac{x}{q_s^n} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log q_s^n}{\log x}}} \leq \sum_{\sqrt{x} < q_s^n \leq q_s^{\lfloor r \rfloor}} \frac{C_{s-1} \sqrt{x}}{\sqrt{\log(q_1 \cdots q_{s-1} q_s)}} \\
&\leq r C_{s-1} \frac{1}{\sqrt{\log(q_1 \cdots q_{s-1} q_s)}} \sqrt{x} = o \left(\frac{x}{\sqrt{\log x}} \right)
\end{aligned} \tag{3.14}$$

since $r \sim \frac{\log x}{\log q_s}$.

Substituting equations (3.14), (3.13), (3.11) and (3.9) into equation (3.8) (see also (3.10)) we obtain

$$\left| \frac{C_s(x)}{\frac{x}{\sqrt{\log x}}} - A_s \right| \leq \epsilon + \epsilon A_{s-1} + \epsilon + \epsilon C_{s-1} \sqrt{2} + \epsilon \quad (x \geq x_\epsilon)$$

That is, equation (3.4), since ϵ can be arbitrarily small. The theorem is proved. $\square$

Theorem 3.3. *Let $Q_{q_1, q_2, \dots, q_m}(x)$ be the number of square-free relatively prime to the square-free sum of two squares $q_1 q_2, \dots, q_m$ ($m \geq 1$) not exceeding x and representable as the sum of two squares. The following formula holds*

$$Q_{q_1, q_2, \dots, q_m}(x) \sim \frac{q_1 q_2 \cdots q_m}{(q_1 + 1)(q_2 + 1) \cdots (q_m + 1)} \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \quad (3.15)$$

Proof. By the inclusion exclusion principle, we have

$$\begin{aligned} Q_{q_1, q_2, \dots, q_m}(x) &= N(x) - \sum_{1 \leq i \leq m} N_{q_i}(x) + \sum_{1 \leq i < j \leq m} N_{q_i, q_j}(x) - \cdots \\ &= \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \prod_{i=1}^m \left(1 - \frac{1}{q_i + 1} \right) + o\left(\frac{x}{\sqrt{\log x}} \right) \end{aligned}$$

That is, equation (3.15). The theorem is proved. $\square$

Let $A_h(x)$ be the number of h -full numbers not exceeding x . We recall (see [6]) that

$$A_h(x) = \frac{6}{\pi^2} C_h x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

where C_h is a positive constant depending of h and if $s \geq 1$ is an arbitrary but fixed integer then the following series converges.

$$\sum_{q_1^{s_1} \cdots q_r^{s_r}} \frac{1}{(q_1^{s_1} \cdots q_r^{s_r})^{\frac{1}{s}}},$$

where the sum runs on all $(s+1)$ -full numbers $q_1^{s_1} \cdots q_r^{s_r}$.

Theorem 3.4. *Let $q_1^{s_1} \cdots q_r^{s_r}$ be a square-full number such that q_i ($1 \leq i \leq t$) is the prime 2 or a prime of the form $4n+1$ and the rest of the primes (if there is some) are primes of the form $4n-1$ with even exponent. If $t=0$ then all the primes are primes of the form $4n-1$. If $t>0$ we put $A(q_1^{s_1} \cdots q_r^{s_r}) = \frac{q_1 \cdots q_t}{(q_1+1) \cdots (q_t+1)}$ and if $t=0$ we put $A(q_1^{s_1} \cdots q_r^{s_r}) = 1$. We have*

$$1 + \sum_{q_1^{s_1} \cdots q_r^{s_r}} A(q_1^{s_1} \cdots q_r^{s_r}) \frac{1}{q_1^{s_1} \cdots q_r^{s_r}} = \frac{\pi^2}{6} \quad (3.16)$$

Proof. We have

$$1 + \sum_{q_1^{s_1} \dots q_r^{s_r}} A(q_1^{s_1} \dots q_r^{s_r}) \frac{1}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} = \prod_p \left(1 + \frac{p}{p+1} \sum_{h=2}^{\infty} \frac{1}{p^h} \right)$$

$$\prod_q \left(1 + \sum_{h=1}^{\infty} \frac{1}{q^{2h}} \right) = \prod_p \frac{1}{1 - \frac{1}{p^2}} \prod_q \frac{1}{1 - \frac{1}{q^2}} = \frac{\pi^2}{6}$$

where p denotes the prime 2 or a prime of the form $4n+1$ and q denotes a prime of the form $4n-1$. The theorem is proved. $\square$

Theorem 3.5. *Let s be an odd number. Let us consider the s -full numbers not exceeding x representable as the sum of two squares. The number of these s -full numbers not exceeding x will be denote $C_s(x)$. The following formula holds.*

$$C_s(x) \sim C \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \quad (3.17)$$

where C is a positive constant and its value is

$$C = \sqrt{s} \frac{6}{\pi^2} B \left(1 + \frac{1}{3} \frac{1}{2^{\frac{1}{s}} - 1} \right) \prod_{p \equiv 1, \text{mod}(4)} \left(1 + \frac{1}{p+1} \frac{1}{p^{\frac{1}{s}} - 1} \right) \prod_{p \equiv 3, \text{mod}(4)} \left(1 + \frac{p^{\frac{1}{s}}}{p} \frac{1}{p^{\frac{2}{s}} - 1} \right)$$

Proof. Let q be a square-free sum of two squares. If the s -full number sum of two squares is of the form q^s then the number of s -full numbers that satisfy the inequality $q^s \leq x$ will be (see Theorem 3.1)

$$\frac{6}{\pi^2} B \sqrt{s} \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} + o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) \quad (3.18)$$

If the s -full number sum of two squares is of the form $q^s q_1^{s_1} \dots q_r^{s_r}$, where $q_1^{s_1} \dots q_r^{s_r}$ is a $(s+1)$ -full number sum of two squares, and consequently the primes of the form $4n-1$ (if there is some) have even exponent, then the number of s -full numbers that satisfy the inequality $q^s q_1^{s_1} \dots q_r^{s_r} \leq x$ will be

$$\frac{6}{\pi^2} B A(q_1^{s_1} \dots q_r^{s_r}) \frac{x^{\frac{1}{s}}}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \frac{1}{\sqrt{\log \left(\frac{x^{\frac{1}{s}}}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \right)}} + o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log \left(\frac{x^{\frac{1}{s}}}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \right)}} \right)$$

$$= \frac{6}{\pi^2} B \sqrt{s} A(q_1^{s_1} \dots q_r^{s_r}) \frac{1}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} + o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) \quad (3.19)$$

Let $\epsilon > 0$. There exists M depending of ϵ such that

$$\sum_{q_1^{s_1} \dots q_r^{s_r} \geq M+1} \frac{1}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} < \epsilon \quad (3.20)$$

Equations (3.18), (3.19) and (3.20) give

$$\begin{aligned}
C_s(x) &= \frac{6}{\pi^2} B \sqrt{s} \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \\
&+ \sum_{q_1^{s_1} \dots q_r^{s_r} \leq M} \left(\frac{6}{\pi^2} B \sqrt{s} A(q_1^{s_1} \dots q_r^{s_r}) \frac{1}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) + o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) \\
&+ E_s(x) \\
&= C \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} - \frac{6}{\pi^2} B \sqrt{s} \left(\sum_{q_1^{s_1} \dots q_r^{s_r} \geq M+1} A(q_1^{s_1} \dots q_r^{s_r}) \frac{1}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}}} \right) \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \\
&+ o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) + E_s(x) \tag{3.21}
\end{aligned}$$

Note that there exists a positive constant c_1 such that (see (3.3))

$$N(x) < c_1 \frac{x}{\sqrt{\log x}} \quad (x \geq 2) \tag{3.22}$$

Let $\alpha = \frac{2s+1}{2s+2}$. We have (see equations (3.21) and (3.22))

$$\begin{aligned}
0 \leq E_s(x) &\leq \sum_{M+1 \leq q_1^{s_1} \dots q_r^{s_r} \leq x^\alpha} c_1 \frac{\left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}} \right)^{\frac{1}{s}}}{\sqrt{\log \left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}} \right)^{\frac{1}{s}}}} \\
&+ \sum_{x^\alpha < q_1^{s_1} \dots q_r^{s_r} \leq \frac{x}{2^s}} c_1 \frac{\left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}} \right)^{\frac{1}{s}}}{\sqrt{\log \left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}} \right)^{\frac{1}{s}}}} + \sum_{\frac{x}{2^s} < q_1^{s_1} \dots q_r^{s_r} \leq x} 1 \tag{3.23}
\end{aligned}$$

Note that (see Lemma 1.13 in the reference [6]) there exists a positive constant c_2 such that $A_{s+1}(x) < c_2 x^{\frac{1}{s+1}}$. Therefore (see (3.23)) we have

$$\sum_{\frac{x}{2^s} < q_1^{s_1} \dots q_r^{s_r} \leq x} 1 \leq c_2 x^{\frac{1}{s+1}} = o \left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \right) \tag{3.24}$$

Note that the inequality $x^\alpha < q_1^{s_1} \dots q_r^{s_r}$ implies the inequality $\left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}} \right)^{\frac{1}{s}} < x^{\frac{1-\alpha}{s}}$. Besides the function $f(x) = \frac{x}{\sqrt{\log x}}$ is positive and strictly increasing since $f'(x) >$

0 ($x \geq 2$). Consequently we have (see equation (3.23))

$$\begin{aligned} & \sum_{x^\alpha < q_1^{s_1} \dots q_r^{s_r} \leq \frac{x}{2^s}} c_1 \frac{\left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}}\right)^{\frac{1}{s}}}{\sqrt{\log \left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}}\right)^{\frac{1}{s}}}} \leq \sum_{x^\alpha < q_1^{s_1} \dots q_r^{s_r} \leq \frac{x}{2^s}} c_1 \frac{x^{\frac{1-\alpha}{s}}}{\sqrt{\log x^{\frac{1-\alpha}{s}}}} \\ & \leq c_1 \frac{x^{\frac{1-\alpha}{s}}}{\sqrt{\log x^{\frac{1-\alpha}{s}}}} c_2 x^{\frac{1}{s+1}} = o\left(\frac{x^{\frac{1}{s}}}{\sqrt{\log x}}\right) \end{aligned} \quad (3.25)$$

Finally, note that the inequality $q_1^{s_1} \dots q_r^{s_r} \leq x^\alpha$ implies the inequality $\frac{1}{\sqrt{1 - \frac{\log(q_1^{s_1} \dots q_r^{s_r})}{\log x}}} \leq \frac{1}{\sqrt{1-\alpha}}$ and hence we have (see equations (3.23) and (3.20))

$$\begin{aligned} & \sum_{M+1 \leq q_1^{s_1} \dots q_r^{s_r} \leq x^\alpha} c_1 \frac{\left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}}\right)^{\frac{1}{s}}}{\sqrt{\log \left(\frac{x}{q_1^{s_1} \dots q_r^{s_r}}\right)^{\frac{1}{s}}}} \\ & = \sum_{M+1 \leq q_1^{s_1} \dots q_r^{s_r} \leq x^\alpha} \frac{c_1 \sqrt{s}}{(q_1^{s_1} \dots q_r^{s_r})^{\frac{1}{s}} \sqrt{\log x}} \frac{x^{\frac{1}{s}}}{\sqrt{1 - \frac{\log(q_1^{s_1} \dots q_r^{s_r})}{\log x}}} \\ & \leq \frac{c_1 \sqrt{s}}{\sqrt{1-\alpha}} \epsilon \frac{x^{\frac{1}{s}}}{\sqrt{\log x}} \end{aligned} \quad (3.26)$$

Substituting equations (3.26), (3.25) and (3.24) into (3.23) and then substituting (3.23) into (3.21) (see also (3.20)) we obtain

$$\left| \frac{C_s(x)}{\frac{x^{\frac{1}{s}}}{\sqrt{\log x}}} - C \right| \leq \frac{6}{\pi^2} B \sqrt{s} \epsilon + \epsilon + \epsilon + \epsilon + \frac{c_1 \sqrt{s}}{\sqrt{1-\alpha}} \epsilon \quad (x \geq x_\epsilon)$$

That is, equation (3.17), since ϵ can be arbitrarily small. Note that

$$\begin{aligned} C &= \sqrt{s} \frac{6}{\pi^2} B \left(1 + \frac{2}{3} \left(\left(\frac{1}{2^{\frac{1}{s}}} \right)^{s+1} + \left(\frac{1}{2^{\frac{1}{s}}} \right)^{s+2} + \dots \right) \right) \\ & \prod_{p \equiv 1, \text{mod}(4)} \left(1 + \frac{p}{p+1} \left(\left(\frac{1}{p^{\frac{1}{s}}} \right)^{s+1} + \left(\frac{1}{p^{\frac{1}{s}}} \right)^{s+2} + \dots \right) \right) \\ & \prod_{p \equiv 3, \text{mod}(4)} \left(1 + \left(\frac{1}{p^{\frac{1}{s}}} \right)^{s+1} + \left(\frac{1}{p^{\frac{1}{s}}} \right)^{s+3} + \dots \right) \end{aligned} \quad (3.27)$$

The theorem is proved. $\square$

Remark 3.6. If $s = 1$ then we have the 1-full numbers representable as sum of two squares, that is, the set of all positive integers representable as sum of two squares. In this case $C = B$ (see equation (3.27) and Theorem 3.4) and consequently equation (3.17) becomes equation (3.2) as must be.

Theorem 3.7. *Let s be an even number. Let us consider the s -full numbers not exceeding x representable as the sum of two squares. The number of these s -full numbers not exceeding x will be denote $C_s(x)$. The following formula holds.*

$$C_s(x) \sim Cx^{\frac{1}{s}} \quad (3.28)$$

where C is a positive constant and its value is

$$C = \frac{6}{\pi^2} \left(1 + \frac{1}{3} \frac{1}{2^{\frac{1}{s}} - 1}\right) \prod_{p \equiv 1, \text{mod}(4)} \left(1 + \frac{1}{p+1} \frac{1}{p^{\frac{1}{s}} - 1}\right) \prod_{p \equiv 3, \text{mod}(4)} \left(1 + \frac{1}{p+1} \frac{1}{p^{\frac{2}{s}} - 1}\right)$$

Proof. Let q be any square-free. If the s -full number is of the form q^s then the number of s -full numbers sum of two squares that satisfy the inequality $q^s \leq x$ will be

$$\frac{6}{\pi^2} x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right)$$

If the s -full number sum of two squares is of the form $q^s q_1^{s_1} \cdots q_r^{s_r}$, where $q_1^{s_1} \cdots q_r^{s_r}$ is a $(s+1)$ -full number sum of two squares, and consequently the primes of the form $4n-1$ (if there is some) have even exponent, then the number of s -full numbers that satisfy the inequality $q^s q_1^{s_1} \cdots q_r^{s_r} \leq x$ will be (note that the square-free q is not necessarily sum of two squares)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1) \cdots (q_r+1)} \frac{x^{\frac{1}{s}}}{(q_1^{s_1} \cdots q_r^{s_r})^{\frac{1}{s}}} + o\left(x^{\frac{1}{s}}\right)$$

The rest of the proof is as the former theorems. Note that

$$\begin{aligned} C &= \frac{6}{\pi^2} \left(1 + \frac{2}{3} \left(\left(\frac{1}{2^{\frac{1}{s}}}\right)^{s+1} + \left(\frac{1}{2^{\frac{1}{s}}}\right)^{s+2} + \cdots \right) \right) \\ &\quad \prod_{p \equiv 1, \text{mod}(4)} \left(1 + \frac{p}{p+1} \left(\left(\frac{1}{p^{\frac{1}{s}}}\right)^{s+1} + \left(\frac{1}{p^{\frac{1}{s}}}\right)^{s+2} + \cdots \right) \right) \\ &\quad \prod_{p \equiv 3, \text{mod}(4)} \left(1 + \frac{p}{p+1} \left(\left(\frac{1}{p^{\frac{1}{s}}}\right)^{s+2} + \left(\frac{1}{p^{\frac{1}{s}}}\right)^{s+4} + \cdots \right) \right) \end{aligned}$$

The theorem is proved. $\square$

We now give the second proof of Theorem 3.1 using equation (3.2) proved by Landau by use of a contour integral [7]

Theorem 3.8. *Let $N(x)$ be the number of square-free not exceeding x representable as the sum of two squares. The following formula holds*

$$N(x) \sim \frac{6}{\pi^2} B \frac{x}{\sqrt{\log x}} \quad (3.29)$$

Proof. We need the prime number theorem, namely, if $\pi(x)$ is the prime counting function then $\pi(x) \sim \frac{x}{\log x}$.

If $B(x) = N_1(x)$ is the number of numbers not exceeding x which are sum of two squares we have (see equation (3.2))

$$B(x) = B \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right)$$

Therefore we have

$$B(x) < CB \frac{x}{\sqrt{\log x}} \quad (x \geq 2)$$

where we can suppose $C > \sqrt{2}$ and

$$B(x) = 1 \quad (1 \leq x < 2)$$

The numbers representable as sum of two squares are the numbers such that in their prime factorization the primes (if there is some) $p \equiv 3(mod 4)$ have even exponent. Therefore the number of numbers representable as the sum of two squares multiples of the square n^2 is $B\left(\frac{x}{n^2}\right)$. That is

$$\begin{aligned} B\left(\frac{x}{n^2}\right) &= B \frac{x}{n^2} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log n^2}{\log x}}} + o\left(\frac{x}{n^2} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log n^2}{\log x}}}\right) \\ &= B \frac{x}{n^2} \frac{1}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) \end{aligned} \quad (3.30)$$

Let $A_{p_h}(x)$ be the number of numbers representable as the sum of two squares such that the prime p_h is the least prime with exponent greater than 1 in their prime factorization. The inclusion-exclusion principle and (3.30) give

$$\begin{aligned} A_{p_h}(x) &= B\left(\frac{x}{p_h^2}\right) - \sum_{1 \leq i \leq h-1} B\left(\frac{x}{p_h^2 p_i^2}\right) + \cdots + (-1)^{h-1} B\left(\frac{x}{p_h^2 p_1^2 \cdots p_{h-1}^2}\right) \\ &= B \frac{x}{\sqrt{\log x}} \left(\frac{1}{p_h^2} \prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i^2}\right)\right) + o\left(\frac{x}{\sqrt{\log x}}\right) \end{aligned} \quad (3.31)$$

We choose m such that

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^2} < \epsilon$$

and consequently also

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^2} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2}\right) < \epsilon$$

Besides, we put

$$\sum_{j=1}^{\infty} \frac{1}{p_j^2} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2}\right) = A$$

Let $A(x)$ be the number of not square-free not exceeding x representable as the sum of two squares. We have

$$\begin{aligned}
 A(x) &= \sum_{p_h^2 \leq x} A_{p_h}(x) \geq \sum_{j=1}^m A_{p_j}(x) \\
 &= B \frac{x}{\sqrt{\log x}} \left(\sum_{j=1}^m \frac{1}{p_j^2} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2} \right) \right) + o\left(\frac{x}{\sqrt{\log x}}\right) \\
 &\geq B \frac{x}{\sqrt{\log x}} (A - \epsilon) + o\left(B \frac{x}{\sqrt{\log x}}\right) \geq B \frac{x}{\sqrt{\log x}} (A - 2\epsilon) \\
 &\geq \left(A - 2 \frac{C}{\sqrt{2}} \epsilon \right) B \frac{x}{\sqrt{\log x}}
 \end{aligned} \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
 A(x) &= \sum_{p_h^2 \leq x} A_{p_h}(x) \leq \sum_{j=1}^m A_{p_j}(x) + \sum_{p_m^2 < p_h^2 \leq \sqrt{x}} A_{p_h}(x) + \sum_{\sqrt{x} < p_h^2 \leq \frac{x}{2}} A_{p_h}(x) \\
 &+ \sum_{\frac{x}{2} < p_h^2 \leq x} A_{p_h}(x)
 \end{aligned} \tag{3.33}$$

Now

$$\sum_{j=1}^m A_{p_j}(x) \leq B \frac{x}{\sqrt{\log x}} A + o\left(\frac{x}{\sqrt{\log x}}\right) \tag{3.34}$$

If $p_m^2 < p_h^2 \leq \sqrt{x}$ then we have $\frac{1}{\sqrt{1 - \frac{\log p_h^2}{\log x}}} \leq \frac{1}{\sqrt{2}}$. Hence

$$\begin{aligned}
 \sum_{p_m^2 < p_h^2 \leq \sqrt{x}} A_{p_h}(x) &\leq \sum_{p_m^2 < p_h^2 \leq \sqrt{x}} B \left(\frac{x}{p_h^2} \right) \\
 &\leq \sum_{p_m^2 < p_h^2 \leq \sqrt{x}} CB \frac{x}{p_h^2} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log p_h^2}{\log x}}} \leq \frac{CB}{\sqrt{2}} \frac{x}{\sqrt{\log x}} \sum_{p_m^2 < p_h^2 \leq \sqrt{x}} \frac{1}{p_h^2} \\
 &\leq \epsilon \frac{C}{\sqrt{2}} B \frac{x}{\sqrt{\log x}}
 \end{aligned} \tag{3.35}$$

If $\sqrt{x} < p_h^2 \leq \frac{x}{2}$ then we have $\frac{1}{\sqrt{1 - \frac{\log p_h^2}{\log x}}} \leq \frac{\sqrt{\log x}}{\sqrt{\log 2}}$ and $\frac{x}{p_h^2} < \sqrt{x}$. Hence

$$\begin{aligned}
 \sum_{\sqrt{x} < p_h^2 \leq \frac{x}{2}} A_{p_h}(x) &\leq \sum_{\sqrt{x} < p_h^2 \leq \frac{x}{2}} B \left(\frac{x}{p_h^2} \right) \leq \sum_{\sqrt{x} < p_h^2 \leq \frac{x}{2}} CB \frac{x}{p_h^2} \frac{1}{\sqrt{\log x}} \frac{1}{\sqrt{1 - \frac{\log p_h^2}{\log x}}} \\
 &\leq \frac{CB}{\sqrt{\log 2}} \sqrt{x} \sum_{\sqrt{x} < p_h^2 \leq \frac{x}{2}} 1 \leq \frac{CB}{\sqrt{\log 2}} \sqrt{x} \pi(\sqrt{x}) = \frac{CB}{\sqrt{\log 2}} \sqrt{x} \frac{\sqrt{x}}{\log \sqrt{x}} (1 + o(1)) \\
 &= o\left(\frac{x}{\sqrt{\log x}}\right)
 \end{aligned} \tag{3.36}$$

Finally, if $\frac{x}{2} < p_h^2 \leq x$ then we have $B\left(\frac{x}{p_h^2}\right) = 1$. Therefore

$$\sum_{\frac{x}{2} < p_h^2 \leq x} A_{p_h}(x) \leq \sum_{\frac{x}{2} < p_h^2 \leq x} B\left(\frac{x}{p_h^2}\right) = \sum_{\frac{x}{2} < p_h^2 \leq x} 1 \leq \pi(\sqrt{x}) = o\left(\frac{x}{\sqrt{\log x}}\right) \quad (3.37)$$

Substituting (3.34), (3.35), (3.36) and (3.37) into (3.33) we obtain

$$\begin{aligned} A(x) &\leq B \frac{x}{\sqrt{\log x}} A + \frac{\epsilon C}{\sqrt{2}} B \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) \\ &\leq \left(A + 2 \frac{C}{\sqrt{2}} \epsilon\right) B \frac{x}{\sqrt{\log x}} \end{aligned} \quad (3.38)$$

Equations (3.32) and (3.38) give

$$\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = A$$

Now, note that we have the following equality (use mathematical induction)

$$\sum_{j=1}^n \frac{1}{p_j^2} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^2}\right) = 1 - \prod_{i=1}^n \left(1 - \frac{1}{p_i^2}\right)$$

Consequently using the well-known limit

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n \left(1 - \frac{1}{p_i^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

we find that $A = 1 - \frac{6}{\pi^2}$. Therefore the following limit holds

$$\lim_{x \rightarrow \infty} \frac{B(x) - A(x)}{B(x)} = \frac{6}{\pi^2}$$

where $B(x) - A(x)$ is the number of square-free not exceeding x representable as the sum of two squares. The theorem is proved. $\square$

Now, we prove the following theorem. Let $P(n)$ be the greatest prime factor of n . We denote r a number representable as the sum of two squares and $B(x)$ is their counting function (see (3.2)).

Theorem 3.9. *The following asymptotic formula holds*

$$\sum_{r \leq x} P(r) = \frac{1}{4} E_{m,r} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (3.39)$$

where the positive constant $E_{m,r}$ is

$$E_{m,r} = \sum_{k=1}^{\infty} B(k) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \quad (3.40)$$

Proof. In this proof p denotes a prime of the form $4n+1$. The prime number theorem is

$$\sum_{p \leq x} 1 = \frac{1}{2} \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad (3.41)$$

Therefore Abel summation gives

$$\sum_{p \leq x} p = \frac{1}{4} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (3.42)$$

and if k is a positive integer

$$\sum_{\frac{x}{k+1} < p \leq \frac{x}{k}} p = \frac{1}{4} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (3.43)$$

Let s be a positive integer. Let us consider the numbers $k = 1, 2, \dots, s$ and the inequalities

$$\frac{x}{k+1} < p \leq \frac{x}{k} \quad (3.44)$$

Then the multiple of p not exceeding x are $p.1, p.2, \dots, p.k$ and if x is sufficiently large p is the greatest prime factor with exponent 1 of these numbers and consequently p is a prime of the form $4n+1$. The number of numbers r in the set $1, 2, \dots, k$ is $B(k)$. Now, the proof follows as the proof of Theorem 1.19 in the reference [6]. The theorem is proved. $\square$

Theorem 3.10. *Let us consider the numbers r not exceeding x . Their counting function is $B(x)$. Let $N(x)$ be the number of numbers r not exceeding x such that $P(r)$ has exponent 1 in the prime factorization of r and consequently $P(r)$ is a prime of the form $4n+1$. Then $N(x) \sim B(x) \sim B\frac{x}{\sqrt{\log x}}$, that is, almost all greatest prime factor has exponent 1 in the numbers r and it is of the form $4n+1$. Therefore the numbers r such that $P(r)$ is a prime of the form $4n-1$ or of the form $4n+1$ with exponent greater than 1 are in quantity $o\left(\frac{x}{\sqrt{\log x}}\right)$, that is, negligible.*

Proof. The proof is the same as the proof of Theorem 3.5 with $s = 1$ (see also Remark 3.6 and the proof of Theorem 1.21 in the reference [6]). The theorem is proved. $\square$

The numbers not exceeding x representable as the sum of two squares such that all their prime factors are of the form $4n+3$ are squares. Therefore their number is negligible, namely $O(\sqrt{x})$. Perhaps we can think that also the numbers sum of two squares with some prime factor of the form $4n+3$ are negligible. This is not true, as we shall prove in the following theorem.

Theorem 3.11. *Suppose that p_i is the sequence of primes of the linear form $4n+3$. Then the number $N(x)$ of numbers sum of two squares not exceeding x and multiple of some prime p_i ($i = 1, \dots, n$) has positive density $0 < 1-a < 1$ such that the value of a is*

$$\prod_{p \equiv 3, \text{mod}(4)} \left(1 - \frac{1}{p^2}\right)$$

That is

$$N(x) \sim B(1-a) \frac{x}{\sqrt{\log x}} \sim (1-a)B(x)$$

Proof. By inclusion-exclusion principle the number of numbers sum of two squares and multiple of some prime p_i ($i = 1, \dots, n$) is

$$\begin{aligned} & \sum_{p_i \leq p_n} B\left(\frac{x}{p_i^2}\right) - \sum_{p_i < p_j \leq p_n} B\left(\frac{x}{p_i^2 p_j^2}\right) + \dots \\ &= B \frac{x}{\sqrt{\log x}} \left(1 - \prod_{i=1}^n \left(1 - \frac{1}{p_i^2}\right)\right) + o\left(\frac{x}{\sqrt{\log x}}\right) + F(x) \\ &= B \frac{x}{\sqrt{\log x}} (1 - a) - \left(\prod_{i=1}^n \left(1 - \frac{1}{p_i^2}\right) - a\right) B \frac{x}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right) + F(x) \end{aligned}$$

where by use of the inequality $B(x) < c \frac{x}{\sqrt{\log x}}$ ($x \geq 2$) and if we put $\frac{1}{2} < \beta < 1$ we find that

$$\begin{aligned} 0 \leq F(x) &\leq c \frac{x}{\sqrt{\log x}} \sum_{p_{n+1}^2 \leq p_i^2 \leq x^\beta} \frac{1}{p_i^2} \frac{1}{\sqrt{1 - \frac{\log p_i^2}{\log x}}} \\ &+ c \frac{x}{\sqrt{\log x}} \sum_{x^\beta \leq p_i^2 \leq \frac{x}{2}} \frac{1}{p_i^2} \frac{1}{\sqrt{1 - \frac{\log p_i^2}{\log x}}} + \sum_{\frac{x}{2} \leq p_i^2 \leq x} \frac{x}{p_i^2} \end{aligned}$$

Now, the proof follows as the former theorems. The theorem is proved. $\square$

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