

k -FREE AND k -FULL NUMBERS WITH PRIME FACTORS IN ARITHMETIC PROGRESSION

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ABSTRACT. We apply a classical Wirsing's theorem and study k -free numbers, k -full numbers and perfect powers with prime factors in an arithmetic progression.

1. k -FREE NUMBERS, k -FULL NUMBERS, PERFECT POWERS, ARITHMETIC PROGRESSION

Let us consider the prime factorization of a positive integer $n = q_1^{s_1} \cdots q_r^{s_r}$, where the q_i ($i = 1, \dots, r$) are the distinct primes and the s_i ($i = 1, \dots, r$) are the multiplicity or exponents. The positive integer is called k -free number ($k \geq 2$) if and only if $s_i \leq k - 1$ ($i = 1, \dots, r$). If $k = 2$ then the numbers are called square-free. The positive integer is called h -full ($h \geq 2$) if and only if $s_i \geq h$ ($i = 1, \dots, r$). If $h = 2$ then the numbers are called square-full or powerful.

Theorem 1.1. *Let $n_{k,x}$ be a k -free number such that in their prime factorization all prime factor does not exceed x and it is of the linear form $mn + r$, where $\gcd(m, r) = 1$. We have the asymptotic formula*

$$\sum_{n_{k,x}} \frac{1}{n_{k,x}} \sim \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^k}\right) \exp \left(A_{m,r} + \frac{1}{\varphi(m)} \gamma \right) (\log x)^{\frac{1}{\varphi(m)}} \quad (1.1)$$

In particular, for square-free numbers equation (1.1) becomes

$$\sum_{n_{2,x}} \frac{1}{n_{2,x}} \sim \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^2}\right) \exp \left(A_{m,r} + \frac{1}{\varphi(m)} \gamma \right) (\log x)^{\frac{1}{\varphi(m)}} \quad (1.2)$$

Let us consider the fixed square-free $q = q_1 \cdots q_r$, where q_i ($i = 1, \dots, r$) are the distinct prime factors in the prime factorization, these prime factors are of the linear form $mn + r$, where $\gcd(m, r) = 1$. If we consider only the square-free $n_{2,x}$

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relatively prime to q then equation (1.2) becomes

$$\sum_{n_{2,x}} \frac{1}{n_{2,x}} \sim \left(\prod_{i=1}^r \frac{q_i}{q_i + 1} \right) \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^2} \right) X \exp \left(A_{m,r} + \frac{1}{\varphi(m)} \gamma \right) (\log x)^{\frac{1}{\varphi(m)}} \quad (1.3)$$

Let n_x be a positive integer such that in their prime factorization all prime factor does not exceed x and it is of the linear form $mn + r$, where $\gcd(m, r) = 1$. We have the asymptotic formula

$$\sum_{n_x} \frac{1}{n_x} \sim \exp \left(A_{m,r} + \frac{1}{\varphi(m)} \gamma \right) (\log x)^{\frac{1}{\varphi(m)}} \quad (1.4)$$

where the constant $A_{m,r}$ was defined in Example 1.5 of [8].

Proof. We have (see equation (1.22) in reference [8])

$$\begin{aligned} \sum_{n_{k,x}} \frac{1}{n_{k,x}} &= \prod_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{k-1}} \right) = \prod_{\substack{p \leq x \\ p \equiv r, \text{mod}(m)}} \frac{1 - \frac{1}{p^k}}{1 - \frac{1}{p}} \\ &\sim \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^k} \right) \exp \left(A_{m,r} + \frac{1}{\varphi(m)} \gamma \right) (\log x)^{\frac{1}{\varphi(m)}} \end{aligned}$$

That is, equation (1.1). The proof of equation (1.4) is the same. In this case we consider the geometric formula

$$1 + \frac{1}{p} + \frac{1}{p^2} + \cdots = \frac{1}{1 - \frac{1}{p}}$$

The theorem is proved. $\square$

Definition 1.2. Let us consider a set A of infinite primes p where the set of exponents of p depends of p and it is a certain subset of the positive integers denoted $E(p)$. Now consider the positive integers whose prime factorization is of the form $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$, where the distinct primes q_i ($i = 1, \dots, k$) are in the set A and the exponent a_i is in the set $E(q_i)$ ($i = 1, \dots, k$). The set of these positive integers n will be denoted B . The set of primes p in the set A such that $E(p)$ contains the number 1 and consequently p is in the set B will be denoted A_1 and the set of the rest of the primes p such that 1 is not in the set $E(p)$ will be denoted A_2 . The set B of positive integers n defined in this way is called multiplicative set.

Theorem 1.3. Let us consider a multiplicative set B and consider the positive integers n in the set B such that all their prime factors does not exceed x . We denote these elements in the set B in the form n_x . The following asymptotic formula holds

$$\sum_{n_x} \frac{1}{n_x} \sim C_B e^{\sum_{p \leq x} \frac{1}{p}} \quad (1.5)$$

where the sum run over all n_x and C_B is a positive constant similar to the constants obtained in section 1 of the reference [8].

Proof. We have

$$\begin{aligned} \sum_{n_x} \frac{1}{n_x} &= \prod_{\substack{p \leq x \\ p \in A}} \left(1 + \sum_{k \in E_p} \frac{1}{p^k} \right) = \prod_{\substack{p \leq x \\ p \in A_2}} \left(1 + \sum_{k \in E_p} \frac{1}{p^k} \right) \prod_{\substack{p \leq x \\ p \in A_1}} \left(1 + \frac{1}{p} + \sum_{\substack{k \in E_p \\ k \geq 2}} \frac{1}{p^k} \right) \\ &\sim \left(\prod_{p \in A_2} \left(1 + \sum_{k \in E_p} \frac{1}{p^k} \right) \right) \prod_{\substack{p \leq x \\ p \in A_1}} \left(1 + \frac{1}{p} + \sum_{\substack{k \in E_p \\ k \geq 2}} \frac{1}{p^k} \right) \end{aligned}$$

The rest of the proof is as the proofs of the theorems in section 1 of reference [8]. The theorem is proved. $\square$

Example 1.4. Let us consider the positive integers of the form $n = a^2 + b^2$ where a and b are nonnegative integers. That is, the numbers such that their prime factors of the form $4n + 3$ have even exponent in their prime factorization [9]. Let $B(x)$ be the number of these numbers not exceeding x . It is well-known [9] the asymptotic formula

$$B(x) \sim B \frac{x}{\sqrt{\log x}} \quad (1.6)$$

where B is Landau-Ramanujan constant and its value is [1]

$$B = \frac{1}{\sqrt{2}} \prod_{p \equiv 3, \text{mod}(4)} \left(1 - \frac{1}{p^2} \right)^{-\frac{1}{2}} = \frac{\pi}{4} \prod_{p \equiv 1, \text{mod}(4)} \left(1 - \frac{1}{p^2} \right)^{\frac{1}{2}} \quad (1.7)$$

Equation (1.6) and Abel summation give

$$\sum_{n=a^2+b^2 \leq x} \frac{1}{n} \sim 2B \sqrt{\log x} \quad (1.8)$$

On the other hand, it is well-known that [1]

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv 3, \text{mod}(4)}} \frac{1}{p} &= \frac{1}{2} \log \log x \\ &+ \left(\ln \left(\frac{2B}{\sqrt{\pi}} \right) + \frac{\gamma}{2} + \sum_{p \equiv 3, \text{mod}(4)} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \right) + o(1) \\ \sum_{\substack{p \leq x \\ p \equiv 1, \text{mod}(4)}} \frac{1}{p} &= \frac{1}{2} \log \log x \\ &+ \left(\ln \left(\frac{\sqrt{\pi}}{4B} \right) + \frac{\gamma}{2} + \sum_{p \equiv 1, \text{mod}(4)} \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \right) + o(1) \end{aligned}$$

Let us consider the positive integers of the form $n_x = a^2 + b^2$ where a and b are nonnegative integers and such that all their prime factors does not exceed x . Then we obtain (compare with (1.8))

$$\sum_{n_x=a^2+b^2} \frac{1}{n_x} \sim e^{\frac{\gamma}{2}} \sqrt{\pi} B \sqrt{\log x}$$

We shall need the following general theorem.

Theorem 1.5. *Let $k \geq 1$ fixed. Let us consider the positive real numbers r_i ($i = 1, 2, \dots, k$) and consider the inequality*

$$r_1 x_1 + r_2 x_2 + \dots + r_k x_k \leq x$$

Let $S_k(x)$ be the number of solutions $(x_1, x_2, \dots, x_k)$ where x_i ($i = 1, 2, \dots, k$) are positive integers. Then the following asymptotic formula holds

$$S_k(x) \sim \frac{1}{k!} \frac{x^n}{r_1 \dots r_k}$$

and the following inequality holds

$$S_k(x) \leq \frac{1}{k!} \frac{x^n}{r_1 \dots r_k} \quad (x \geq 0)$$

Proof. See [5], [6]. The theorem is proved. $\square$

Let us consider a set A of infinite primes p such that the prime counting function is denoted $C(x)$ and where the set of exponents of p are all positive integers, that is $E(p)$ is the set of all positive integers. Therefore the multiplicative set B is formed by the numbers a such that their prime factorization is $a = q_1^{a_1} \dots q_k^{a_k}$, where the distinct primes q_i ($i = 1, \dots, k$) are in the set A and the exponents a_i are arbitrary. The counting function of the set B will be denoted $A(x)$. We have

$$\lim_{x \rightarrow \infty} \frac{A(x)}{(\log x)^\alpha} = \infty \quad (1.9)$$

for all $\alpha > 0$. Since given k fixed distinct primes $q_1, \dots, q_k$ in the infinite set A the number of numbers $a = q_1^{a_1} \dots q_k^{a_k}$ in the set B , denoted $A_{q_1, \dots, q_k}(x)$ will be by Theorem 1.5,

$$A_{q_1, \dots, q_k}(x) \sim \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \dots \log q_k}$$

and if $x \geq 1$ we have the inequality

$$A_{q_1, \dots, q_k}(x) \leq \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \dots \log q_k}$$

Therefore an upper bound for $A(x)$ is

$$A(x) = \sum_{q_1 \dots q_k \leq x} A_{q_1, \dots, q_k}(x) \leq \sum_{q_1 \dots q_k \leq x} \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \dots \log q_k} \quad (1.10)$$

where the sum run over all square-free $q_1 \dots q_k$ not exceeding x in the set B .

The square-free numbers are an important number in the set of positive integers since, as it is well-known, they have positive density $\frac{6}{\pi^2}$ [2]. On the other hand,

the square-full numbers are negligible, since their number is $o(x) = O(\sqrt{x})$, as it is well-known [3] [4].

In the multiplicative sets such that the number of primes $C(x)$ is very small, for example, $C(x) \leq \log \log x$ the contrary case occur. Since the number of square-free numbers in the multiplicative set B not exceeding x does not exceed

$$1 + \binom{C(x)}{1} + \binom{C(x)}{2} + \cdots + \binom{C(x)}{C(x)} = 2^{C(x)} < e^{\log \log x} = \log x$$

On the other hand, the number of square-full numbers not exceeding x in the multiplicative set B is greater than $(\log x)^\alpha$ for all positive α . Since given k arbitrarily large and k distinct primes $q_1, \dots, q_k$ in the set A the number of squares of the form $q_1^{2a_1} \cdots q_k^{2a_k}$ in the set B is by Theorem 1.5 asymptotically equal to

$$\frac{1}{k!} \frac{(\log x)^k}{\log q_1^2 \cdots \log q_k^2} = \frac{1}{2^k} \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \cdots \log q_k}$$

Therefore in the multiplicative set B the number of square-free is negligible compared with the number of squares and consequently square-full numbers. We have, by Theorem 1.5, the following upper bound for the number of perfect powers of the form $q_1^{a_1} \cdots q_k^{a_k}$, where q_i ($i = 1, \dots, k$) are fixed,

$$\frac{1}{2^k} \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \cdots \log q_k} + \frac{1}{3^k} \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \cdots \log q_k} + \cdots = (\zeta(k) - 1) \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \cdots \log q_k}$$

If $C(x)$ is small then we have the following upper bound for the number of elements $A(x)$ in the multiplicative set B

$$\begin{aligned} A(x) &= \sum_{q_1 \cdots q_k \leq x} A_{q_1, \dots, q_k}(x) \leq \sum_{q_1 \cdots q_k \leq x} \frac{1}{k!} \frac{(\log x)^k}{\log q_1 \cdots \log q_k} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{q \leq x} \frac{1}{\log q} \right)^k \frac{1}{k!} (\log x)^k \\ &\leq \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{\log x} \sqrt{\sum_{q \leq x} \frac{1}{\log q}} \right)^k \right)^2 = \exp \left(2 \sqrt{\log x} \sqrt{\sum_{q \leq x} \frac{1}{\log q}} \right) \end{aligned}$$

where the sum run over all prime q not exceeding x in the set A . We study the following cases,

Case 1) $C(x) \leq (\log x)^\alpha$, where $0 < \alpha < 1$. In this case Abel summation gives $\sum_q \frac{1}{\log q} = C > 0$, that is, the series converges. Therefore

$$A(x) \leq \exp \left(2 \sqrt{\log x} \sqrt{C} \right) = o(x^\epsilon)$$

for all $\epsilon > 0$.

Case 2) $C(x) \leq \log x$. In this case Abel summation gives $\sum_{q \leq x} \frac{1}{\log q} < C \log \log x$, where C is a positive constant. Therefore

$$A(x) \leq \exp \left(2 \sqrt{\log x} \sqrt{C \log \log x} \right) = o(x^\epsilon)$$

for all $\epsilon > 0$.

Case 3) $C(x) \leq (\log x)^\alpha$, where $1 < \alpha < 2$. In this case Abel summation gives $\sum_{q \leq x} \frac{1}{\log q} < C(\log x)^{\alpha-1}$, where C is a positive constant. Therefore

$$A(x) \leq \exp \left(2\sqrt{C}(\log x)^{\frac{\alpha}{2}} \right) = o(x^\epsilon)$$

for all $\epsilon > 0$.

Now, we shall prove the following theorem about composite numbers, in the sequence of all positive integers. As usual, p denotes a positive prime.

Theorem 1.6. *Let $B(x)$ be the number of composite numbers not exceeding x such that their prime factorization is of the form $q_1^{a_1} \cdots q_k^{a_k}$ ($q_i \leq x^{\frac{1}{k}}$) ($k = 2, 3, \dots$). The number of these composite numbers is negligible, more precise, their magnitude order is $\frac{x}{\log^2 x}$.*

Proof. Abel summation gives

$$\sum_{p \leq x} \frac{1}{\log p} = \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right) < C \frac{x}{\log^2 x} \quad (x \geq 2)$$

Now, we have

$$B(x) \geq \left(\pi \left(x^{\frac{1}{2}} \right) \right) > C_2 \frac{x}{\log^2 x}$$

where C_2 is a positive constant. On the other hand, we have

$$\begin{aligned} B(x) &\leq \sum_{k=2}^{\infty} \frac{1}{k!} \left(\sum_{p \leq x^{\frac{1}{k}}} \frac{1}{\log p} \right)^k \frac{1}{k!} (\log x)^k \\ &\leq \sum_{k=2}^{\infty} \frac{1}{(k!)^2} \left(C \frac{x^{\frac{1}{k}}}{\log^2 x^{\frac{1}{k}}} \right)^k (\log x)^k = x \sum_{k=2}^{\infty} \frac{(k^k)^2}{(k!)^2} \left(\frac{C}{\log x} \right)^k \\ &\leq x \sum_{k=2}^{\infty} \left(\frac{eC}{\log x} \right)^k = x \left(\frac{eC}{\log x} \right)^2 \frac{1}{1 - \frac{eC}{\log x}} < C_1 \frac{x}{\log^2 x} \end{aligned}$$

where C_1 is a positive constant. Therefore $C_2 \frac{x}{\log^2 x} < B(x) < C_1 \frac{x}{\log^2 x}$. The theorem is proved. $\square$

If $C_{m,r}(x)$ is the number of primes not exceeding x in the arithmetic progression $mn + r$, where $\gcd(m, r) = 1$, then the prime number theorem for arithmetic progressions is [9]

$$C_{m,r}(x) = \frac{1}{\varphi(m)} \int_2^x \frac{1}{\log t} dt + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right) \sim \frac{1}{\varphi(m)} \frac{x}{\log x}$$

where a is a positive constant.

We have the following fundamental result of Wirsing [10]

Theorem 1.7. *Let us consider the set of numbers such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. A generic number of this set will be denoted $n_{m,r}$. Let $A_{m,r}(x)$ be the number of these numbers not exceeding x . Then*

$$A_{m,r}(x) = \sum_{n_{m,r} \leq x} 1 \sim C_{m,r} \frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \quad (1.11)$$

where $C_{m,r}$ is a positive constant and its value is $C_{m,r} = \frac{e^{A_{m,r}}}{\Gamma(\frac{1}{\varphi(m)})}$, where $\Gamma(s)$ is the Gamma function and the constant $A_{m,r}$ was defined in equation (1.20) of the reference [8].

The following asymptotic formula holds.

$$\sum_{n_{m,r} \leq x} \frac{1}{n_{m,r}} \sim \varphi(m) C_{m,r} (\log x)^{\frac{1}{\varphi(m)}} \quad (1.12)$$

Proof. Equation (1.12) is an immediate consequence of equation (1.11) and Abel summation. Equation (1.11) was proved by Wirsing [10]. The theorem is proved. $\square$

Remark 1.8. Compare equation (1.12) with equation (1.4). They have the same magnitude order, namely $(\log x)^{\frac{1}{\varphi(m)}}$.

Theorem 1.9. *Suppose that the primes in the set A have a counting function $C(x)$ of magnitude order $\frac{x}{(\log x)^\delta}$, where $\delta > 1$. That is, there exist two positive constants D_1 and D_2 such that*

$$D_1 \frac{x}{(\log x)^\delta} < C(x) < D_2 \frac{x}{(\log x)^\delta} \quad (1.13)$$

If $A(x) = \sum_{a \leq x} 1$, where the a belong to the multiplicative set B , we have also

$$D_3 \frac{x}{(\log x)^\delta} < A(x) < D_4 \frac{x}{(\log x)^\delta} \quad (1.14)$$

where D_3 and D_4 are positive constants.

Proof. The numbers a in the multiplicative set B can be divided in three disjoint sets A_1 , A_2 and A_3 . The set A_1 contains the numbers a which are square-free, we shall denote q these square-free and $A_1(x)$ the number of these square-free not exceeding x . The set A_3 contains the numbers a which are square-full, we shall denote P these square-full and $A_3(x)$ the number of these square-full not exceeding x . The set A_2 contains the numbers a of the form qP , where q and P are relatively prime, that is $(q, P) = 1$. We shall denote $A_2(x)$ the number of these numbers not exceeding x . We have (p denotes a positive prime)

$$C(x) = \sum_{p \leq x, p \in A} 1 \leq c \frac{x}{(\log x)^\delta} \quad (x \geq 2) \quad (1.15)$$

and consequently the series of the reciprocal converges, that is

$$\sum_{p \in A} \frac{1}{p} = C_1 > 0 \quad (1.16)$$

Let us consider a square-free $q \in A_1$ with k prime factors, that is, $q = p_1 \cdots p_{k-1}p_k$, where $p_1 < \cdots < p_{k-1} < p_k$ are distinct primes. The number of these square-free not exceeding x will be denoted $D_k(x)$. Let $\beta = 1 - \frac{1}{2k} < 1$. Note that there are not square-free $p_1 \cdots p_{k-1}p_k$ not exceeding x such that $p_1 \cdots p_{k-1} > x^\beta$. Since if $p_1 \cdots p_{k-1} > x^\beta$ we obtain $p_k > p_{k-1} > x^{\frac{\beta}{k-1}}$ and consequently $p_1 \cdots p_{k-1}p_k > x^\beta x^{\frac{\beta}{k-1}} > x$. Note also that the inequality $p_1 \cdots p_{k-1} \leq x^\beta$ implies the inequality $\frac{1}{1 - \frac{\log(p_1 \cdots p_{k-1})}{\log x}} \leq \frac{1}{1-\beta}$. Therefore equations (1.15) and (1.16) give

$$\begin{aligned} D_k(x) &\leq \sum_{\substack{p_1 \cdots p_{k-1} \leq x^\beta \\ p_i \in A, (i=1, \dots, k-1)}} C \left(\frac{x}{p_1 \cdots p_{k-1}} \right) \\ &\leq \sum_{\substack{p_1 \cdots p_{k-1} \leq x^\beta \\ p_i \in A, (i=1, \dots, k-1)}} c \frac{x}{p_1 \cdots p_{k-1}} \frac{1}{\left(\log \left(\frac{x}{p_1 \cdots p_{k-1}} \right) \right)^\delta} \leq \frac{c}{(1-\beta)^\delta} \frac{x}{(\log x)^\delta} \frac{C_1^{k-1}}{(k-1)!} \\ &\leq c \frac{x}{(\log x)^\delta} \frac{(C_2 C_1)^{k-1}}{(k-1)!} \end{aligned} \quad (1.17)$$

Note that $(2k)^{\delta \frac{1}{k-1}} \rightarrow 1$ and consequently if $k \geq 2$ we have $(2k)^{\delta \frac{1}{k-1}} < C_2$, where $C_2 > 0$.

Therefore we have

$$\begin{aligned} A_1(x) &\leq \sum_{k=1}^{\infty} c \frac{x}{(\log x)^\delta} \frac{(C_2 C_1)^{k-1}}{(k-1)!} = c \frac{x}{(\log x)^\delta} e^{C_2 C_1} \\ &\leq C_3 \frac{x}{(\log x)^\delta} \quad (x \geq 2) \end{aligned} \quad (1.18)$$

Equation (1.18) gives

$$\begin{aligned} A_2(x) &\leq \sum_{P \leq \frac{x}{2}} \left(\sum_{q \leq \frac{x}{P}} 1 \right) \leq \sum_{P \leq \frac{x}{2}} C_3 \frac{x}{P} \frac{1}{\log^\delta \left(\frac{x}{P} \right)} \\ &= C_3 \frac{x}{\log^\delta x} \sum_{P \leq \frac{x}{2}} \frac{1}{P} \frac{1}{\left(1 - \frac{\log P}{\log x} \right)^\delta} \end{aligned} \quad (1.19)$$

Let us consider the function

$$\sum_{P \leq \frac{x}{2}} \frac{1}{P} \frac{1}{\left(1 - \frac{\log P}{\log x} \right)^\delta}$$

Let P_n be the n -th square-full. If we put $x = 2P_n$ then we obtain

$$\begin{aligned} \sum_{P \leq P_n} \frac{1}{P} \frac{1}{\left(1 - \frac{\log P}{\log 2P_n}\right)^\delta} &= \sum_{i=1}^n \frac{1}{P_i} \frac{1}{\left(1 - \frac{\log P_i}{\log 2P_n}\right)^\delta} \\ &\leq \sum_{i=1}^n \frac{1}{P_i} \frac{1}{\left(1 - \frac{\log P_i}{\log 2P_i}\right)^\delta} \leq \sum_{i=1}^n \frac{1}{P_i} \left(1 + \frac{\log P_i}{\log 2}\right)^{[\delta]+1} < C_4 \end{aligned}$$

where C_4 is a positive constant. Note that the series $\sum_{P_i} \frac{\log^\alpha P_i}{P_i}$ converges for all $\alpha \geq 0$.

Note also that if $x \in [2P_n, 2P_{n+1})$ the function $(P_i \text{ fixed}) \frac{1}{P_i} \frac{1}{\left(1 - \frac{\log P_i}{\log x}\right)^\delta}$ is strictly decreasing. Therefore we have

$$\sum_{P \leq \frac{x}{2}} \frac{1}{P} \frac{1}{\left(1 - \frac{\log P}{\log x}\right)^\delta} < C_4 \quad (1.20)$$

and consequently (see (1.19) and (1.20))

$$A_2(x) \leq C_5 \frac{x}{\log^\delta x} \quad (1.21)$$

where C_5 is a positive constant. On the other hand, we have [3][4]

$$A_3(x) \leq C_6 \sqrt{x} \quad (1.22)$$

Therefore (see (1.18), (1.21) and (1.22))

$$A(x) = A_1(x) + A_2(x) + A_3(x) \leq D_4 \frac{x}{(\log x)^\delta} \quad (1.23)$$

where D_4 is a positive constant. On the other hand, we have the trivial inequality $C(x) \leq A(x)$. This proves equation (1.14). The theorem is proved. $\square$

Theorem 1.10. *Let us consider the set $S_{m,r,k}$ of k -free numbers such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let $S_{m,r,k}(x)$ be the number of these numbers not exceeding x . Then*

$$S_{m,r,k}(x) \sim S_{m,r,k} \frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \quad (1.24)$$

where $S_{m,r,k}$ is the positive constant

$$S_{m,r,k} = C_{m,r} \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^k}\right) \quad (1.25)$$

Note that $C_{m,r}$ is the constant in Theorem 1.7.

Proof. We have (Theorem 1.7)

$$A_{m,r}(x) = C_{m,r} \frac{x}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \quad (1.26)$$

where for sake of simplicity we put $\alpha = 1 - \frac{1}{\varphi(m)}$.

Therefore we have

$$A_{m,r}(x) < C \frac{x}{(\log x)^\alpha} \quad (x \geq 2) \quad (1.27)$$

where C is a positive constant.

Let p_n be the n -th prime in the linear form $mn+r$. We work with this sequence of primes.

Let $A_{p_h}(x)$ be the number of numbers with their prime factors of the linear form $mn+r$ such that the prime p_h is the least prime with exponent greater than or equal to k in their prime factorization. The inclusion-exclusion principle and (1.26) give

$$\begin{aligned} A_{p_h}(x) &= A_{m,r} \left(\frac{x}{p_h^k} \right) - \sum_{1 \leq i \leq h-1} A_{m,r} \left(\frac{x}{p_h^k p_i^k} \right) \\ &+ \cdots + (-1)^{h-1} A_{m,r} \left(\frac{x}{p_h^k p_1^k \cdots p_{h-1}^k} \right) \\ &= C_{m,r} \frac{x}{(\log x)^\alpha} \left(\frac{1}{p_h^k} \prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i^k} \right) \right) + o \left(\frac{x}{(\log x)^\alpha} \right) \end{aligned} \quad (1.28)$$

We choose m such that

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^k} < \epsilon \quad (1.29)$$

and consequently also

$$\sum_{j=m+1}^{\infty} \frac{1}{p_j^k} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^k} \right) < \epsilon \quad (1.30)$$

Besides, we put

$$C_{m,r} \sum_{j=1}^{\infty} \frac{1}{p_j^k} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^k} \right) = A \quad (1.31)$$

Let $A(x)$ be the number of not k -free numbers not exceeding x with their prime factors of the linear form $mn+r$. We have (see (1.28), (1.30) and (1.31))

$$\begin{aligned} A(x) &= \sum_{p_h^k \leq x} A_{p_h}(x) = \sum_{j=1}^m A_{p_j}(x) + F(x) \\ &= C_{m,r} \frac{x}{(\log x)^\alpha} \left(\sum_{j=1}^m \frac{1}{p_j^k} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^k} \right) \right) + o \left(\frac{x}{(\log x)^\alpha} \right) + F(x) \\ &= A \frac{x}{(\log x)^\alpha} - C_{m,r} \frac{x}{(\log x)^\alpha} \left(\sum_{j=m+1}^{\infty} \frac{1}{p_j^k} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^k} \right) \right) + o \left(\frac{x}{(\log x)^\alpha} \right) \\ &+ F(x) \end{aligned} \quad (1.32)$$

Now (see (1.32) and (1.27))

$$\begin{aligned}
 0 \leq F(x) &= \sum_{p_{m+1} \leq p_h \leq x^{\frac{1}{k}}} A_{p_h}(x) \leq \sum_{p_{m+1} \leq p_h \leq \frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}}} C \frac{x}{p_h^k} \frac{1}{\left(\frac{x}{p_h^k}\right)^\alpha} \\
 &+ \sum_{\frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}} < p_h \leq x^{\frac{1}{k}}} \frac{x}{p_h^k}
 \end{aligned} \tag{1.33}$$

where

$$0 \leq \sum_{\frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}} < p_h \leq x^{\frac{1}{k}}} \frac{x}{p_h^k} \leq 2 \sum_{p_h \leq x^{\frac{1}{k}}} 1 \leq 2x^{\frac{1}{k}} = o\left(\frac{x}{(\log x)^\alpha}\right) \tag{1.34}$$

and if we put $\beta = \frac{1}{k} + \frac{1}{2k}$ then we obtain

$$\begin{aligned}
 &\sum_{p_{m+1} \leq p_h \leq x^{\frac{1}{k}}} A_{p_h}(x) \leq \sum_{p_{m+1} \leq p_h \leq \frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}}} C \frac{x}{p_h^k} \frac{1}{\left(\log\left(\frac{x}{p_h^k}\right)\right)^\alpha} \\
 &= \frac{x}{(\log x)^\alpha} C \sum_{p_{m+1} \leq p_h \leq x^{\frac{\beta}{k}}} \frac{1}{p_h^k} \frac{1}{\left(1 - \frac{\log p_h^k}{\log x}\right)^\alpha} \\
 &+ \frac{x}{(\log x)^\alpha} C \sum_{x^{\frac{\beta}{k}} < p_h \leq \frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}}} \frac{1}{p_h^k} \frac{1}{\left(1 - \frac{\log p_h^k}{\log x}\right)^\alpha} \leq \frac{C}{(1-\beta)^\alpha} \frac{x}{(\log x)^\alpha} \sum_{p_h \geq p_{m+1}} \frac{1}{p_h^k} \\
 &+ \frac{Cx}{(\log 2)^\alpha} \sum_{x^{\frac{\beta}{k}} < p_h \leq \frac{x^{\frac{1}{k}}}{2^{\frac{1}{k}}}} \frac{1}{p_h^k} \leq \left(\epsilon \frac{C}{(1-\beta)^\alpha} + o(1)\right) \frac{x}{(\log x)^\alpha}
 \end{aligned} \tag{1.35}$$

Equations (1.32), (1.29), (1.30), (1.33), (1.34) and (1.35) give (as in Theorem 2.2 of the reference [8])

$$A(x) = A \frac{x}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \tag{1.36}$$

Now, note that we have the following equality (use mathematical induction)

$$\sum_{j=1}^n \frac{1}{p_j^k} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i^k}\right) = 1 - \prod_{i=1}^n \left(1 - \frac{1}{p_i^k}\right)$$

Therefore

$$A = C_{m,r} \left(1 - \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^k}\right)\right) = C_{m,r} \left(1 - \prod_{p \equiv r} \left(1 - \frac{1}{p_i^k}\right)\right)$$

Now $S_{m,r,k}(x) = A_{m,r}(x) - A(x)$. The theorem is proved. $\square$

Corollary 1.11. *Let us consider the set $S_{m,r,2}$ of square-free numbers such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let $S_{m,r,2}(x)$ be the number of these numbers not exceeding x . Then*

$$S_{m,r,2}(x) \sim S_{m,r,2} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.37)$$

where $S_{m,r,2}$ is the positive constant

$$S_{m,r,2} = C_{m,r} \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^2}\right) \quad (1.38)$$

Theorem 1.12. *Let us consider the fixed square-free $q = q_1 \cdots q_r$, where q_i ($i = 1, \dots, r$) are the distinct prime factors in the prime factorization, these prime factors are of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let us consider the set $S_{q,m,r}$ of square-free numbers relatively prime to q such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let $A_{q,m,r}(x)$ be the number of these numbers not exceeding x . Then*

$$S_{q,m,r}(x) \sim S_{m,r,2} \prod_{i=1}^r \frac{q_i}{q_i + 1} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.39)$$

where $S_{m,r,2}$ is the positive constant of the former theorem.

Proof. We have (Theorem 1.7)

$$A_{m,r}(x) = \sum_{n_{m,r} \leq x} 1 \sim C_{m,r} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.40)$$

By use of the inclusion-exclusion principle, we eliminate the primes q_i ($i = 1, \dots, r$) and obtain

$$\begin{aligned} & A_{m,r}(x) - \sum_{1 \leq i \leq r} A_{m,r} \left(\frac{x}{q_i} \right) + \sum_{1 \leq i < j \leq r} A_{m,r} \left(\frac{x}{q_i q_j} \right) \\ & - \sum_{1 \leq i < j < k \leq r} A_{m,r} \left(\frac{x}{q_i q_j q_k} \right) + \cdots = C_{m,r} \prod_{i=1}^r \left(1 - \frac{1}{q_i} \right) \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \\ & + o \left(\frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \right) \end{aligned} \quad (1.41)$$

Now, the same proof as the proof of Theorem 1.10 gives

$$\begin{aligned}
S_{q,m,r}(x) &= C_{m,r} \prod_{i=1}^r \left(1 - \frac{1}{q_i}\right) \prod_{\substack{p \equiv r, \text{mod}(m) \\ p \neq q_i}} \left(1 - \frac{1}{p^2}\right) \frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \\
&+ o\left(\frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}}\right) = S_{m,r,2} \prod_{i=1}^r \frac{q_i}{q_i + 1} \frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \\
&+ o\left(\frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}}\right)
\end{aligned} \tag{1.42}$$

that is, equation (1.39). The theorem is proved. $\square$

Let $s \geq 1$ be and let $A_s(x)$ be the number of s -full numbers not exceeding x . It was proved by Ivić and Shiu (see [4], Chapter 14, or [3])

$$A_s(x) = \gamma_{0,s} x^{\frac{1}{s}} + \gamma_{1,s} x^{\frac{1}{s+1}} + \cdots + \gamma_{s-1,s} x^{\frac{1}{2s-1}} + \Delta_s(x), \tag{1.43}$$

where $\Delta_s(x) = O(x^\rho)$ for ρ small.

We need the weaker lemma.

Lemma 1.13. *The following asymptotic formula holds*

$$A_s(x) = \gamma_{0,s} x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right), \tag{1.44}$$

where

$$\gamma_{0,s} = \frac{6}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{s}} - 1)}\right) \tag{1.45}$$

Note that if $s = 1$ then we obtain the trivial formula $A_1(x) = x + o(x)$.

Proof. Equation (1.44) is a weak consequence of (1.43). For equation (1.45) see the reference [7]. The lemma is proved. $\square$

Lemma 1.14. *Let $s \geq 1$ be an arbitrary but fixed integer. The following series converges.*

$$\sum_Q \frac{1}{Q^{\frac{1}{s}}},$$

where the sum runs on all $(s+1)$ -full numbers Q .

Proof. Let a_n be the n -th $(s+1)$ -full number and let $A_{s+1}(x)$ be the number of $(s+1)$ -full numbers not exceeding x . We have (see Lemma 1.13) $A_{s+1}(x) \sim \gamma_{0,s} \sqrt[s+1]{x}$. Therefore if $x = a_n$ we obtain $n = A_{s+1}(a_n) \sim \gamma_{0,s} \sqrt[s+1]{a_n}$, that is, $a_n \sim \frac{n^{s+1}}{\gamma_{0,s}^{s+1}}$. Now, the lemma follows by the Comparison Criterion since the series $\sum \frac{1}{n^{\frac{s+1}{s}}}$ converges. The lemma is proved. $\square$

Theorem 1.15. *Let us consider the set $H_{m,r}$ of s -full numbers such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let $H_{m,r}(x)$ be the number of these numbers not exceeding x . Then*

$$H_{m,r}(x) \sim H_{m,r} \frac{x^{\frac{1}{s}}}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \quad (1.46)$$

where $H_{m,r}$ is the positive constant

$$\begin{aligned} H_{m,r} &= S_{m,r,2} s^{1 - \frac{1}{\varphi(m)}} \prod_{p \equiv r, \text{mod}(m)} \left(1 + \frac{1}{(p+1) \left(p^{\frac{1}{s}} - 1 \right)} \right) \\ &= C_{m,r} s^{1 - \frac{1}{\varphi(m)}} \prod_{p \equiv r, \text{mod}(m)} \left(1 - \frac{1}{p^2} \right) \prod_{p \equiv r, \text{mod}(m)} \left(1 + \frac{1}{(p+1) \left(p^{\frac{1}{s}} - 1 \right)} \right) \end{aligned} \quad (1.47)$$

Proof. Let us consider the s -full numbers of the form q^s , where q is a square-free product of primes of the linear form $mn + r$, where $\gcd(m, r) = 1$. Then $q^s \leq x$ if and only if $q \leq x^{\frac{1}{s}}$ and consequently by Corollary 1.11 the number of these s -full numbers not exceeding x will be

$$\begin{aligned} &S_{m,r,2} s^{1 - \frac{1}{\varphi(m)}} \frac{x^{\frac{1}{s}}}{(\log x)^{1 - \frac{1}{\varphi(m)}}} + o \left(\frac{x^{\frac{1}{s}}}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \right) \\ &= S_{m,r,2} s^{\alpha} \frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} + o \left(\frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} \right) \end{aligned} \quad (1.48)$$

For sake of simplicity we put $\alpha = 1 - \frac{1}{\varphi(m)}$.

The rest of the s -full numbers is of the form $q^s Q$, where q is a square-free product of primes of the linear form $mn + r$ and Q is a $(s+1)$ -full number product of primes of the linear form $mn + r$. Note that q and Q are relatively prime.

Let us consider the prime factorization of the $(s+1)$ -full number Q , that is, $Q = q_1^{r_1} \cdots q_t^{r_t}$. Let us consider the numbers of the form $q^s Q$, where Q is fixed, that is, the numbers with the same $Q = q_1^{r_1} \cdots q_t^{r_t}$. The number of these numbers not exceeding x , that is, $q^s Q \leq x$ will be since $q^s Q \leq x$ if and only if $q \leq \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}$ (see Theorem 1.12)

$$\begin{aligned} &S_{m,r,2} s^{\alpha} \left(\prod_{i=1}^r \frac{q_i}{q_i + 1} \right) \frac{1}{Q^{\frac{1}{s}}} \frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} + o \left(\frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} \right) \\ &= S_{m,r,2} s^{\alpha} \frac{a(Q)}{Q^{\frac{1}{s}}} \frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} + o \left(\frac{x^{\frac{1}{s}}}{(\log x)^{\alpha}} \right) \end{aligned} \quad (1.49)$$

where for sake of simplicity we put $a(Q) = \prod_{i=1}^r \frac{q_i}{q_i + 1}$.

Given $\epsilon > 0$ there exists M such that (Lemma 1.14)

$$\sum_{Q>M} \frac{1}{Q^{\frac{1}{s}}} < \epsilon \quad (1.50)$$

Equations (1.48), (1.49) and (1.50) give

$$\begin{aligned} H_{m,r}(x) &= \left(S_{m,r,2}s^\alpha + \sum_{Q \leq M} S_{m,r,2}s^\alpha \frac{a(Q)}{Q^{\frac{1}{s}}} \right) \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \\ + F(x) &= H_{m,r} \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} - \left(\sum_{Q>M} S_{m,r,2}s^\alpha \frac{a(Q)}{Q^{\frac{1}{s}}} \right) \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \\ + o\left(\frac{x}{(\log x)^\alpha}\right) &+ F(x) \end{aligned}$$

where

$$\begin{aligned} H_{m,r} &= S_{m,r,2}s^\alpha + \sum_Q S_{m,r,2}s^\alpha \frac{a(Q)}{Q^{\frac{1}{s}}} \\ &= S_{m,r,2}s^\alpha \prod_{p \equiv r} \left(1 + \frac{p}{p+1} \left(\frac{1}{\left(p^{\frac{1}{s}}\right)^{s+1}} + \frac{1}{\left(p^{\frac{1}{s}}\right)^{s+2}} + \dots \right) \right) \\ &= S_{m,r,2}s^{1-\frac{1}{\varphi(m)}} \prod_{p \equiv r} \left(1 + \frac{1}{(p+1)\left(p^{\frac{1}{s}}-1\right)} \right) \end{aligned}$$

and $F(x)$ is the contribution of the numbers $q^s Q \leq x$ such that $M < Q \leq x$.

Note that there exists a positive constant c such that the following inequality holds (Corollary 1.11) $S_{m,r,2}(x) < c \frac{x}{(\log x)^\alpha}$ ($x \geq 2$).

Therefore we have

$$\begin{aligned} 0 \leq F(x) &\leq \sum_{M < Q \leq \frac{x}{2^s}} c \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} \frac{1}{\left(\log \left(\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}\right)\right)^\alpha} + \sum_{\frac{x}{2^s} < Q \leq x} \left\lfloor \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} \right\rfloor \\ &\leq cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \sum_{M < Q \leq \frac{x}{2^s}} \frac{1}{Q^{\frac{1}{s}}} \frac{1}{\left(1 - \frac{\log Q}{\log x}\right)^\alpha} + \sum_{\frac{x}{2^s} < Q \leq x} \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} \end{aligned}$$

If $Q > \frac{x}{2^s}$ and consequently $\frac{1}{Q^{\frac{1}{s}}} < \frac{2}{x^{\frac{1}{s}}}$ then we obtain

$$0 \leq \sum_{\frac{x}{2^s} < Q \leq x} \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} \leq 2 \sum_{Q \leq x} 1 \leq c_1 x^{\frac{1}{s+1}} = o\left(\frac{x^{\frac{1}{s}}}{(\log x)^\alpha}\right)$$

On the other hand, if we put $\frac{s}{s+1} < \beta < 1$, then we obtain

$$\begin{aligned}
 0 &\leq cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \sum_{M < Q \leq \frac{x}{2^s}} \frac{1}{Q^{\frac{1}{s}}} \frac{1}{\left(1 - \frac{\log Q}{\log x}\right)^\alpha} \\
 &= cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \sum_{M < Q \leq x^\beta} \frac{1}{Q^{\frac{1}{s}}} \frac{1}{\left(1 - \frac{\log Q}{\log x}\right)^\alpha} \\
 &+ cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \sum_{x^\beta < Q \leq \frac{x}{2^s}} \frac{1}{Q^{\frac{1}{s}}} \frac{1}{\left(1 - \frac{\log Q}{\log x}\right)^\alpha} \\
 &\leq cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \frac{1}{(1-\beta)^\alpha} \sum_{Q > M} \frac{1}{Q^{\frac{1}{s}}} \\
 &+ cs^\alpha \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \sum_{Q \leq x} \frac{1}{x^{\frac{\beta}{s}}} \frac{(\log x)^\alpha}{s^\alpha (\log 2)^\alpha} \leq cs^\alpha \frac{1}{(1-\beta)^\alpha} \epsilon \frac{x^{\frac{1}{s}}}{(\log x)^\alpha} \\
 &+ \frac{c}{(\log 2)^\alpha} x^\delta < \epsilon' \frac{x^{\frac{1}{s}}}{(\log x)^\alpha}
 \end{aligned}$$

where $0 < \delta < \frac{1}{s}$. The rest of proof is as the proof of Theorem 2.2 in the reference [8]. The theorem is proved. $\square$

Theorem 1.16. *Let us consider the set $P_{m,r}$ of s -th powers such that in their prime factorization all prime factor is of the linear form $mn+r$, where $\gcd(m, r) = 1$. Let $P_{m,r}(x)$ be the number of these numbers not exceeding x . Then*

$$P_{m,r}(x) \sim P_{m,r} \frac{x^{\frac{1}{s}}}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \quad (1.51)$$

where $P_{m,r}$ is the positive constant

$$P_{m,r} = C_{m,r} s^{1 - \frac{1}{\varphi(m)}} \quad (1.52)$$

Proof. The proof is the same as the proof of Theorem 1.15. In this case we have

$$\begin{aligned}
 P_{m,r} &= S_{m,r,2} s^\alpha + \sum_Q S_{m,r,2} s^\alpha \frac{a(Q)}{Q^{\frac{1}{s}}} \\
 &= S_{m,r,2} s^\alpha \prod_{p \equiv r, \text{mod}(m)} \left(1 + \frac{p}{p+1} \left(\frac{1}{\left(p^{\frac{1}{s}}\right)^{2s}} + \frac{1}{\left(p^{\frac{1}{s}}\right)^{3s}} + \dots \right) \right) \\
 &= S_{m,r,2} s^{1 - \frac{1}{\varphi(m)}} \prod_{p \equiv r, \text{mod}(m)} \frac{1}{1 - \frac{1}{p^2}} = C_{m,r} s^{1 - \frac{1}{\varphi(m)}} \quad (1.53)
 \end{aligned}$$

where the sum runs over all $(s+1)$ -full and s -th power numbers Q . The theorem is proved. $\square$

We have proved Corollary 1.11 on square-free from Wirsing's Theorem 1.7. Is possible to prove Wirsing's Theorem 1.7 from Corollary 1.11. That is, both theorems are equivalent. The theorem is as follows,

Theorem 1.17. *Let $S_{m,r,2}(x)$ be the number of square-free not exceeding x such that all prime factor in their prime factorization is of the linear form $mn + r$. Suppose that*

$$S_{m,r,2}(x) \sim S_{m,r,2} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.54)$$

Then, if $A_{m,r}(x)$ is the number of numbers not exceeding x such that all prime factor in their prime factorization is of the linear form $mn + r$ we have

$$A_{m,r}(x) \sim C_{m,r} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.55)$$

where $C_{m,r} = S_{m,r,2} \prod_{p \equiv r, \text{mod}(m)} \frac{1}{1-\frac{1}{p^2}}$

Proof. We have the formula

$$\begin{aligned} A_{m,r}(x) &= \sum_{n^2 \leq x} S_{m,r,2} \left(\frac{x}{n^2} \right) \\ &= \sum_{n^2 \leq M} S_{m,r,2} \frac{x}{n^2} \frac{1}{(\log x)^\alpha} + o \left(\frac{x}{(\log x)^\alpha} \right) + F(x) \\ &= S_{m,r,2} \frac{x}{(\log x)^\alpha} \sum_{n^2} \frac{1}{n^2} - S_{m,r,2} \frac{x}{(\log x)^\alpha} \sum_{n^2 > M} \frac{1}{n^2} + o \left(\frac{x}{(\log x)^\alpha} \right) \\ &+ F(x) \end{aligned} \quad (1.56)$$

where n is either 1 or their prime factors are of the form $mn + r$ and consequently

$$\sum_{n^2} \frac{1}{n^2} = \prod_{p \equiv r, \text{mod}(m)} \frac{1}{1-\frac{1}{p^2}} \quad (1.57)$$

The rest of the proof is as the proof of the former theorems. The theorem is proved. $\square$

Theorem 1.18. *Let $P(n)$ be the greatest prime factor of n . Let us consider the set of numbers such that in their prime factorization all prime factor is of the linear form $mn + r$, where $\gcd(m, r) = 1$. Let $A_{m,r}(x)$ be the number of these numbers $n_{m,r}$ not exceeding x and let $A_\beta(x)$ be the number of these numbers $n_{m,r}$ not exceeding x such that their greatest prime factor $P(n_{m,r})$ satisfies $x^\beta < P(n_{m,r}) \leq x$, where $0 < \beta < 1$. Then the magnitude order of $A_\beta(x)$ is the same as the magnitude order of $A_{m,r}(x)$, that is, $\frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}}$ (see Theorem 1.7).*

Proof. By Theorem 1.7 we have

$$A_{m,r}(x) = \sum_{n_{m,r} \leq x} 1 \sim C_{m,r} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (1.58)$$

where $C_{m,r}$ is a positive constant. Therefore there exists a positive constant c_1 such that

$$A_{m,r}(x) > c_1 \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \quad (x \geq q_1)$$

where q_1 is the first prime of the linear form $mn + r$.

Besides equation (1.58) gives

$$A_\beta(x) \leq A_{m,r}(x) < c_2 \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}}$$

where c_2 is a positive constant. We have prove the another inequality. It is sufficient to prove

$$A_\beta(x) > c_3 \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}}$$

for $\frac{1}{2} \leq \beta < 1$ and c_3 a positive constant, since if $0 < \beta < \frac{1}{2}$ we clearly have $A_\beta(x) \geq A_{1/2}(x)$. Suppose then that $\frac{1}{2} \leq \beta < 1$. Let p be a prime of the linear form $mn + r$. Since $\sqrt{x} \leq x^\beta < p \leq x$ the multiples of p not exceeding x are $p.1, p.2, \dots, p. \left\lfloor \frac{x}{p} \right\rfloor$ and p is the greatest prime factor of these multiples. Therefore we have

$$\begin{aligned} A_\beta(x) &\geq \sum_{\substack{x^\beta < p \leq \frac{x}{q_1} \\ p \equiv r, \text{mod}(m)}} A_{m,r}\left(\frac{x}{p}\right) \geq \sum_{\substack{x^\beta < p \leq \frac{x}{q_1} \\ p \equiv r, \text{mod}(m)}} c_1 \frac{x}{p} \frac{1}{\left(\log \frac{x}{p}\right)^{1-\frac{1}{\varphi(m)}}} \\ &\geq c_1 \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \sum_{\substack{x^\beta < p \leq \frac{x}{q_1} \\ p \equiv r, \text{mod}(m)}} \frac{1}{p} > c_4 \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} \end{aligned}$$

This proves the another inequality. The theorem is proved. $\square$

Now, let $P(n)$ be the greatest prime factor of n . It is well-known the formula

$$\sum_{n \leq x} P(n) = \frac{\pi^2}{12} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (1.59)$$

If we consider the $n_{m,r}$ numbers then we have the following theorem.

Theorem 1.19. *The following asymptotic formula holds*

$$\sum_{n_{m,r} \leq x} P(n_{m,r}) = \frac{1}{2} \frac{1}{\varphi(m)} D_{m,r} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (1.60)$$

where the positive constant $D_{m,r}$ is

$$D_{m,r} = \sum_{k=1}^{\infty} A_{m,r}(k) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \quad (1.61)$$

Proof. In this proof p denotes a prime such that $p \equiv r \pmod{m}$. The prime number theorem is

$$\sum_{p \leq x} 1 = \frac{1}{\varphi(m)} \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad (1.62)$$

Therefore Abel summation gives

$$\sum_{p \leq x} p = \frac{1}{\varphi(m)} \frac{1}{2} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (1.63)$$

and if k is a positive integer

$$\sum_{\frac{x}{k+1} < p \leq \frac{x}{k}} p = \frac{1}{\varphi(m)} \frac{1}{2} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \quad (1.64)$$

Let s be a positive integer. Let us consider the numbers $k = 1, 2, \dots, s$ and the inequalities

$$\frac{x}{k+1} < p \leq \frac{x}{k} \quad (1.65)$$

Then the multiple of p not exceeding x are $p.1, p.2, \dots, p.k$ and if x is sufficiently large p is the greatest prime factor with exponent 1 of these numbers. The number of $n_{m,r}$ numbers in the set $1, 2, \dots, k$ is $A_{m,r}(k)$. Therefore

$$\begin{aligned} \sum_{n_{m,r} \leq x} P(n_{m,r}) &= \left(\sum_{k=1}^s A_{m,r}(k) \frac{1}{\varphi(m)} \frac{1}{2} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \frac{x^2}{\log x} \right) \\ &+ o\left(\frac{x^2}{\log x}\right) = \frac{1}{2} \frac{1}{\varphi(m)} D_{m,r} \frac{x^2}{\log x} \\ &- \left(\frac{1}{\varphi(m)} \frac{1}{2} \sum_{k=s+1}^{\infty} A_{m,r}(k) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \right) \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right) \\ &+ F(x) \end{aligned} \quad (1.66)$$

Note that $\sum_{p \leq x} 1 < c \frac{x}{\log x}$ for $x \geq 2$, where c is a positive constant. Hence

$$F(x) \leq \sum_{p \leq \frac{x}{s+1}} p \left\lfloor \frac{x}{p} \right\rfloor \leq x \sum_{p \leq \frac{x}{s+1}} 1 \leq 2c \frac{1}{s+1} \frac{x^2}{\log x} \quad (1.67)$$

If we choose s sufficiently large then by combination of these equations we obtain equation (1.60). The theorem is proved. $\square$

Remark 1.20. Theorem 1.15 on s -full numbers is clearly also valid for 1-full numbers, that is, all numbers $n_{m,r}$. In this case

$$H_{m,r}(x) \sim H_{m,r} \frac{x}{(\log x)^{1 - \frac{1}{\varphi(m)}}} \quad (1.68)$$

where $H_{m,r}$ is the positive constant

$$H_{m,r} = C_{m,r} \prod_{p \equiv r \pmod{m}} \left(1 - \frac{1}{p^2} \right) \prod_{p \equiv r} \left(1 + \frac{1}{(p+1)(p-1)} \right) = C_{m,r} \quad (1.69)$$

as must be. Since in this case $H_{m,r}(x) = A_{m,r}(x)$.

Theorem 1.21. *Let us consider the $n_{m,r}$ numbers not exceeding x . Their counting function is $A_{m,r}(x)$. Let $T_{m,r}(x)$ be the number of numbers $n_{m,r}$ not exceeding x such that $P(n_{m,r})$ has exponent 1 in the prime factorization of $n_{m,r}$. Then $T_{m,r}(x) \sim A_{m,r}(x)$, that is, almost all greatest prime factor has exponent 1 in the $n_{m,r}$ numbers.*

Proof. We follow the proof of Theorem 1.15 with $s = 1$ (see Remark 1.20). Let us consider the 1-full numbers of the form q , where q is a square-free product of primes of the linear form $mn + r$, where $\gcd(m, r) = 1$. Note that $P(q)$ has exponent 1 since q is square-free. The number of these 1-full numbers not exceeding x will be

$$\begin{aligned} & S_{m,r,2} \frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}} + o\left(\frac{x}{(\log x)^{1-\frac{1}{\varphi(m)}}}\right) \\ &= S_{m,r,2} \frac{x}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \end{aligned} \quad (1.70)$$

For sake of simplicity we put $\alpha = 1 - \frac{1}{\varphi(m)}$.

The rest of the 1-full numbers is of the form qQ , where q is a square-free product of primes of the linear form $mn + r$ and Q is a square-full number product of primes of the linear form $mn + r$. Note that q and Q are relatively prime.

Let us consider the prime factorization of the square-full number Q , that is, $Q = q_1^{r_1} \cdots q_t^{r_t}$. Let us consider the numbers of the form qQ , where Q is fixed, that is, the numbers with the same $Q = q_1^{r_1} \cdots q_t^{r_t}$. Then, except by a finite number of cases the greatest prime factor of qQ is in q and hence it has exponent 1. Therefore the number of these numbers qQ not exceeding x , such that their greatest prime factor have exponent 1 will be since $q^s Q \leq x$ if and only if $q \leq \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}$

$$\begin{aligned} & S_{m,r,2} \left(\prod_{i=1}^r \frac{q_i}{q_i + 1} \right) \frac{1}{Q} \frac{x}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \\ &= S_{m,r,2} \frac{a(Q)}{Q} \frac{x}{(\log x)^\alpha} + o\left(\frac{x}{(\log x)^\alpha}\right) \end{aligned} \quad (1.71)$$

where for sake of simplicity we put $a(Q) = \prod_{i=1}^r \frac{q_i}{q_i + 1}$. Now, the proof is identical to the proof of Theorem 1.15 (with $s = 1$). See also Remark 1.20. The theorem is proved. $\square$

Now, we give a very simple proof that k -free numbers have density $\frac{1}{\zeta(k)}$ and we generalize this simple proof and obtain a more general theorem. Finally we prove a more general theorem on multiplicative sets, k -free numbers and s -full numbers are particular cases of this more general theorem.

Theorem 1.22. *Let $k \geq 2$. Let $Q_k(x)$ be the number of k -free numbers not exceeding x . Then*

$$Q_k(x) \sim \frac{1}{\zeta(k)} x \quad (1.72)$$

Consider the following multiplicative set. Let us consider any sequence k_i of positive integers. Suppose that the possible exponents of the prime p_i are $1, \dots, k_i$, that is, the set $E(p_i)$ is formed by $1, \dots, k_i$. Let $N(x)$ be the number of positive integers in this multiplicative set not exceeding x . Then

$$N(x) \sim cx \quad (1.73)$$

where

$$c = \prod_{p_i} \left(1 - \frac{1}{p_i^{k_i+1}}\right) \quad (1.74)$$

Proof. By inclusion-exclusion principle the number of numbers not exceeding x in the complementary set of k -free numbers is

$$\begin{aligned} & \sum_{p_i^k \leq \log x} \left\lfloor \frac{x}{p_i^k} \right\rfloor - \sum_{p_i^k < p_j^k \leq \log x} \left\lfloor \frac{x}{p_i^k p_j^k} \right\rfloor + \dots \\ &= x \left(1 - \prod_{p_i^k \leq \log x} \left(1 - \frac{1}{p_i^k}\right)\right) + O\left(2^{\sqrt[k]{\log x}}\right) + F(x) \\ &= x \left(1 - \prod_p \left(1 - \frac{1}{p^k}\right) + o(1)\right) + O\left(2^{\sqrt[k]{\log x}}\right) + F(x) \end{aligned} \quad (1.75)$$

where $0 \leq F(x) \leq \sum_{\log x < p_i^k \leq x} \left\lfloor \frac{x}{p_i^k} \right\rfloor = o(x)$. This prove equation (1.72) on k -free numbers. The proof of the more general formulas (1.73) and (1.74) is the same. The theorem is proved. $\square$

Theorem 1.23. Consider the following multiplicative set. Suppose that the sets $E(p)$ of possible exponents of the primes p have the same least element s , in general the finite or infinite set $E(p)$ depends of p . Let $N(x)$ be the number of positive integers in this multiplicative set not exceeding x . Then

$$N(x) \sim cx^{\frac{1}{s}} \quad (1.76)$$

where

$$c = \frac{6}{\pi^2} \prod_p \left(1 + \frac{p}{p+1} \left(\sum_{\substack{k \in E(p) \\ k > s}} \frac{1}{p^{\frac{k}{s}}}\right)\right) \quad (1.77)$$

Proof. The number of numbers q^s not exceeding x , where q is a square-free is

$$\frac{6}{\pi^2} x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right)$$

The rest of the numbers is of the form $q^s Q$. The prime factorization of Q is of the form $Q = q_1^{s_1} \cdots q_r^{s_r}$, where q_i are the distinct primes and $s_i \in E_{q_i}$ and $s_i > s$

($i = 1, 2, \dots, r$). Therefore the number of numbers of the form $q^s Q$ not exceeding x (Q fixed) is

$$\frac{6}{\pi^2} \prod_{i=1}^r \frac{q_i}{q_i + 1} \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} + o\left(x^{\frac{1}{s}}\right)$$

The rest of the proof is as the proof of former theorems. The theorem is proved. $\square$

It is well-known if n is large the number of positive integers multiple of some prime p_i ($i = 1, \dots, n$) has positive density near of 1.

To Finish, we prove a similar theorem for $n_{m,r}$ numbers.

Theorem 1.24. *If p_i is the sequence of primes of the linear form $mn + r$. Then if n is large the number of $n_{m,r}$ numbers multiple of some prime p_i ($i = 1, \dots, n$) has positive density near of 1.*

Proof. By inclusion-exclusion principle the number of $n_{m,r}$ numbers multiple of some prime p_i ($i = 1, \dots, n$) is

$$\begin{aligned} & \sum_{p_i \leq p_n} A_{m,r} \left(\frac{x}{p_i} \right) - \sum_{p_i < p_j \leq p_n} A_{m,r} \left(\frac{x}{p_i p_j} \right) + \dots \\ &= C_{m,r} \frac{x}{(\log x)^{\alpha_m}} \left(1 - \prod_{i=1}^n \left(1 - \frac{1}{p_i} \right) \right) + o \left(\frac{x}{(\log x)^{\alpha_m}} \right) \end{aligned} \quad (1.78)$$

and $\prod_{i=1}^n \left(1 - \frac{1}{p_i} \right) \rightarrow 0$ as $n \rightarrow \infty$. The theorem is proved. $\square$

Theorem 1.25. *Consider the following multiplicative set. Suppose that the set of primes p is the set of primes in the linear form $mn + r$, where $\gcd(m, r) = 1$. Suppose that the sets $E(p)$ of possible exponents of the primes p have the same least element $s \geq 1$, in general the finite or infinite set $E(p)$ depends of p . Let $N(x)$ be the number of positive integers in this multiplicative set not exceeding x . That is, the number of positive integers n not exceeding x whose prime factorization is of the form $n = q_1^{s_1} \cdots q_r^{s_r}$, where the q_i ($i = 1, \dots, r$) are primes of the linear form $mn + r$, $\gcd(m, r) = 1$, and $s_i \in E_{q_i}$ ($i = 1, \dots, r$). Then*

$$N(x) \sim C \frac{x^{\frac{1}{s}}}{(\log x)^{1 - \frac{1}{\varphi(m)}}}$$

where the constant C can be obtained as in Theorem 1.15.

Proof. The proof is the same as the proof of Theorem 1.15. Theorem 1.15 is a particular case of this more general theorem. The theorem is proved. $\square$

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REFERENCES

1. S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2005.
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1960.
3. A. Ivić and P. Shiu, *The distribution of powerful integers*, Illinois J. Math., **26** (1982), Issue 4, 576 - 590. <https://doi.org/10.1215/ijm/1256046597>
4. A. Ivić, *The Riemann Zeta-Function*, Dover, 2003.
5. R. Jakimczuk, *Integers of the form $p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ where $p_1, p_2, \dots, p_k$ are primes fixed*, International Journal of Contemporary Mathematical Sciences, **2** (2007), Issue 27, 1327-1333. <https://doi.org/10.12988/ijcms.2007.07137>
6. R. Jakimczuk, *Numbers with the same kernel*, Notes on Number Theory and Discrete Mathematics, **25** (2019), Issue 3, 44-64. <https://doi.org/10.7546/nntdm.2019.25.3.44-64>
7. R. Jakimczuk and M. Lalín, *The number of prime factors on average in certain integer sequences*, J. Integer Seq., **25** (2022), Article 22.2.3.
8. R. Jakimczuk, *Mertens's formula, k -full numbers and sum of two squares*, Gulf Journal of Mathematics, **17**(2) (2024), 263-291. <https://doi.org/10.56947/gjom.v17i2.2290>
9. W. J. LeVeque, *Topics in Number Theory*, Volume 2, Addison-Wesley, 1958.
10. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann., **143** (1961), 75 - 102. <https://doi.org/10.1007/BF01351892>

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