Gulf Journal of Mathematics ISSN: 2309-4966
Vol 19, Issue 1 (2025) 16-39 https://doi.org/10.56947/gjom.v19i1.2317

OUTPUT TRACKING PROBLEM FOR A CLASS OF 3-D HYPERBOLIC PDES

MUSTAPHA EL HASSANI*

Abstract. The output tracking problem (OTP) refers to the procedure of determining an input control that ensures the system output follows a specified trajectory. In this paper, we investigate the OTP for controlled systems governed by a class of 3-D hyperbolic partial differential equations (PDEs), aiming for an exact alignment between the system output and the desired trajectory. Utilizing backstepping techniques, semigroup theory, the Hahn-Banach theorem, Laplace transforms, and Green's functions, our approach begins by establishing the OTP solution as a bounded feedback law, with Green's functions allowing the solution of a Cauchy-Euler equation. This is followed by deriving a detailed formulation for the complete state of the corresponding control problem. Finally, the approach provides an explicit formula for the OTP solution, using the desired trajectory and the solutions of specific kernel PDEs.

1. Introduction and problem formulation

The problem of designing an input signal to guide system output toward a specific target configuration is relevant in various real-world applications, particularly for automated devices like robots, which must move from a starting position to a target position along a predefined trajectory. This typical problem is often referred in literature as the *motion planning problem*, or *trajectory (path) tracking problem*. In this work, we investigate the OTP for dynamical systems governed by the following PDEs:

$$\begin{cases} u_{tt}(t,\mathbf{x}) = \epsilon \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} u(t,\mathbf{x}) + q u(t,\mathbf{x}), & \text{in } (0,\infty)\times\Omega, \\ u_{tt}(t,y_i,0) = \alpha_i \frac{\partial}{\partial x_i} u(t,y_i,0), & \text{in } (0,\infty)\times\Upsilon_i, \quad \text{for } i=1,2,3, \\ u(t,y_i,1) = U_i(t,y_i), & \text{in } (0,\infty)\times\Gamma_i, \quad \text{for } i=1,2,3, \end{cases} \tag{1.1}$$

where $\mathbf{x} := \{x_1, x_2, x_3\}$, $y_i := \{x_1, x_{i-1}, x_{i+1}, x_3\}$, and $\Omega := (0,1)\times(0,1)\times(0,1)$ represents the spatial domain. The boundary $\partial\Omega$ is divided into the subsets Υ and Γ, defined by $\partial\Omega := \Upsilon \bigcup \Gamma$, where $\Upsilon := \bigcup_{i=1}^{3} \Upsilon_i$ and $\Gamma := \bigcup_{i=1}^{3} \Gamma_i$. Here, the

Date: Received: Oct 7, 2024; Accepted: Nov 17, 2024.
* Corresponding author.
2020 *Mathematics Subject Classification.* Primary 93C05, 93C20; Secondary 47D06.
Key words and phrases. PDEs, feedback and closed-loop system, generator of a C0-semigroup, Laplace transform, Green's function.

boundary subsets Γ_i, which designate the locations of controls, and subsets Υ_i are given by:

$$\Upsilon_1 := \{(0, x_2, x_3),\ 0 \le x_2 < 1,\ 0 < x_3 \le 1\}, \Gamma_1 := \{(1, x_2, x_3),\ 0 \le x_2 < 1,\ 0 < x_3 \le 1\},$$
$$\Upsilon_2 := \{(x_1, 0, x_3),\ 0 < x_1 \le 1,\ 0 \le x_3 < 1\}, \Gamma_2 := \{(x_1, 1, x_3),\ 0 < x_1 \le 1,\ 0 \le x_3 < 1\},$$
$$\Upsilon_3 := \{(x_1, x_2, 0),\ 0 \le x_1 < 1,\ 0 < x_2 \le 1\}, \Gamma_3 := \{(x_1, x_2, 1),\ 0 \le x_1 < 1,\ 0 < x_2 \le 1\}.$$

The constants ϵ, q, and α_i, $i = 1, 2, 3$, are not negative. Additionally, without loss of generality in the control problem under consideration, we consider that the initial conditions $u^0 = u(0, .)$ and $u_t^0 = u_t(0, .)$ are identically zero functions.

System (1.1) is described by a second-order hyperbolic equation defined on the 3-D domain $[0, 1]$, where the coefficients ε and q represent the diffusion and multiplication coefficient, respectively. Boundary equations of parabolic type occur at Υ, and control inputs U_i are applied at Γ_i. This type of control system is derived by linearizing the controlled wave equations, as demonstrated, for example, in [21, 29]. From a physical perspective, by incorporating certain approximations, the system (1.1) can model the transverse vibrations of a nonhomogeneous string that is loaded at one extremity and controlled from the other [20, 29], while the boundary condition on Υ describes the transverse displacement of the string's endpoint. This boundary condition is commonly known as a *dynamic boundary condition* since it corresponds to Newton's second law (the fundamental principle of dynamics) applied to the tip where the string is loaded, as described in [1] and related references.

Our control objective is to ensure that the output system tracks a required trajectory $\varphi(t)$ by employing $U(t)$ as an input control. More precisely, we seek to solve the following problem:

Control objective. Let $\varphi_i(t, .)$, $i = 1, 2, 3$, be a (sufficiently smooth) target trajectories. We are looking for feedback controllers $U_i(t, .)$, $i = 1, 2, 3$, in (1.1) such that the state response $u(t, \mathbf{x})$ to this controller verifies the output equation

$$u_{|\Upsilon_i}(t, y_i, 0) = \varphi_i(t, y_i, 0), \text{ for } i = 1, 2, 3. \tag{1.2}$$

Typically, when the condition (1.2) is difficult to satisfy with any controller, one often refers to the concept of *trajectory asymptotic tracking*. This involves the process of identifying a controller that ensures the system output converges to the intended trajectory.

The OTP problem for multi-dimensional control systems remains a challenging area of research, while various approaches have been explored. In [12], a purely differential-geometric approach is utilized. Continuous feedbacks [27] and piecewise constant functions [16] have also been proposed as input controls. A numerical method for approximating the planning control for a hyperbolic PDE is developed in [24]. Stable trajectory tracking is tackled in [11] by combining the passivity-based method with semigroup theory. In [10], [13], [14], [17], [19], [23], the concept of flatness for systems governed by ordinary differential equations offers a solution for this problem by parameterizing trajectories through Gevrey functions and translating operations of the reference trajectory. For one-dimensional hyperbolic PDEs, significant contributions include the works in [5], [7], [9], and [20].

The control problem in [5] is formulated for a one-dimensional spatial domain and was solved using C_0-semigroups with the Laplace transform. However, from both practical and theoretical perspectives, it is advantageous to study OTP for systems with multi-dimensional spatial domains. This paper expands on the work in [5] by applying it to a three-dimensional spatial domain, which brings considerable technical challenges. The extension requires specific structural conditions for the domain's geometry and appropriate spatial distribution of controls. Additionally, improvements to the techniques developed in [5] are necessary. In contrast to the literature on similar control problems, system (1.1) contains a lower-order term in the differential operator, namely $qu(t, \mathbf{x})$, which creates significant challenges for solving the OTP. Inspired by the approach in [5] for the one-dimensional case, the primary idea to address these challenges is to convert system (1.1) into an auxiliary w-system using integral transformations with two kernels, which are solutions to appropriate PDEs. This transformation shifts the lower-order term to the control boundary and the kernels' PDEs, transforming the challenges related to this term in the control problem into the difficulty of solving the kernels' PDEs and those associated with the target system.

We have structured this document as follows: Section (2) outlines the procedure for converting system (1.1) into a target w-system using an integral transformation with three kernels that satisfy specific PDEs. Additionally, we establish the existence of solutions to these kernels' PDEs. We further reformulate the target w-system as an abstract semigroup control system and demonstrate its well-posedness.

Section (3) presents a solution of the closed-loop system through a boundary feedback law, employing the Laplace transform and Green's functions (see Theorem 3.2). Sections (4) and (5) provide a detailed formula for the state response and the planning controller for system (1.1), utilizing the kernels and reference trajectories. The main contributions of this work are encapsulated in Theorems 4.1 and 5.1.

2. Preliminaries.

2.1. **Backstepping transformation.** The primary challenge in analyzing the OTP for system (1.1) arises from the disturbance term $qu(t, \mathbf{x})$. Therefore, the first step involves transforming system (1.1) into a target one by eliminating the perturbation term $qu(t, \mathbf{x})$. To achieve this, inspired by the approach in [3], we consider u as a solution of (1.1) and introduce a new state variable z through the following Volterra-type integral transformation:

$$z(t, \mathbf{x}) = (\mathcal{B}u(t))(\mathbf{x}) := u(t, \mathbf{x}) - \sum_{i=1}^{3} \int_0^{x_i} k_i(x_i, r_i) u(t, y_i, r_i) dr_i, \tag{2.1}$$

$\mathbf{x} := \{x_1, x_2, x_3\}$, $y_i := \{x_1, x_{i-1}, x_{i+1}, x_3\}$, for $t > 0$ and x_i in $[0, 1]$, for $i = 1, 2, 3$. It should be noted here that the transformation (2.1) was first introduced in [15] for a system with a one-dimensional spatial domain. The kernels k_i are functions that need to be determined. By applying integration by parts, the second derivative of (2.1) results in the following expression:

$$
\begin{aligned}
z_{tt}(t,\mathbf{x}) = u_{tt}(t,\mathbf{x}) - \sum_{i=1}^{3}\int_0^{x_i} k_i(x_i,r_i)u_{tt}(t,y_i,r_i)dr_i = \epsilon\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}u(t,\mathbf{x}) + qu(\mathbf{x})\\
-\varepsilon\Big[k_1(x_1,x_1)\frac{\partial}{\partial x_1}u(t,\mathbf{x}) - k_1(x_1,0)\frac{\partial}{\partial x_1}u(t,y_1,0)\Big] + \varepsilon\Big[\frac{\partial}{\partial r_1}k_1(x_1,x_1)u(t,\mathbf{x})\\
-\frac{\partial}{\partial r_1}k_1(x_1,0)u(t,y_1,0)\Big] - \varepsilon\int_0^{x_1}\frac{\partial^2}{\partial r_1^2}k_1(x_1,r_1)u(t,y_1,r_1)dr_1\\
-\varepsilon\int_0^{x_1}k_1(x_1,r_1)\Big[\frac{\partial^2}{\partial x_2^2}u(t,y_1,r_1) + \frac{\partial^2}{\partial x_3^2}u(t,y_1,r_1)\Big]dr_1 - \varepsilon\Big[k_2(x_2,x_2)\frac{\partial}{\partial x_2}u(t,\mathbf{x})\\
-k_2(x_2,0)\frac{\partial}{\partial x_2}u(t,y_2,0)\Big] + \varepsilon\Big[\frac{\partial}{\partial r_2}k_2(x_2,x_2)u(t,\mathbf{x}) - \frac{\partial}{\partial r_2}k_2(x_2,0)u(t,y_2,0)\Big]\\
-\varepsilon\int_0^{x_2}\frac{\partial^2}{\partial r_2^2}k_2(x_2,r_2)u(t,y_2,r_2)dr_2 - \varepsilon\int_0^{x_2}k_2(x_2,r_2)\Big[\frac{\partial^2}{\partial x_1^2}u(t,y_2,r_2)\\
+\frac{\partial^2}{\partial x_3^2}u(t,y_2,r_2)\Big]dr_2 - \varepsilon\Big[k_3(x_3,x_3)\frac{\partial}{\partial x_3}u(t,\mathbf{x}) - k_3(x_3,0)\frac{\partial}{\partial x_3}u(t,y_3,0)\Big]\\
+\varepsilon\Big[\frac{\partial}{\partial r_3}k_3(x_3,x_3)u(t,\mathbf{x}) - \frac{\partial}{\partial r_3}k_3(x_3,0)u(t,y_3,0)\Big] - \varepsilon\int_0^{x_3}\frac{\partial^2}{\partial r_3^2}k_3(x_3,r_3)u(t,y_3,r_3)dr_3\\
-\varepsilon\int_0^{x_3}k_3(x_3,r_3)\Big[\frac{\partial^2}{\partial x_1^2}u(t,y_3,r_3) + \frac{\partial^2}{\partial x_2^2}u(t,y_3,r_3)\Big]dr_3.
\end{aligned}
$$

Conversely, taking the second derivative of (2.1) with respect to x_1 gives

$$
\begin{aligned}
\frac{\partial^2}{\partial x_1^2}z(t,\mathbf{x}) = \frac{\partial^2}{\partial x_1^2}u(t,\mathbf{x}) - \int_0^{x_1}\frac{\partial^2}{\partial x_1^2}k_1(x_1,r_1)u(t,y_1,r_1)dr_1 - \frac{\partial}{\partial x_1}k_1(x_1,x_1)u(t,\mathbf{x})\\
-\frac{d}{dx_1}k_1(x_1,x_1)u(t,\mathbf{x}) - k_1(x_1,x_1)\frac{\partial}{\partial x_1}u(t,\mathbf{x}) - \int_0^{x_2}k_2(x_2,r_2)\frac{\partial^2}{\partial x_1^2}u(t,y_2,r_2)dr_2\\
-\int_0^{x_3}k_3(x_3,r_3)\frac{\partial^2}{\partial x_1^2}u(t,y_3,r_3)dr_3,
\end{aligned}
$$

with respect to x_2

$$
\begin{aligned}
\frac{\partial^2}{\partial x_2^2}z(t,\mathbf{x}) = \frac{\partial^2}{\partial x_2^2}u(t,\mathbf{x}) - \int_0^{x_2}\frac{\partial^2}{\partial x_2^2}k_2(x_2,r_2)u(t,y_2,r_2)dr_2 - \frac{\partial}{\partial x_2}k_2(x_2,x_2)u(t,\mathbf{x})\\
-\frac{d}{dx_2}k_2(x_2,x_2)u(t,\mathbf{x}) - k_2(x_2,x_2)\frac{\partial}{\partial x_2}u(t,\mathbf{x}) - \int_0^{x_1}k_1(x_1,r_1)\frac{\partial^2}{\partial x_2^2}u(t,y_1,r_1)dr_1\\
-\int_0^{x_3}k_3(x_3,r_3)\frac{\partial^2}{\partial x_2^2}u(t,y_3,r_3)dr_3,
\end{aligned}
$$

and with respect to x_3

$$
\begin{aligned}
\frac{\partial^2}{\partial x_3^2}z(t,\mathbf{x}) = \frac{\partial^2}{\partial x_3^2}u(t,\mathbf{x}) - \int_0^{x_3}\frac{\partial^2}{\partial x_3^2}k_3(x_3,r_3)u(t,y_3,r_3)dr_3 - \frac{\partial}{\partial x_3}k_3(x_3,x_3)u(t,\mathbf{x})\\
-\frac{d}{dx_3}k_3(x_3,x_3)u(t,\mathbf{x}) - k_3(x_3,x_3)\frac{\partial}{\partial x_3}u(t,\mathbf{x}) - \int_0^{x_1}k_1(x_1,r_1)\frac{\partial^2}{\partial x_3^2}u(t,y_1,r_1)dr_1\\
-\int_0^{x_2}k_2(x_2,r_2)\frac{\partial^2}{\partial x_3^2}u(t,y_2,r_2)dr_2,
\end{aligned}
$$

Thus, $z_{tt}(t,\mathbf{x}) = \epsilon \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} z(t,\mathbf{x})$ in $(0,\infty)\times\Omega$ if and only if the kernels k and p satisfy

$$\sum_{i=1}^{3} \frac{d}{dx_i} k_i(x_i,x_i) = \frac{-q}{2\varepsilon}, \tag{2.2}$$

$$\begin{cases} \frac{\partial}{\partial^2 x_i^2} k_i(x_i,r_i) - \frac{\partial^2}{\partial r_i^2} k_i(x_i,r_i) = \frac{q}{\varepsilon} k_i(x_i,r_i), & 0 \le r_i \le x_i \le 1, \\ k_i(x_i,0) = \frac{\partial}{\partial r_i} k_i(x_i,0) = 0 & 0 \le x_i \le 1, \end{cases} \tag{2.3}$$

for $i = 1,2,3$. The sum of the functions in (2.2) equals a constant. Therefore, all terms must also be constant. Supposing one term equals $-\frac{c_1}{2\varepsilon}$ and the other equals $-\frac{c_2}{2\varepsilon}$, we derive the following results:

$$k_1(x_1,x_1) = -\frac{c_1}{2\varepsilon} x_1, \quad k_2(x_2,x_2) = -\frac{c_2}{2\varepsilon} x_2 \quad \text{and} \quad k_3(x_3,x_3) = -\frac{c_3}{2\varepsilon} x_3 \tag{2.4}$$

where c_1 and c_2 is arbitrary constants, and $c_3 := q - c_1 - c_2$.
On the other hand, taking the boundary conditions in (1.1) into consideration, we obtain from (2.1)

$$z_{tt}(t,y_i,0) = \alpha_i \frac{\partial}{\partial x_i} z(t,y_i,0), \quad \text{for } i = 1,2,3. \tag{2.5}$$

In view of (2.2)-(2.5), if the kernels k_i are chosen to be solutions of the PDEs

$$\begin{cases} \frac{\partial}{\partial^2 x_i^2} k_i(x_i,r_i) - \frac{\partial}{\partial^2 r_i^2} k_i(x_i,r_i) = \frac{q}{\varepsilon} k_i(x_i,r_i), & 0 \le r_i \le x_i \le 1, \\ k_i(x_i,x_i) = -\frac{c_i}{2\varepsilon} x_i, & 0 \le x_i \le 1, \\ k_i(x_i,0) = \frac{\partial}{\partial r_i} k_i(x_i,0) = 0, & 0 \le x_i \le 1, \end{cases} \tag{2.6}$$

for $i = 1,2,3$. Then the solutions of the PDEs (2.6) are given in [26] by

$$k_i(x_i,r_i) = -\frac{c_i}{\varepsilon} r_i \frac{I_1\left(\sqrt{\frac{q}{\varepsilon}(x_i^2 - r_i^2)}\right)}{\sqrt{\frac{q}{\varepsilon}(x_i^2 - r_i^2)}}, \qquad \text{for } i = 1,2,3$$

where I_1 is the modified Bessel function of order one.

Therefore, the new state z satisfies the following auxiliary system

$$\begin{cases} z_{tt}(t,\mathbf{x}) = \varepsilon \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} z(t,\mathbf{x}), & \text{in } (0,\infty)\times\Omega, \\ z_{tt}(t,y_i,0) = \alpha_i \frac{\partial}{\partial x_i} z(t,y_i,0), & \text{in } (0,\infty)\times\Upsilon_i, \quad \text{for } i = 1,2,3, \\ z(t,y_i,1) =: V_i(t,y_i), & \text{in } (0,\infty)\times\Gamma_i, \quad \text{for } i = 1,2,3, \end{cases} \tag{2.7}$$

where

$$
\begin{aligned}
V_1(t,y_1) =& U_1(t,y_1) - \int_0^1 k_1(1,r_1)u(t,y_1,r_1)dr_1 - \int_0^{x_2} k_2(x_2,r_2)U_1(t,r_2,x_3)dr_2 \\
& - \int_0^{x_3} k_3(x_3,r_3)U_1(t,x_2,r_3)dr_3, \qquad (2.8)\\
V_2(t,y_2) =& U_2(t,y_2) - \int_0^{x_1} k_1(x_1,r_1)U_2(t,r_1,x_3)dr_1 - \int_0^1 k_2(1,r_2)u(t,y_2,r_2)dr_2 \\
& - \int_0^{x_3} k_3(x_3,r_3)U_2(t,x_1,r_3)dr_3, \qquad (2.9)\\
V_3(t,y_3) =& U_3(t,y_3) - \int_0^{x_1} k_1(x_1,r_1)U_3(t,r_1,x_2)dr_1 - \int_0^{x_2} k_2(x_2,r_2)U_3(t,x_1,r_2)dr_2 \\
& - \int_0^1 k_3(1,r_3)U_1(t,x_2,r_3)dr_3, \qquad (2.10)
\end{aligned}
$$

and choosing

$$V_1(t,x_2,0) = V_1(t,0,x_3) = V_2(t,x_1,0) = V_2(t,0,x_3) = V_3(t,x_1,0) = V_3(t,0,x_2) = 0. \qquad (2.11)$$

2.2. **Abstract control system.** Inspired by the method in [6] for a 1-D spacial domain system, we shall rewrite system (2.7) as an abstract control system in the semigroup framework [25]. To this end, we denote

$$H^1_\Gamma(\Omega) := \{f \in H^1(\Omega),\ f_{/\Gamma} = 0\},$$

$\Gamma = \bigcup_{i=1}^{3}\Gamma_i$, and we define the state space as $\mathcal{X} := H^1_\Gamma(\Omega) \times L^2(\Omega) \times L^2(\Upsilon_1) \times L^2(\Upsilon_2) \times L^3(\Upsilon_3)$ equipped with the inner product

$$\left\langle \begin{pmatrix} f\\ g\\ h_1\\ h_2\\ h_3\end{pmatrix}, \begin{pmatrix} \varphi\\ \phi\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}\right\rangle = \int_\Omega \left(\varepsilon\sum_{i=1}^{3}\frac{\partial f}{\partial x_i}.\frac{\partial \varphi}{\partial x_i} + g\phi\right)dx_1dx_2dx_3 + \sum_{i=1}^{3}\frac{\varepsilon}{\alpha_i}\int_0^1\int_0^1 h_i(y_i)\psi_i(y_i)dy_i.$$

Moreover, for $t \geq 0$, we introduce

$$X(t) := \begin{pmatrix} z(t,\cdot) - \sum_{i=1}^{3} V_i(t,\cdot)\mathbb{1}_{\Gamma_i}\\ z_t(t,\cdot,\cdot)\\ z_t(t,0,\cdot)\\ z_t(t,\cdot,0,\cdot)\\ z_t(t,\cdot,\cdot,0)\end{pmatrix},\quad V(t) := \begin{pmatrix} V_1(t,\cdot)\\ V_2(t,\cdot)\\ V_3(t,\cdot)\end{pmatrix},$$

and we consider the evolution operator[1]

$$\mathcal{D}(\mathcal{A}) := \{X := (f,g,h_1,h_2,h_3)^\top \in \mathcal{X} : f \in H^2(\Omega),\ g \in H^1_\Gamma(\Omega),\ h_i(y_i) = g(y_i),\ i = 1,2,3\},$$

[1] Z_{-1} denotes the completion of Z with respect the norm $||z||_{-1} := ||z||_Z + ||(\lambda - A)^{-1}||_Z$, for a fixed λ in the resolvent set $\varrho(A)$. A_{-1} is the extension of A on Z, precisely A is the part of A_{-1} on X, see for instance [8, Ch. V]. Moreover, we will denote $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$, and $\mathbb{1}_E$ will stand for the characteristic function of the set E.

$$\mathcal{A}X = \mathcal{A}\begin{pmatrix} f \\ g \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} g \\ \varepsilon \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2} \\ \alpha_1 \frac{\partial f}{\partial x_1}(0,\cdot,\cdot) \\ \alpha_2 \frac{\partial f}{\partial x_2}(\cdot,0,\cdot) \\ \alpha_3 \frac{\partial f}{\partial x_3}(\cdot,\cdot,0) \end{pmatrix}, \tag{2.12}$$

and the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{U}; \mathcal{X}_{-1}), \mathbb{U} := \prod_{i=1}^{3} L^2(\Gamma_i)$,

$$\mathcal{B}\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_2 \end{pmatrix} := \mathcal{A}_{-1} \sum_{i=1}^{3} (\mu_i \otimes b_i) \tag{2.13}$$

where

$$b_i := \begin{pmatrix} 0 \\ -\mathbb{1}_{\Gamma_i} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mu_i \otimes b_i := \begin{pmatrix} 0 \\ -\mu_i(\cdot)\mathbb{1}_{\Gamma_i} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{for } i = 1, 2, 3.$$

Consequently, system (2.7) simplifies on $\mathcal{X}$ to the following abstract control system

$$\dot{X}(t) = \mathcal{A}_{-1}X(t) + \mathcal{B}\dot{V}(t), \ t \geq 0, \ X(0) = 0_{\mathcal{X}} \text{ the zero of } \mathcal{X}. \tag{2.14}$$

In the subsequent lemma, we shall establish the well-posedness of the free system corresponding to (2.14). This means that system (2.14), with $\dot{V} \equiv 0$, possesses a unique solution that depends continuously on the initial conditions $X_0 = X(0)$. By virtue of [8, p. 151], this is equivalent to proving that $\mathcal{A}$ generates a strongly continuous semigroup.

Lemma 2.1. *The operator $\mathcal{A}$ defined by (2.12) is skew adjoint. Consequently, $\mathcal{A}$ generates a group of isometries $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$.*

Proof. By virtue of [28, Prop. 3.7], it suffices to show that $\mathcal{A}$ is onto and skew-symmetric operator. Indeed, let $X = (f, g, g(0,\cdot,\cdot), g(\cdot,0,\cdot), g(\cdot,\cdot,0))^\top$ and $Y = (\varphi, \phi, \phi(0,\cdot,\cdot), \phi(\cdot,0,\cdot), \phi(\cdot,\cdot,0))^\top$ tow elements in $\mathcal{D}(A)$. Using Fibuni's theorem and Green's formula we get

$$
\begin{aligned}
< \mathcal{A}X, Y > &= \varepsilon \int_\Omega \Big[\frac{\partial g}{\partial x_1}\cdot\frac{\partial \varphi}{\partial x_1} + \frac{\partial g}{\partial x_2}\cdot\frac{\partial \varphi}{\partial x_2} + \frac{\partial g}{\partial x_3}\cdot\frac{\partial \varphi}{\partial x_3}\Big] dx_1dx_2dx_3 \\
&+ \int_\Omega \varepsilon\Big[\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\Big]\cdot \phi dx_1dx_2dx_3 + \varepsilon\int_0^1 \frac{\partial f}{\partial x_1}(0,x_2,x_3)\phi(0,x_2,x_3)dx_2dx_3 \\
&+\varepsilon\int_0^1 \frac{\partial f}{\partial x_2}(x_1,0,x_3)\phi(x_1,0,x_3)dx_1dx_3 + \varepsilon\int_0^1 \frac{\partial f}{\partial x_3}(x_1,x_2,0)\phi(x_1,x_2,0)dx_1dx_2 \\
&= -\varepsilon\int_\Omega g\Big[\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}\Big]dx_1dx_2dx_3 \\
&- \int_\Omega \varepsilon\Big[\frac{\partial f}{\partial x_1}\cdot\frac{\partial \phi}{\partial x_1} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial \phi}{\partial x_2} + \frac{\partial f}{\partial x_3}\cdot\frac{\partial \phi}{\partial x_3}\Big] dx_1dx_2dx_3 \\
&-\varepsilon\int_0^1 f(0,x_2,x_3)\frac{\partial\varphi}{\partial x_1}(0,x_2,x_3)dx_2dx_3 - \varepsilon\int_0^1 f(x_1,0,x_3)\frac{\partial\varphi}{\partial x_2}(x_1,0,x_3)dx_1dx_3 \\
&- \varepsilon\int_0^1 f(x_1,x_2,0)\frac{\partial\varphi}{\partial x_3}(x_1,x_2,0)dx_1dx_2 \\
&= - < X, \mathcal{A}Y > .
\end{aligned}
$$

It follows that $\mathcal{A}^*X = -\mathcal{A}X$ for all $X \in \mathcal{D}(A)$. Therefore $\mathcal{A}$ is skew-symmetric operator.

To show that A is onto, let $X = (f, g, g(0,\cdot,\cdot), g(\cdot,0,\cdot), g(\cdot,\cdot,0))^\top$ in $D(A)$ and $Y = (\varphi, \phi, \gamma, \delta, \chi)^\top$ in $\mathcal{X}$. The equation

$$\mathcal{A}X = Y,\ \ X \in \mathcal{D}(\mathcal{A}) \tag{2.15}$$

is equivalent, in a expended form, to the following system

$$
\begin{aligned}
&g \in H^1_\Gamma(\Omega),\ \ g = \varphi, \\
&\begin{cases} f \in H^2(\Omega),\ \ f_{|\Gamma} = 0, \\ \varepsilon\sum_{i=1}^3 \frac{\partial^2 f}{\partial x_i^2} = \phi, \\ \alpha_i \frac{\partial f}{\partial x_i}(y_i, 0) = \gamma(y_i),\ \text{ in } \Upsilon_i,\ \ i = 1,2,3. \end{cases}
\end{aligned} \tag{2.16}
$$

The PDE (2.16) corresponds to Poisson's equation with mixed (Dirichlet-Neuman) boundary conditions. It therefore has a unique solution $f_0 \in H^2(\Omega)$, see for instance [22, p. 782]. Consequently, the equation (2.15) is solvable by

$$\theta = \begin{pmatrix} f_0 \\ h \\ h(0,\cdot,\cdot) \\ h(\cdot,0,\cdot) \\ h(\cdot,\cdot,0) \end{pmatrix}.$$

This demonstrates that $\mathcal{A}$ is onto. The rest is a consequence of a standard Stone's theorem for a skew-adjoint operator on a Hilbert space, see e.g. [8, Thm 3.24]. $\square$

In the presence of the control, i.e., when $\mathcal{B} \neq 0$, we recall that the (mild) solution of system (2.14), with $X_0 = 0$, is defined by

$$X(t) = \int_0^t e^{\mathcal{A}_{-1}(t-\sigma)}\mathcal{B}\dot{V}(\sigma)d\sigma.$$

It is crucial to highlight that $X(t)$ belongs to $\mathcal{X}_{-1}$. To ensure that $X(t)$ is in $\mathcal{X}$, for all $t \geq 0$, it suffices to demonstrate that $\mathcal{B}$ is an admissible control operator for $\mathcal{T}$ in the sense defined in [28], i.e. there exists $\tau > 0$ such that

$$\mathcal{K}_\tau u := \int_0^\tau e^{\mathcal{A}_{-1}(\tau-\sigma)}\mathcal{B}v(\sigma)d\sigma \in \mathcal{X}$$

for all $v \in L^2([0,\tau];\mathbb{U})$.

Lemma 2.2. *The control operator $\mathcal{B}$ defined by (2.13) is admissible for $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$.*

Proof. Let $\tau > 0$ and let $\nu : [0,\tau] \longrightarrow L^2(\Gamma_1)$, $\mu : [0,\tau] \longrightarrow L^2(\Gamma_2)$, and $\varrho : [0,\tau] \longrightarrow L^2(\Gamma_3)$ be a step functions ;

$$\mu(s) = \sum_{i=0}^{n} \mu_i(1,\cdot)\mathbb{1}_{[\tau_i,\tau_{i+1}]}(s), \quad \nu(s) = \sum_{i=0}^{n} \nu_i(1,\cdot)\mathbb{1}_{[\tau_i,\tau_{i+1}]}(s) \quad \text{and}$$

$$\varrho(s) = \sum_{i=0}^{n} \varrho_i(1,\cdot)\mathbb{1}_{[\tau_i,\tau_{i+1}]}(s).$$

Since $\mu_i \otimes b_1$ and $\nu_i \otimes b_2$ being in $\mathcal{X}$, we obtain

$$\begin{aligned}
\mathcal{K}_\tau \begin{pmatrix} \mu \\ \nu \\ \varrho \end{pmatrix} &= \int_0^\tau e^{\mathcal{A}_{-1}(\tau-s)}\mathcal{B}\begin{pmatrix} \mu(s) \\ \nu(s) \\ \varrho(s) \end{pmatrix} ds \\
&= \sum_{i=0}^{n} \int_{\tau_i}^{\tau_{i+1}} e^{\mathcal{A}_{-1}(\tau-s)}\mathcal{A}_{-1}\left(\mu_i \otimes b_1 + \nu_i \otimes b_2 + \varrho_i \otimes b_3\right) ds \\
&= \sum_{i=0}^{n} \mathcal{A} \int_{\tau_i}^{\tau_{i+1}} e^{\mathcal{A}(\tau-s)}\left(\mu_i \otimes b_1 + \nu_i \otimes b_2 + \varrho_i \otimes b_3\right) ds \\
&= \sum_{i=0}^{n} \left(e^{\mathcal{A}_{-1}(\tau-\tau_{i+1})} - e^{\mathcal{A}_{-1}(\tau-\tau_i)}\right)\left(\mu_i \otimes b_1 + \nu_i \otimes b_2 + \varrho_i \otimes b_3\right) ds.
\end{aligned}$$

It follows that $\mathcal{K}_\tau \begin{pmatrix} \mu \\ \nu \\ \varrho \end{pmatrix} \in \mathcal{X}$. Moreover, it is not difficult to see that there exists a nonnegative constant $C = C(\tau, b_1, b_2, b_3)$ such that

$$\left|\left|\mathcal{K}_\tau \begin{pmatrix} \mu \\ \nu \\ \varrho \end{pmatrix}\right|\right|_{\mathcal{X}} \leq C \cdot \left(||\mu||_{L^2(\Gamma_1)} + ||\nu||_{L^2(\Gamma_2)} + ||\varrho||_{L^2(\Gamma_3)}\right).$$

Since the set of step functions is dense in $L^2([0,\tau];\mathbb{U})$, the operator $\mathcal{K}_\tau$ defines a bounded linear operator from $L^2([0,\tau];\mathbb{U})$ into $\mathcal{X}$. This shows the admissibility of $\mathcal{B}$ for $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$. ☐

By virtue of [28, Prop. 4.2.5], the admissibility of $\mathcal{B}$ for $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$ implies that for $V \in H^1_{loc}((0,\infty);\mathbb{U})$, hence $\dot{V} \in L^2_{loc}((0,\infty);\mathbb{U})$, the unique (mild) solution $X(\cdot)$ of the abstract control system (2.14) is given by

$$X(t) = \mathcal{K}_t\dot{V} = \int_0^t e^{\mathcal{A}_{-1}(t-s)}\mathcal{B}\dot{V}(t)ds \tag{2.17}$$

and it satisfies $X(\cdot) \in C([0,\infty),\mathcal{X}) \cap H^1_{loc}((0,\infty),\mathcal{X}_{-1})$.

3. The controllers in the form of feedback laws

In the sequel, we assume that

$$V_1(t,x_2,0) = V_1(t,0,x_3) = V_2(t,x_1,0) = V_2(t,0,x_3) = V_3(t,x_1,0) = V_3(t,0,x_2) = 0. \tag{3.1}$$

Since $\mathcal{A}$ is skew adjoint, then $\sigma(\mathcal{A}) \subseteq i\mathbb{R}$. This implies that the growth bound $\omega_0(\mathcal{T})$ of $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$ is negative. Therefore for each $X \in \mathcal{X}$, the Laplace transform $\widehat{e^{\mathcal{A}\cdot}X}(\lambda)$[2] exists and equals $R(\lambda,\mathcal{A})X$ for all $\lambda > 0$. Applying the Laplace transform to both side of (2.17) we get

$$\widehat{X}(\lambda) = [\lambda^2 R(\lambda,\mathcal{A}) - \lambda]\left(\widehat{V_1}(\lambda)\otimes b_1 + \widehat{V_2}(\lambda)\otimes b_2 + \widehat{V_3}(\lambda)\otimes b_3\right), \tag{3.2}$$

for every $\lambda > 0$, bearing in mind that $V_1(0,\cdot) = V_2(0,\cdot) = V_3(0,\cdot) = 0$. For the rest, it will be convenient to put

$$R(\lambda,\mathcal{A})\left(\widehat{V}_i(\lambda)\otimes b_i\right) = Y_i := \begin{pmatrix} \Phi_i \\ f_i \\ f_i(0,\cdot,\cdot) \\ f_i(\cdot,0,\cdot) \\ f_i(\cdot,\cdot,0) \end{pmatrix}, \; i = 1,2,3. \tag{3.3}$$

We must point out here that the functions Φ_i, $i = 1,2,3$, depend on λ and $\mathbf{x}$, but for simplicity we will write $\Phi_i(\mathbf{x})$, where $\mathbf{x} = \{x_1,x_2,x_3\}$.
Now, using the representation (3.3), we obtain from (3.2)

$$\widehat{X}(\lambda) = \lambda^2\sum_{i=1}^{3} Y_i + \lambda\sum_{i=1}^{3}\widehat{V}_i(\lambda)\otimes b_i. \tag{3.4}$$

By comparing the first component of the both sides of (3.4) one gets

$$\widehat{\omega}(\lambda,\mathbf{x}) = \lambda^2\sum_{i=1}^{3}\Phi_i(\mathbf{x}) + \sum_{i=1}^{3}\widehat{V}_i(\lambda,y_i)\mathbb{1}_{\Gamma_i}(\mathbf{x}). \tag{3.5}$$

---

[2]For a locally Bochner integrable function $f : \mathbb{R} \to \mathcal{X}$, $\widehat{f}(\lambda)$ denotes the Laplace transform of $f(t)$,

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-\lambda t}f(t)dt$$

whenever this limit exists.

$y_i := \{x_1, x_{i-1}, x_{i+1}, x_3\}$. Moreover

$$\widehat{\omega}(\lambda, y_i, 0) = \lambda^2 \left[\Phi_1(y_i, 0) + \Phi_2(y_i, 0) + \Phi_3(y_i, 0)\right], \qquad \text{for } i = 1, 2, 3. \tag{3.6}$$

The next steep is to determine the functions Φ_i, $i = 1, 2, 3$. To get Φ_1 we should solve (3.3). Indeed, equation (3.3) can be written as

$$(\lambda - \mathcal{A}) \begin{pmatrix} \Phi_1 \\ f_1 \\ f_1(0, \cdot, \cdot) \\ f_1(\cdot, 0, \cdot) \\ f_1(\cdot, \cdot, 0) \end{pmatrix} = \widehat{V_1}(\lambda) \otimes b_1, \tag{3.7}$$

is equivalent to the following system

$$\begin{cases} f_1 = \lambda \Phi_1, \quad \Phi_1 \in H^2(\Omega), \quad \Phi_{1|_\Gamma} = 0, \\ \varepsilon \sum_{i=1}^3 \frac{\partial^2 \Phi_1}{\partial x_i^2} - \lambda^2 \Phi_1 = \widehat{V_1}(\lambda, \cdot), \quad \text{in } \Omega, \\ \alpha_i \frac{\partial^2 \Phi_1}{\partial x_i^2}(y_i, 0) - \lambda^2 \Phi_1(y_i, 0) = 0, \quad \text{in } \Gamma_i, \qquad \text{for } i = 1, 2, 3. \end{cases} \tag{3.8}$$

Using the method of Green's function [2, 18], the solution of system (3.8) is given by:

$$\Phi_1(\mathbf{x}) = \int_\Omega \mathcal{G}(\mathbf{x}, \mathbf{r}) \widehat{V_1}(\lambda, r_2, r_3) dr_1 dr_2 dr_3, \tag{3.9}$$

where $\mathbf{r} = \{r_1, r_2, r_3\}$, and the Green's function $\mathcal{G}$ satisfies the differential equation

$$\varepsilon \sum_{i=1}^3 \frac{\partial^2 \mathcal{G}}{\partial x_i^2}(\mathbf{x}, \mathbf{r}) - \lambda^2 \mathcal{G}(\mathbf{x}, \mathbf{r}) = \delta(\mathbf{x} - \mathbf{r}), \tag{3.10}$$

where $\delta(\mathbf{x} - \mathbf{r}) := \delta(x_1 - r_1)\delta(x_2 - r_2)\delta(x_3 - r_3)$, and the boundary conditions of (3.8), for all $x_i, r_i \in [0, 1]$, for $i = 1, 2, 3$. Here δ is the Dirac function.
To determine $\mathcal{G}$, we use the method of eigenfunction expansions, see [2, Chp.10]. The idea of this method is to construct $\mathcal{G}$ via the eigenfunctions end eigenvalues of the unbounded operator Λ defined on the Hilbert space $L^2(\Omega)$ by

$$\Lambda f := \varepsilon \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} f,$$

$$\mathcal{D}(\Lambda) := \{f \in H^2(\Omega) : \ f_{|\Gamma} = 0, \ \alpha_i \frac{\partial f}{\partial x_i}(y_i, 0) = \lambda^2 f(y_i, 0), \text{ for } i = 1, 2, 3\}.$$

We must first establish the following preliminary result:

Lemma 3.1.

i) *The operator $(\Lambda, D(\Lambda))$ is self-adjoint.*

ii) *The eigenvalues of $(\Lambda, D(\Lambda))$ are given by*

$$\kappa_{m,n,p} = -\varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2), \ \textit{for} \ \ n, m, p = 1, 2, 3, \cdots,$$

with corresponding normalized eigenfunctions given by

$$\varpi_{n,m,p} = \frac{\upsilon_{n,m,p}}{\|\upsilon_{n,m,p}\|_{L^2(\Omega)}}$$

where

$$\begin{aligned} v_{n,m,p}(\mathbf{x}) =& C \cdot \Big(\mu_n \cos(\mu_n x_1) + \frac{\lambda^2}{\alpha_1}\sin(\mu_n x_1)\Big) \times \Big(\nu_n \cos(\nu_n x_2) + \frac{\lambda^2}{\alpha_2}\sin(\nu_n x_2)\Big) \\ & \times \Big(\sigma_p \cos(\sigma_p x_3) + \frac{\lambda^2}{\alpha_3}\sin(\sigma_p x_3)\Big), \end{aligned} \tag{3.11}$$

and the parameters μ_n, ν_m, σ_p are determined as roots of the following transcendental equations: $\mu\cos(\mu)+\frac{\lambda^2}{\alpha_1}\sin(\mu) = 0$, $\nu\cos(\nu)+\frac{\lambda^2}{\alpha_2}\sin(\nu) = 0$, and $\sigma\cos(\sigma) + \frac{\lambda^2}{\alpha_3}\sin(\sigma) = 0$, respectively.

iii) *The eigenfunctions $(\varpi_{n,m,p})_{n,m,p\in\mathbb{N}}$ form an orthonormal basis of $L^2(\Omega)$.*

Proof. (i) For $f, g \in D(\Lambda)$ we have

$$\begin{aligned} \varepsilon^{-1}\langle \Lambda f, g\rangle_{L^2} =& \int_\Omega \Big[\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(x_1,x_2,x_3)\Big] g dx_1dx_2dx_3 \\ =& -\int_\Omega \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_3} dx_1dx_2dx_3 - \frac{\partial f}{\partial x_1}(0,x_2,x_3)g(0,x_2,x_3) \\ & - \frac{\partial f}{\partial x_2}(x_1,0,x_3)g(x_1,0,x_3) - \frac{\partial f}{\partial x_3}(x_1,x_2,0)g(x_1,x_2,0), \\ =& -\int_\Omega \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_3} dx_1dx_2dx_3 - \frac{\lambda^2}{\alpha_1} f(0,x_2,x_3)g(0,x_2,x_3) \\ & - \frac{\lambda^2}{\alpha_2} f(x_1,0,x_3)g(x_1,0,x_3) - \frac{\lambda^2}{\alpha_3} f(x_1,x_2,0)g(x_1,x_2,0), \\ =& \int_\Omega \Big[\sum_{i=1}^{3} f\frac{\partial^2 g}{\partial x_i^2}(x_1,x_2,x_3)\Big] dx_1dx_2dx_3, \\ =& \varepsilon^{-1}\langle f, \Lambda g\rangle_{L^2}. \end{aligned}$$

This shows that Λ is self-adjoint.

To show (ii), observe first that since Λ is self-adjoint, then all the eigenvalues are real. Let $\kappa \in \mathbb{R}$. The eigenproblem

$$\varpi \in D(\Lambda),\ \Lambda\varpi = \kappa\varpi,$$

is equivalent to the system

$$\begin{cases} \varepsilon\Delta\varpi(\mathbf{x}) - \kappa\varpi(\mathbf{x}) = 0, \\ \alpha_i \frac{\partial}{\partial x_i}\varpi(y_i,0) - \lambda^2\varpi(y_i,0) = 0, \quad y_i \in \Upsilon_i,\ \text{for } i = 1,2,3 \quad \varpi_{/\Gamma} = 0. \end{cases} \tag{3.12}$$

To solve (3.12), we look for a solution in the following form:

$$\varpi(\mathbf{x}) = X(x_1)Y(x_2)Z(x_3).$$

Substituting this expression in (3.12) we find

$$\frac{Y''}{Y} + \frac{Z''}{Z} - \frac{\kappa}{\varepsilon} = -\frac{X''}{X}.$$

Which means that

$$-\frac{X''}{X} = \mu,$$

where $\mu := \mu(x_2, x_3)$ is an (x_2, x_3)-depending constant. Moreover, in view of the boundary conditions of (3.12) the function $X(x)$ satisfies the following system

$$\begin{cases} X''(x_1) = -\mu X(x_1), \quad 0 < x_1 < 1, \\ \alpha_1 X'(0) - \lambda^2 X(0) = 0, \\ X(1) = 0. \end{cases} \tag{3.13}$$

If $\mu \leq 0$, then it is not difficult to see that equation (3.13) has a unique solution $X \equiv 0$. Therefore system (3.12) is uniquely solvable by $\varpi \equiv 0$.
If $\mu > 0$, the solution of (3.13) has the following form

$$X(x_1) = C_1 \cos(\sqrt{\mu} x_1) + D_1 \sin(\sqrt{\mu} x_1),$$

where C_1 and D_1 are constants. On account of the boundary conditions of (3.13) we get

$$\begin{cases} C_1 \cos(\sqrt{\mu}) + D_1 \sin(\sqrt{\mu}) = 0, \\ D_1 = C_1 \frac{\lambda^2}{\alpha_1 \sqrt{\mu}}. \end{cases}$$

This implies that

$$\sqrt{\mu} \cos(\sqrt{\mu}) + \frac{\lambda^2}{\alpha_1} \sin(\sqrt{\mu}) = 0 \tag{3.14}$$

The equation (3.14) has a countable set of roots $\mu = \mu_n^2$, $n \in \mathbb{N}$. This can be justified by the fact that the curve of $\mu \mapsto \sqrt{\mu} \cos(\sqrt{\mu}) + \frac{\lambda^2}{\alpha_1} \sin(\sqrt{\mu})$ and the abscissa axis has a countable points of intersection for $\mu > 0$. Consequently, for each $n \in \mathbb{N}$, the solution of (3.13) corresponding to the root $\mu = \mu_n^2$ is given by

$$X_n(x_1) = C_1 \cdot \left(\mu_n \cos(\mu_n x_1) + \frac{\lambda^2}{\beta} \sin(\mu_n x_1) \right), \tag{3.15}$$

We now turn to solve the Y and Z-PDE :

$$\begin{cases} Y''(x_2) = -\nu^2 Y(x_2), \quad 0 < x_2 < 1, \\ \alpha_2 Y'(0) - \lambda^2 Y(0) = 0, \\ Y(1) = 0, \end{cases} \tag{3.16}$$

and

$$\begin{cases} Z''(x_3) = -\sigma^2 Z(x_3), \quad 0 < x_3 < 1, \\ \alpha_3 Z'(0) - \lambda^2 Z(0) = 0, \\ Z(1) = 0. \end{cases} \tag{3.17}$$

If $\kappa = \kappa_{m,n,p} = -\varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)$, it follows from the first equation of (3.16) and (3.17) that $Y''(x_2) = -\nu_m^2 Y(x_2)$ and $Z''(x_3) = -\sigma_p^2 Z(x_3)$. Then, by arguing as for X, if ν_m and $\sigma_p, m, p \in \mathbb{N}$ are respectively roots of the transcendental equation $\nu \cos(\nu) + \frac{\lambda^2}{\alpha_2} \sin(\nu) = 0$ and $\sigma \cos(\sigma) + \frac{\lambda^2}{\alpha_3} \sin(\sigma) = 0$, then for each m and p the solutions of (3.16)-(3.17) corresponding to ν_m^2 and σ_p^2 are given by

$$Y_m(x_2) = C_2 \cdot \left(\nu_m \cos(\nu_m x_2) + \frac{\lambda^2}{\alpha_2} \sin(\nu_m x_2) \right), \tag{3.18}$$

and

$$Z_p(x_3) = C_3 \cdot \left(\sigma_p \cos(\sigma_p x_3) + \frac{\lambda^2}{\alpha_3} \sin(\sigma_p x_3) \right). \tag{3.19}$$

Therefore the solutions of (3.12) corresponding to the eigenvalues $\kappa_{m,n,p} = -\varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)$ are given by

$$\begin{aligned} v_{n,m,p}(\mathbf{x}) =& C \cdot \left(\mu_n \cos(\mu_n x_1) + \frac{\lambda^2}{\alpha_1} \sin(\mu_n x_1)\right) \times \left(\nu_n \cos(\nu_n x_2) + \frac{\lambda^2}{\alpha_2} \sin(\nu_n x_2)\right) \\ & \times \left(\sigma_p \cos(\sigma_p x_3) + \frac{\lambda^2}{\alpha_3} \sin(\sigma_p x_3)\right), \end{aligned} \tag{3.20}$$

for $n, m, p \in \mathbb{N}$. Thus, the normalized eigenfunctions are given by $\varpi_{n,m,p} = \frac{v_{n,m,p}}{\|v_{n,m,p}\|}$, $n, m, p \in \mathbb{N}$. Finally, the statement (iii) is a consequence of (i) and a well-known result of a self-adjoint operator given on a Hilbert space. □

Now, let us write the Green's function $\mathcal{G}$ in the orthonormal basis $(\varpi_{n,m,p})$,

$$\mathcal{G}(\mathbf{x}, \mathbf{r}) = \sum_{n,m=0}^{\infty} a_{n,m,p} \varpi_{n,m,p}(\mathbf{x}).$$

Using that $\left[\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + (\mu_n^2 + \nu_m^2 + \sigma_p^2)\right] \varpi_{n,m,p}(\mathbf{x}) = 0$ and observing that

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{r}) &= \sum_{m,n=0}^{\infty} \varpi_{n,m,p}(\mathbf{x}) \int_\Omega \delta(r_1 - \xi)\delta(r_2 - \eta)\delta(r_3 - \rho)\varpi_{n,m,p}(\xi, \eta, \rho) d\xi d\eta d\rho \\ &= \sum_{m,n=0}^{\infty} \varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r}) \end{aligned}$$

we find from (3.10) that

$$\sum_{n,m=0}^{\infty} a_{n,m,p}\big(\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)\big)\varpi_{n,m,p}(\mathbf{x}) = -\sum_{n=0}^{\infty} \varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r}).$$

If $\lambda^2 \neq -\varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)$, then

$$a_{n,m,p} = \frac{-1}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)} \varpi_{n,m,p}(\mathbf{r}).$$

The Green's function becomes

$$\mathcal{G}(\mathbf{x}, \mathbf{r}) := \sum_{n,m,p=0}^{\infty} \frac{-\varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r})}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)}.$$

Further, in view of (3.9) the solution of (3.8) is given by

$$\Phi_1(\mathbf{x}) = \int_\Omega \sum_{n,m,p=0}^{\infty} \frac{-\varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r})}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)} \widehat{V_1}(\lambda, r_2, r_3) dr_1 dr_2 dr_3, \tag{3.21}$$

for x_i in $[0, 1]$, $i = 1, 2, 3$. In the same way, we also obtain

$$\Phi_2(\mathbf{x}) = \int_\Omega \sum_{n,m,p=0}^{\infty} \frac{-\varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r})}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)} \widehat{V_2}(\lambda, r_1, r_3) dr_1 dr_2 dr_3, \tag{3.22}$$

and

$$\Phi_3(\mathbf{x}) = \int_\Omega \sum_{n,m,p=0}^{\infty} \frac{-\varpi_{n,m,p}(\mathbf{x})\varpi_{n,m,p}(\mathbf{r})}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)} \widehat{V_3}(\lambda, r_1, r_2) dr_1 dr_2 dr_3. \tag{3.23}$$

The next steep is to determine $V_1(t, y_1)$. Indeed, substituting (3.21)-(3.23) in (3.6) to obtain

$$\widehat{\omega}(\lambda, y_1, 0) = \sum_{n,m,p=0}^{\infty} \frac{-\lambda^2 \cdot \varpi_{n,m,p}(y_1, 0)}{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)} \int_\Omega \varpi_{n,m,p}(\mathbf{r}) \Big[\widehat{V_1}(\lambda, r_2, r_3) + \widehat{V_2}(\lambda, r_1, r_3) + \widehat{V_3}(\lambda, r_1, r_2)\Big] dr_1 dr_2 dr_3. \tag{3.24}$$

From (3.24) and the uniqueness of the decomposition of the vector $\widehat{\omega}(\lambda, y_1, 0)$ in the basis $(\varpi_{n,m,p})$ it follows that

$$\int_\Omega \varpi_{n,m,p}(\mathbf{r}) \Big[\widehat{V_1}(\lambda, r_2, r_3) + \widehat{V_2}(\lambda, r_1, r_3) + \widehat{V_3}(\lambda, r_1, r_2)\Big] dr_1 dr_2 dr_3 = \frac{\lambda^2 + \varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2)}{-\lambda^2} \int_\Omega \varpi_{n,m,p}(\mathbf{r}) \widehat{\omega}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3.$$

Therefore, by (3.1), i.e $V_2(t, 0, x_3) = V_3(t, 0, x_2) = 0$, we have

$$\begin{aligned} \widehat{V_1}(\lambda, y_1) &= \widehat{V_1}(\lambda, y_1) + \widehat{V_2}(\lambda, 0, x_3) + \widehat{V_3}(\lambda, 0, x_2), \\ &= \sum_{n=0}^{\infty} \varpi_{n,m,p}(0, x_2, x_3) \int_\Omega \varpi_{n,m,p}(\mathbf{r}) \left[\widehat{V_1}(\lambda, r_2, r_3) + \widehat{V_2}(\lambda, r_2, r_3) + \widehat{V_3}(\lambda, r_1, r_2)\right] dr_1 dr_2 dr_3, \\ &= -\sum_{n,m,p=0}^{\infty} \frac{\varepsilon(\mu_n^2 + \nu_m^2 + \sigma_p^2) + \lambda^2}{\lambda^2} \varpi_{n,m,p}(0, x_2, x_3) \int_\Omega \varpi_{n,m,p}(\mathbf{r}) \widehat{z}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3. \end{aligned} \tag{3.25}$$

Observing that

$$\begin{aligned} \widehat{z}(\lambda, y_1, 0) &= \sum_{n,m=0}^{\infty} \varpi_{n,m,p}(\mathbf{x}) \int_\Omega \varpi_{n,m,p}(0, r_2, r_3) \widehat{z}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3 d\eta, \\ &= \int_\Omega \delta(x_1 - 0)\delta(x_2 - r_2)\delta(x_3 - r_3) \widehat{z}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3, \\ &= \int_\Omega \delta(0 - r_1)\delta(x_2 - r_2)\delta(x_3 - r_3) \widehat{z}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3, \end{aligned}$$

we easily conclude

$$\begin{aligned} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \widehat{z}(\lambda, y_1, 0) &= \frac{\partial^2}{\partial x_2^2} \widehat{z}(\lambda, y_1, 0) + \frac{\partial^2}{\partial x_3^2} \widehat{z}(\lambda, y_1, 0) \\ &= \sum_{n,m,p=0}^{\infty} -(\mu_n^2 + \nu_m^2 + \sigma_p^2) \varpi_{n,m,p}(0, x_2, x_3) \int_\Omega \varpi_{n,m,p}(\mathbf{r}) \widehat{z}(\lambda, 0, r_2, r_3) dr_1 dr_2 dr_3. \end{aligned} \tag{3.26}$$

From (3.25)–(3.26) we obtain

$$\begin{aligned}\widehat{V_1}(\lambda, y_1) &= -\sum_{n,m,p=0}^{\infty} \frac{\varepsilon(\mu_n^2+\nu_m^2+\sigma_p^2)+\lambda^2}{\lambda^2}\varpi_{n,m,p}(y_1,0)\int_\Omega \varpi_{n,m,p}(\mathbf{r})\widehat{z}(\lambda,0,r_2,r_3)d\xi d\eta,\\ &= -\sum_{n,m,p=0}^{\infty} \frac{\varepsilon(\mu_n^2+\nu_m^2+\sigma_p^2)}{\lambda^2}\varpi_{n,m,p}(y_1,0)\int_\Omega \varpi_{n,m,p}(\mathbf{r})\widehat{z}(\lambda,0,r_2,r_3)d\xi d\eta\\ &- \widehat{z}(\lambda,y_1,0),\\ &= \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_2^2}\widehat{z}(\lambda,y_1,0)+\frac{\partial^2}{\partial x_3^2}\widehat{z}(\lambda,y_1,0)\Big]-\widehat{z}(\lambda,y_1,0).\end{aligned}$$

Since $\widehat{z}(\lambda, y_1, 0) = \big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)$, then

$$\widehat{V_1}(\lambda, y_1) = \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)\Big]-\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0). \tag{3.27}$$

In the same way, we also obtain

$$\widehat{V_2}(\lambda, y_2) = \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)\Big]-\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0), \tag{3.28}$$

and

$$\widehat{V_3}(\lambda, y_3) = \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)+\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)\Big]-\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0). \tag{3.29}$$

In order to give an explicit expression of the controllers $U_1(t, y_1)$, $U_2(t, y_2)$, and $U_3(t, y_3)$ we perform the following inverse transformation

$$u(t,\mathbf{x}) = (\mathcal{B}^{-1}z(t))(\mathbf{x}) := z(t,\mathbf{x}) - \sum_{i=1}^{3}\int_0^{x_i} p_i(x_i,r_i)z(t,y_i,r_i)dr_i \tag{3.30}$$

where z satisfies (2.7), p_i are functions to be appropriately determined. In fact, when we follow a procedure similar to that performed in section 2, it can be seen that the inverse transformation (3.30) maps (2.7) into (1.1) provided that m and n satisfy respectively the following PDEs :

$$\begin{cases}\frac{\partial}{\partial^2 x_i^2}p_i(x_i,r_i) - \frac{\partial}{\partial^2 r_i^2}p_i(x_i,r_i) = \frac{q}{\varepsilon}p_i(x_i,r_i), & 0\le r_i\le x_i\le 1,\\ p_i(x_i,x_i) = -\frac{c_i}{2\varepsilon}x_i, & 0\le x_i\le 1,\\ p_i(x_i,0) = \frac{\partial}{\partial r_i}p_i(x_i,0) = 0, & 0\le x_i\le 1,\end{cases} \tag{3.31}$$

for $i = 1, 2, 3$. In a very similar way, on can show that the PDEs (3.31) have, respectively, the following solutions

$$p_i(x_i,r_i) = -\frac{c_i}{\varepsilon}r_i\frac{I_1\Big(\sqrt{\frac{q}{\varepsilon}(x_i^2-r_i^2)}\Big)}{\sqrt{\frac{q}{\varepsilon}(x_i^2-r_i^2)}}, \qquad \text{for } i=1,2,3,$$

where I_1 is the modified Bessel function of order one.
Now, if z satisfies (2.7), it results from (3.27)–(3.30) that

$$\begin{aligned}
\widehat{U_1}(\lambda, y_1) &= (\mathcal{B}^{-1}z(t))(y_1,1),\\
&= \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)\Big] - \big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_2} p_2(x_2,r_2)\Big[\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,r_2,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,r_2,x_3)\Big]dr_2\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_3} p_3(x_3,r_3)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,x_2,r_3) + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,x_2,r_3)\Big]dr_3\\
&+ \int_0^{x_2} p_2(x_2,r_2)\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,r_2,x_3)dr_2 + \int_0^{x_3} p_3(x_3,r_3)\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(0,x_2,r_3)dr_3\\
&- \int_0^1 p_1(1,r_1)z(\lambda,r_1,x_2,x_3)dr_1,
\end{aligned}\tag{3.32}$$

$$\begin{aligned}
\widehat{U_2}(\lambda, y_2) &= (\mathcal{B}^{-1}z(t))(y_2,1),\\
&= \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)\Big] - \big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_1} p_1(x_1,r_1)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(r_1,0,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(r_1,0,x_3)\Big]dr_1\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_3} p_3(x_3,r_3)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(x_1,0,r_3) + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(x_1,0,r_3)\Big]dr_3\\
&+ \int_0^{x_1} p_1(x_1,r_1)\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(r_1,0,x_3)dr_1 + \int_0^{x_3} p_3(x_3,r_3)\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(x_1,0,r_3)dr_3\\
&- \int_0^1 p_2(1,r_2)z(\lambda,x_1,r_2,x_3)dr_2,
\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}
\widehat{U_3}(\lambda, y_3) &= (\mathcal{B}^{-1}z(t))(y_3,1),\\
&= \frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)\Big] - \big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_1} p_1(x_1,r_1)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(r_1,x_2,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(r_1,x_2,0)\Big]dr_1\\
&- \frac{\varepsilon}{\lambda^2}\int_0^{x_2} p_2(x_2,r_2)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(x_1,r_2,0) + \frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(x_1,r_2,0)\Big]dr_3\\
&+ \int_0^{x_1} p_1(x_1,r_1)\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(r_1,x_2,0)dr_2 + \int_0^{x_2} p_2(x_2,r_2)\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(x_1,r_2,0)dr_2\\
&- \int_0^1 p_3(1,r_3)z(\lambda,x_1,x_2,r_3)dr_3,
\end{aligned}\tag{3.34}$$

Observing that, when $[\varphi_1\ \varphi_2\ \varphi_3]^\top \in C([0,\infty);H^2(\Upsilon_1)) \times C([0,\infty);H^2(\Upsilon_2)) \times C([0,\infty);H^2(\Upsilon_3))$, the Laplace transform of the maps $t \mapsto \int_0^t (t-\sigma)\big(\mathcal{B}\varphi_1(\sigma)\big)(y_1,0)d\sigma$, $t \mapsto \int_0^t (t-\sigma)\big(\mathcal{B}\varphi_2(\sigma)\big)(y_2,0)d\sigma$, and $t \mapsto \int_0^t (t-\sigma)\big(\mathcal{B}\varphi_3(\sigma)\big)(y_3,0)d\sigma$ are respectively $\lambda \longrightarrow \frac{1}{\lambda^2}\big(\mathcal{B}\varphi_1(\lambda)\big)(y_1,0)$, $\lambda \longrightarrow \frac{1}{\lambda^2}\big(\mathcal{B}\varphi_2(\lambda)\big)(y_2,0)$, and $\lambda \longrightarrow \frac{1}{\lambda^2}\big(\mathcal{B}\varphi_3(\lambda)\big)(y_3,0)$. Taking this remark into account and by applying the Laplace transform to both sides of (3.32)-(3.34) and using (2.11), we get

$$
\begin{cases}
U_1(t,y_1) = & \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(y_1,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(y_1,0)\Big]ds - \big(\mathcal{B}\varphi_1(t)\big)(y_1,0) \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_2} p_2(x_2,r_2)\left[\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3)\right]dr_2 ds \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_3} p_3(x_3,r_3)\left[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3) + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3)\right]dr_3 ds \\
& + \int_0^{x_2} p_2(x_2,r_2)\big(\mathcal{B}\varphi_1(t)\big)(0,r_2,x_3)dr_2 + \int_0^{x_3} p_3(x_3,r_3)\big(\mathcal{B}\varphi_1(t)\big)(0,x_2,r_3)dr_3 \\
& - \int_0^1 p_1(1,r_1)z(t,r_1,x_2,x_3)dr_1, \\
U_1(t,0,x_3) = & -\int_0^1 k_1(1,r_1)\varphi_2(t,r_1,x_3)dr_1, \\
U_1(t,x_2,0) = & -\int_0^1 k_1(1,r_1)\varphi_3(t,r_1,x_2)dr_1,
\end{cases}
\tag{3.35}
$$

$$
\begin{cases}
U_2(t,y_2) = & \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(y_2,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(y_2,0)\Big]ds - \big(\mathcal{B}\varphi_2(t)\big)(y_2,0) \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_1} p_1(x_1,r_1)\left[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3)\right]dr_1 ds \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_3} p_3(x_3,r_3)\left[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3) + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3)\right]dr_3 ds \\
& + \int_0^{x_1} p_1(x_1,r_1)\big(\mathcal{B}\varphi_2(t)\big)(r_1,0,x_3)dr_1 + \int_0^{x_3} p_3(x_3,r_3)\big(\mathcal{B}\varphi_2(t)\big)(x_1,0,r_3)dr_3 \\
& - \int_0^1 p_2(1,r_2)z(t,x_1,r_2,x_3)dr_2, \\
U_2(t,0,x_3) = & -\int_0^1 k_2(1,r_2)\varphi_1(t,r_2,x_3)dr_2, \\
U_2(t,x_1,0) = & -\int_0^1 k_2(1,r_2)\varphi_3(t,x_1,r_2)dr_2,
\end{cases}
\tag{3.36}
$$

and

$$
\begin{cases}
U_3(t,y_2) = & \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(\lambda)\big)(y_3,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(\lambda)\big)(y_3,0)\Big]ds - \big(\mathcal{B}\varphi_3(t)\big)(y_3,0) \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_1} p_1(x_1,r_1)\left[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0)\right]dr_1 ds \\
& -\varepsilon \int_0^t (t-s)\int_0^{x_2} p_2(x_2,r_2)\left[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0) + \frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0)\right]dr_3 ds \\
& + \int_0^{x_1} p_1(x_1,r_1)\big(\mathcal{B}\varphi_3(t)\big)(r_1,x_2,0)dr_2 + \int_0^{x_2} p_2(x_2,r_2)\big(\mathcal{B}\varphi_3(t)\big)(x_1,r_2,0)dr_2 \\
& - \int_0^1 p_3(1,r_3)z(t,x_1,x_2,r_3)dr_3, \\
U_3(t,0,x_3) = & -\int_0^1 k_3(1,r_3)\varphi_1(t,x_2,r_3)dr_3, \\
U_3(t,x_1,0) = & -\int_0^1 k_3(1,r_3)\varphi_2(t,x_1,r_3)dr_3,
\end{cases}
\tag{3.37}
$$

where z satisfies (2.7), k_i and p_i, $i = 1,2,3$, are solutions of (2.6) and (3.31), respectively. We have thus proved the following.

Theorem 3.2. *Assume that* $[\varphi_1\ \varphi_2\ \varphi_3]^\top \in C([0,\infty); H^2(\Upsilon_1))\times C([0,\infty); H^2(\Upsilon_2))\times C([0,\infty); H^2(\Upsilon_3))$*. Then, the solution of the OTP for system* (1.1) *subject to the output equation* (1.2) *is given by the feedback laws* (3.35)–(3.37)*, where z satisfies* (2.7)*, k_i and p_i, $i = 1,2,3$, are solutions of* (2.6) *and* (3.31)*, respectively.*

Formulas (3.35)–(3.37) provide the controllers in the form of feedback laws. Furthermore, the solution of the corresponding closed-loop system perfectly matches the target trajectories φ_i from the boundaries Γ_i, for $i = 1,2,3$, at any coordinates $(t,\mathbf{x})$. To provide the explicit expressions of the controllers U_i, $i = 1,2,3$, we shall first give an explicit expression of the entire state $u(t,\mathbf{x})$, at any time-position $(t,\mathbf{x})$, where u still a solution of the OTP.

4. Explicit expression of the entire state $u(t,\mathbf{x})$

In this subsection, we show how to derive the explicit expression of $u(t,\mathbf{x})$ at any position $(t,\mathbf{x})$. Indeed, from (3.5) and (3.25)-(3.29) we obtain

$$\begin{aligned}
\widehat{z}(\lambda,\mathbf{x}) = \lambda^2 \int_\Omega \mathcal{G}(\mathbf{x},\mathbf{r})\Big[\widehat{V_1}(\lambda,r_2,r_3)+\widehat{V_2}(\lambda,r_1,r_3)+\widehat{V_3}(\lambda,r_1,r_2)\Big]d\mathbf{r} + \widehat{V_1}(\lambda,y_1)\mathbb{1}_{\Gamma_1}(\mathbf{x})\\
+\widehat{V_2}(\lambda,y_2)\mathbb{1}_{\Gamma_2}(\mathbf{x})+\widehat{V_3}(\lambda,y_3)\mathbb{1}_{\Gamma_3}(\mathbf{x}),\\
=\widehat{z}(\lambda,y_1,0)+\widehat{z}(\lambda,y_2,0)+\widehat{z}(\lambda,y_3,0)+\widehat{V_1}(\lambda,y_1)\mathbb{1}_{\Gamma_1}(\mathbf{x})+\widehat{V_2}(\lambda,y_2)\mathbb{1}_{\Gamma_2}(\mathbf{x})\\
+\widehat{V_3}(\lambda,y_3)\mathbb{1}_{\Gamma_3}(\mathbf{x}),\\
=\Big(\frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)\Big]-\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)\Big)\mathbb{1}_{\Gamma_1}(\mathbf{x})\\
+\Big(\frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)\Big]-\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)\Big)\mathbb{1}_{\Gamma_2}(\mathbf{x})\\
+\Big(\frac{\varepsilon}{\lambda^2}\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)+\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)\Big]-\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0)\Big)\mathbb{1}_{\Gamma_3}(\mathbf{x})\\
+\big(\mathcal{B}\widehat{\varphi_1}(\lambda)\big)(y_1,0)+\big(\mathcal{B}\widehat{\varphi_2}(\lambda)\big)(y_2,0)+\big(\mathcal{B}\widehat{\varphi_3}(\lambda)\big)(y_3,0). \qquad (4.1)
\end{aligned}$$

If $[\varphi_1\ \varphi_2\ \varphi_3]^\top \in C([0,\infty);H^2(\Upsilon_1))\times C([0,\infty);H^2(\Upsilon_2))\times C([0,\infty);H^2(\Upsilon_3))$, then

$$\begin{aligned}
z(t,\mathbf{x}) = \Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(y_1,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(y_1,0)\Big]ds-\big(\mathcal{B}\varphi_1(t)\big)(y_1,0)\Big)\mathbb{1}_{\Gamma_1}(\mathbf{x})\\
+\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(y_2,0)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(y_2,0)\Big]ds-\big(\mathcal{B}\varphi_2(t)\big)(y_2,0)\Big)\mathbb{1}_{\Gamma_2}(\mathbf{x})\\
+\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(s)\big)(y_3,0)+\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(s)\big)(y_3,0)\Big]ds-\big(\mathcal{B}\varphi_3(t)\big)(y_3,0)\Big)\mathbb{1}_{\Gamma_3}(\mathbf{x})\\
+\big(\mathcal{B}\varphi_1(t)\big)(y_1,0)+\big(\mathcal{B}\varphi_2(t)\big)(y_2,0)+\big(\mathcal{B}\varphi_3(t)\big)(y_3,0),\\
=\varepsilon\int_0^t (t-s)\sum_{i=1}^3 \mathbb{1}_{\Gamma_i}(\mathbf{x})\frac{\partial^2}{\partial x_i^2}\Big[\sum_{j=1}^3\big(\mathcal{B}\varphi_j(s)\big)(y_j,0)\Big]ds+\sum_{i=1}^3\big(\mathcal{B}\varphi_i(t)\big)(y_i,0)\Big(1-\mathbb{1}_{\Gamma_i}(\mathbf{x})\Big). \qquad (4.2)
\end{aligned}$$

Substituting (4.2) in (3.30), we find

$$\begin{aligned}
u(t,\mathbf{x}) = \varepsilon\int_0^t (t-s)\sum_{i=1}^3 \mathbb{1}_{\Gamma_i}(\mathbf{x})\frac{\partial^2}{\partial x_i^2}\Big[\sum_{j=1}^3\big(\mathcal{B}\varphi_j(s)\big)(y_j,0)\Big]ds+\sum_{i=1}^3\big(\mathcal{B}\varphi_i(t)\big)(y_i,0)\Big(1-\mathbb{1}_{\Gamma_i}(\mathbf{x})\Big)\\
-\sum_{i=1}^3\int_0^{x_i} p_i(x_i,r_i)dr_i\Big\{\Big(\varepsilon\int_0^t (t-s)\frac{\partial^2}{\partial x_i^2}\Big[\big(\mathcal{B}\varphi_i(t)\big)(y_i,0)\Big]ds-\big(\mathcal{B}\varphi_i(t)\big)(y_i,0)\Big)\mathbb{1}_{\Gamma_i}(\mathbf{x})\\
+\big(\mathcal{B}\varphi_i(t)\big)(y_i,0)\Big\}-\int_0^{x_1} p_1(x_1,r_1)\Big\{\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3)\Big]ds-\big(\mathcal{B}\varphi_2(t)\big)(r_1,0,x_3)\Big)\mathbb{1}_{\Gamma_2}(\mathbf{x})\\
&+\Big(\varepsilon\int_0^t(t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0)+\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0)\Big]ds\\
&-\big(\mathcal{B}\varphi_3(t)\big)(r_1,x_2,0)\Big)\mathbb{1}_{\Gamma_3}(\mathbf{x})+\big(\mathcal{B}\varphi_2(t)\big)(r_1,0,x_3)dr_1+\big(\mathcal{B}\varphi_3(t)\big)(r_1,x_2,0)\Bigg\}dr_1\\
&-\int_0^{x_2}p_2(x_2,r_2)\Bigg\{\Big(\varepsilon\int_0^t(t-s)\Big[\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3)+\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3)\Big]ds\\
&-\big(\mathcal{B}\varphi_1(t)\big)(0,r_2,x_3)\Big)\mathbb{1}_{\Gamma_1}(\mathbf{x})+\Big(\varepsilon\int_0^t(t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0)\\
&+\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0)\Big]ds-\big(\mathcal{B}\varphi_3(t)\big)(x_1,r_2,0)\Big)\mathbb{1}_{\Gamma_3}(\mathbf{x})+\big(\mathcal{B}\varphi_1(t)\big)(0,r_2,x_3)dr_2\\
&+\big(\mathcal{B}\varphi_3(t)\big)(x_1,r_2,0)\Bigg\}dr_2-\int_0^{x_3}p_3(x_3,r_3)\Bigg\{\Big(\varepsilon\int_0^t(t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3)\\
&+\frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3)\Big]ds-\big(\mathcal{B}\varphi_1(t)\big)(0,x_2,r_3)\Big)\mathbb{1}_{\Gamma_1}(\mathbf{x})\\
&+\Big(\varepsilon\int_0^t(t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3)+\frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3)\Big]ds\\
&-\big(\mathcal{B}\varphi_2(t)\big)(x_1,0,r_3)\Big)\mathbb{1}_{\Gamma_2}(\mathbf{x})+\big(\mathcal{B}\varphi_1(t)\big)(0,x_2,r_3)dr_3+\big(\mathcal{B}\varphi_2(t)\big)(x_1,0,r_3)\Bigg\}dr_3 \qquad (4.3)
\end{aligned}
$$

where k_i, p_i, $i=1,2,3$, satisfy (2.6) and (3.31), respectively. This yields us the following theorem.

Theorem 4.1. *Assume that $\varphi=[\varphi_1\ \varphi_2\ \varphi_3]^\top\in C([0,\infty);H^2(\Upsilon_1))\times C([0,\infty);H^2(\Upsilon_2))\times C([0,\infty);H^2(\Upsilon_3))$. Then, the solution of the closed-loop system (1.1), (3.35)-(3.36) which also satisfies the output equation (1.2) is given by (4.3), where k_i and p_i, $i=1,2,3$, satisfy (2.6) and (3.31), respectively.*

The formula (4.3) furnishes a detailed formulation of the entire state u in terms of the reference trajectories φ_i, $i=1,2,3$, and the solutions of the kernel PDEs (2.6) and (3.31).

5. Solution of the OTP

Setting $x_1 = 1$ in (4.3) and using that $V_1(t, x_2, 0) = V_1(t, 0, x_3) = 0$ we get

$$\begin{cases} U_1(t,y_1) = & \varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(t)\big)(y_1,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(t)\big)(y_1,0)\Big]ds + \big(\mathcal{B}\varphi_2(t)\big)(1,0,x_3) \\ & +\big(\mathcal{B}\varphi_3(t)\big)(1,x_2,0) - \int_0^1 p_1(1,r_1)dr_1\Big\{\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(t)\big)(y_1,0) \\ & +\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(t)\big)(y_1,0)\Big]ds + \big(\mathcal{B}\varphi_2(t)\big)(1,0,x_3) + \big(\mathcal{B}\varphi_3(t)\big)(1,x_2,0)\Big\} \\ & -\int_0^1 p_1(1,r_1)\Big\{\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3)\Big]ds \\ & -\big(\mathcal{B}\varphi_2(t)\big)(r_1,0,x_3)\Big)\mathbb{1}_{\Gamma_2}(r_1,1,x_3) + \Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0) \\ & +\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0)\Big]ds - \big(\mathcal{B}\varphi_3(t)\big)(r_1,x_2,0)\Big)\mathbb{1}_{\Gamma_3}(r_1,x_2,1) \\ & +\big(\mathcal{B}\varphi_2(t)\big)(r_1,0,x_3) + \big(\mathcal{B}\varphi_3(t)\big)(r_1,x_2,0)\Big\}dr_1 \\ & -\int_0^{x_2} p_2(x_2,r_2)\Big\{\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3)\Big]ds \\ & +\big(\mathcal{B}\varphi_3(t)\big)(1,r_2,0)\Big\}dr_2 - \int_0^{x_3} p_3(x_3,r_3)\Big\{\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3) \\ & +\frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3)\Big]ds + \big(\mathcal{B}\varphi_2(t)\big)(1,0,r_3)\Big\}dr_3 \\ U_1(t,0,x_3) & = -\int_0^1 k_1(1,r_1)\varphi_2(t,r_1,x_3)dr_1, \\ U_1(t,x_2,0) & = -\int_0^1 k_1(1,r_1)\varphi_3(t,r_1,x_2)dr_1, \end{cases} \tag{5.1}$$

for $0 \le x_2 < 1$ and $0 \le x_3 < 1$.
On the other hand, taking $x_2 = 1$ in (4.3) and using that $V_2(t, 0, x_3) = V_2(t, x_1, 0) = 0$ we find

$$\begin{cases} U_2(t,y_2) & = \varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(t)\big)(y_2,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_2(t)\big)(y_2,0)\Big]ds \\ & -\int_0^1 p_2(1,r_2)dr_2\Big\{\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(t)\big)(y_2,0) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(t)\big)(y_2,0)\Big]ds \\ & +\big(\mathcal{B}\varphi_1(t)\big)(0,1,x_3) + \big(\mathcal{B}\varphi_3(t)\big)(x_1,1,0)\Big\} + \big(\mathcal{B}\varphi_1(t)\big)(0,1,x_3) \\ & -\int_0^{x_1} p_1(x_1,r_1)\Big\{\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3) + \frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_2(s)\big)(r_1,0,x_3)\Big]ds \\ & +\big(\mathcal{B}\varphi_3(t)\big)(r_1,1,0)\Big\}dr_1 - \int_0^1 p_2(1,r_2)\Big\{\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3) \\ & +\frac{\partial^2}{\partial x_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,r_2,x_3)\Big]ds - \big(\mathcal{B}\varphi_1(t)\big)(0,r_2,x_3)\Big)\mathbb{1}_{\Gamma_1}(1,r_2,x_3) \\ & +\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0) + \frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0)\Big]ds \\ & -\big(\mathcal{B}\varphi_3(t)\big)(x_1,r_2,0)\Big)\mathbb{1}_{\Gamma_3}(x_1,r_2,0) + \big(\mathcal{B}\varphi_1(t)\big)(0,r_2,x_3)dr_2 \\ & +\big(\mathcal{B}\varphi_3(t)\big)(x_1,r_2,0)\Big\}dr_2 - \int_0^{x_3} p_3(x_3,r_3)\Big\{\Big(\varepsilon\int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3) \\ & +\frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3)\Big]ds + \big(\mathcal{B}\varphi_1(t)\big)(0,1,r_3)\Big\}dr_3 + \big(\mathcal{B}\varphi_3(t)\big)(x_1,1,0), \\ U_2(t,0,x_3) & = -\int_0^1 k_2(1,r_2)\varphi_1(t,r_2,x_3)dr_2, \\ U_2(t,x_1,0) & = -\int_0^1 k_2(1,r_2)\varphi_3(t,x_1,r_2)dr_2, \end{cases} \tag{5.2}$$

for $0 \leq x_1 < 1$ and $0 \leq x_2 < 1$. Taking $x_3 = 1$ in (4.3) and using that $V_3(t,0,x_2) = V_3(t,x_1,0) = 0$ we find

$$\begin{cases} U_3(t,y_3) & = \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(t)\big)(y_3,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(t)\big)(y_3,0)\Big]ds + \big(\mathcal{B}\varphi_1(t)\big)(0,x_2,1) \\ & + \big(\mathcal{B}\varphi_2(t)\big)(x_1,0,1) - \int_0^1 p_3(1,r_3)dr_3 \Big\{ \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(t)\big)(y_3,0) \\ & + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(t)\big)(y_3,0)\Big]ds + \big(\mathcal{B}\varphi_1(t)\big)(0,x_2,1) + \big(\mathcal{B}\varphi_2(t)\big)(x_1,0,1)\Big\} \\ & - \int_0^{x_1} p_1(x_1,r_1)\Big\{ \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial r_1^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0) + \frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_3(s)\big)(r_1,x_2,0)\Big]ds \\ & + \big(\mathcal{B}\varphi_2(t)\big)(r_1,0,1)\Big\}dr_1 - \int_0^{x_2} p_2(x_2,r_2)\Big\{ + \varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0) \\ & + \frac{\partial^2}{\partial r_2^2}\big(\mathcal{B}\varphi_3(s)\big)(x_1,r_2,0)\Big]ds + \big(\mathcal{B}\varphi_1(t)\big)(0,r_2,1)\Big\}dr_2 \\ & - \int_0^1 p_3(1,r_3)\Big\{ \Big(\varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_1^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3) + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_2(s)\big)(x_1,0,r_3)\Big]ds \\ & - \big(\mathcal{B}\varphi_2(t)\big)(x_1,0,r_3)\Big)\mathbb{1}_{\Gamma_2}(x_1,1,r_3) + \Big(\varepsilon \int_0^t (t-s)\Big[\frac{\partial^2}{\partial x_2^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3) \\ & + \frac{\partial^2}{\partial r_3^2}\big(\mathcal{B}\varphi_1(s)\big)(0,x_2,r_3)\Big]ds - \big(\mathcal{B}\varphi_1(t)\big)(0,x_2,r_3)\Big)\mathbb{1}_{\Gamma_1}(0,x_2,r_3) \\ & + \big(\mathcal{B}\varphi_2(t)\big)(x_1,0,r_3) + \big(\mathcal{B}\varphi_1(t)\big)(0,x_2,r_3)\Big\}dr_3 \\ U_3(t,0,x_3) & = -\int_0^1 k_3(1,r_3)\varphi_1(t,x_2,r_3)dr_3, \\ U_3(t,x_1,0) & = -\int_0^1 k_3(1,r_3)\varphi_2(t,x_1,r_3)dr_3. \end{cases} \tag{5.3}$$

This gives the desired explicit expressions of the controllers $U_i(t,\cdot)$, $i = 1,2,3$, and concludes the proof of the following main result.

Theorem 5.1. *Assume that* $[\varphi_1\ \varphi_2\ \varphi_3]^\top \in C([0,\infty);H^2(\Upsilon_1)) \times C([0,\infty);H^2(\Upsilon_2)) \times C([0,\infty);H^2(\Upsilon_3))$. *Then, the solution of the OTP associated with system* (1.1) *and the output equation* (1.2) *is given by* (5.1)*–*(5.3)*, where* k_i *and* p_i*,* $i = 1,2,3$*, are solutions of the kernel PDEs* (2.6) *and* (3.31)*, respectively.*

The controllers $U_i(t,\cdot)$, $i = 1,2,3$, are explicitly expressed via the predefined trajectories φ_i, $i = 1,2,3$, and the solutions of the kernels PDEs (2.6) and (3.31). Moreover, the provided controllers produce a perfect matching of the outputs $u(t,y_i,0)$ with the desired trajectories φ_i, $i = 1,2,3$, at any position $(t,\mathbf{x})$.

6. Conclusion.

The present paper investigates the OTP for the system (1.1)-(1.2). The method begins by transforming the initial system (1.1) into an auxiliary one using a direct backstepping transformation $\mathcal{B}$, where the kernels are chosen to solve an appropriate PDEs. Next, the target system is formulated as an abstract control problem, allowing the solution to be obtained in the form of a boundary feedback law by employing the Laplace transform and Green's functions. Finally, the solution to the OTP of the original system is obtained through the application of the inverse transformation $\mathcal{B}^{-1}$. This process yields explicit expressions for the entire state and tracking controllers, ensuring perfect matching between the outputs and the desired trajectories.

Acknowledgement. The author sincerely thanks the referees for their thorough efforts and for their constructive remarks and recommendations.

References

1. Ammari, K. and Gerbi, S. (2017). Interior feedback stabilization of wave equations with dynamic boundary delay. Journal of Analysis and its Applications, 3(36), 297-327. https://doi.org/10.4171/ZAA/1590
2. Arfken, GB. Weber, HJ. (2012). Mathematical Methods for Physicists, 7th ed., A Comprehensive Guide, Orlando, FL. Academic Press. https://doi.org/10.1112/S0024609306018765
3. Aziz, S. (2015). A Boundary Backstepping Control Design For 2-D, 3-D and N-D Heat Equation, 17th International Conference on Modelling, Identification and Control EngineeringAt: Venice, Italy, 17(8), 13-14. https://www.researchgate.net/publication/281494705
4. Carroll, R. W. and Showalter, R. E. (1976). Singular and degenerate Cauchy problems, Academic Press Harcourt Brace Jovanovich Publishers, Mathematics in Science and Engineering,New York, 127, iii-viii, 1-333. https://www.sciencedirect.com/bookseries/mathematics-in-science-and-engineering/vol/127/suppl/C
5. Elharfi, A. El hassani, M. and El Mourchid, S. (2023). motion planning Problem For a class of hyperbolic PDEs. Journal of Evolution Equation and Control Theory, 12(5), 1268-1286. https://doi.org/10.3934/eect.2023011
6. Elharfi, A. (2016). Exponential stabilization of a class of 1-D hyperbolic PDEs. J. Evol. Equ., 16, 665–679. https://link.springer.com/article/10.1007/s00028-015-0317-z
7. Elharfi, A. (2017). Exponential stabilization and motion planning of an overhead crane system. IMA Journal of Mathematical Control and Information, 34, 1299–1321. https://doi.org/10.1093/imamci/dnw026
8. Engel, K. and Nagel, R. (2000). One-parameter Semigroups for Linear Evolution Equations, Ency-clopedia of Mathematics and its Applications, Springer–Verlag, 63(2), 278-280. https://doi.org/10.1007/s002330010042
9. Grabowski, P. (2009). The motion planning problem and exponential stabilization of a heavy chain. Part I, International Journal of Control, 82(8), 1539–1563. https://doi.org/10.1080/00207170802588188
10. Izelgue, L. Tamoussit, A. (2016). ON On the flatness of $Int(D)$ as a $D[X]$–Module. Gulf Journal of Mathematics, 4(4), 39–47. https://doi.org/10.56947/gjom.v4i4.262
11. Kugi, A. and Thull, D. (2005). Infinite-dimensional decoupling control of the tip position and the tip angle of a composite piezoelectric beam with tip mass, in Control and Observer Design for Nonlinear Finite- and Infinite-Dimensional Systems, of Lecture Notes in Control and Information Sciences, Berlin: Springer-Verlag, 322, 351-368. https://doi.org/10.1007/11529798_22
12. Lafferrier'e, G. and Sussmann, H.J. (1992). A differential geometric approach to motion planning, Nonholonomic Motion Planning, The Kluwer International Series in Engineering and Computer Science, 192. https://doi.org/10.1007/978-1-4615-3176-0_7
13. Laroche, B. and Martin, P. (2003). Motion planning for 1-D linear partial differential equations, in Algebraic Methods in Flatness, Signal Processing and State Estimation, 55–76.
14. Laroche, B. Martin, P. and Rouchon, P. (2000). Motion planning for the heat equation. International Journal of Robust and Nonlinear Control, 10, 629–643. https://doi.org/10.1002/1099-1239(20000715)10:8<629::AID-RNC502>3.0.CO;2-N
15. Liu, W. (2003). Boundary feedback stabilization for an unstable heat equation. SIAM J. Control Optim, 42(3), 1033–1043. https://doi.org/10.1137/S0363012902402414

16. Monaco, S. and Normand-Cyrot, D. (1992). An introduction to motion planning under multirate digital control. 31st IEEE Conf. on Decision and Control, Tucson, AZ, 1780–1785. https://doi.org/10.1109/CDC.1992.371122
17. Ollivier, F. and Sedoglavic, A. (2001). A generalization of flatness to nonlinear systems of partial differential equations, Application to the command of a flexible rod, in Proceedings of the 5th IFAC Symposium on Nonlinear Control Systems, 1, 196–200. https://inria.hal.science/inria-00126994v1
18. Panayappan, S. (2014). Hilbert space frames, Riesz bases and tensorsums. Gulf Journal of Mathematics, 2(3), 81–87. https://doi.org/10.56947/gjom.v2i3.232
19. Petit, N. and Del Vecchio, D. (2005). Boundary control for an industrial under-actuated tubular chemical reactor, Journal of Process Control, 15(7), 771–784. https://doi.org/10.1016/j.jprocont.2005.03.004
20. Petit, N. and Rouchon,P. (2001). Flatness of heavy chain systems. Siam Journal on Control and Optimization, 40(2), 475–492. https://doi.org/10.1137/S0363012900368636
21. Pisano, A. Orlov, Y. and Usai, E. (2011). Tracking control of the uncertain heat and wave equation via power-fractional and slidingmode techniques, SIAM J. Control Optim, 49(2), 363-382. https://doi.org/10.1137/090781140
22. Polyanin, A.D. and Nazaikinskii, V.E. (2001). Handbook of Linear Partial Differential Equations for Engineers and Scientists, 1st ed.; CRC Press: Boca Raton, FL, USA, 800. https://doi.org/10.1201/9781420035322
23. Rouchon, P. (2001). Motion planning, equivalence, infinite dimensional systems. International Journal of Applied Mathematics and Computer Science, 11(1), 165–188. http://eudml.org/doc/207498
24. Rudolph, J. and Woittennek, F. (2008). Motion planning and open loop control design for linear distributed parameter systems with lumped controls, Int. J. Control, 81(3), 445–474. https://doi.org/10.1080/00207170701561435
25. Salamon, D. (1987). Infinite Dimensional Systems with Unbounded Control and Observation: A Functional Analytic Approach, Trans. Amer. Math. Soc., 300(2), 383–431. https://doi.org/10.2307/2000351
26. Smyshlyaev, A. and Krstic, M. (2004). Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations. IEEE Trans. Automat. Control, 49, 2185—2202. https://doi.org/10.1109/TAC.2004.838495
27. Sordalen, O.J. and Canudas, C. (1993). Exponential control law for a mobile robot: Extension to path following. IEEE Trans. on Robotics and Automation, 9(6), 837–842. https://doi.org/10.1109/70.265927
28. Tucsnak, M. and Weiss, G. (2009). Observation and Control for Operator semigroups, Birkhuser, Basel, XI–483. https://doi.org/10.1007/978-3-7643-8994-9
29. Witham, G. B. (1974). Linear and Nonlinear Waves, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney. https://doi.org/10.4249/scholarpedia.4308

[1] Department of mathematics, Laboratory of Mathematical Analysis and Applications, Ibn Zohr University, Agadir, Morocco.

Email address: mustapha.elhassani@edu.uiz.ac.ma