

THE LOGARITHM OF THE LEAST PRIME FACTOR

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ABSTRACT. Let $p(n)$ be the least prime factor of the positive integer n . In this note we prove the equation $\sum_{i=2}^n \log p(i) = o(n \log n)$.

1. INTRODUCTION AND PRELIMINARIES

In this note p_n denotes the n -th prime number and p denotes a positive prime.

Let $P(n)$ be the greatest prime factor of the positive integer n . N. G. de Bruijn [1] proved the following asymptotic formula

$$\sum_{i=2}^n \log P(i) = an \log n + O(n).$$

where a is a positive constant defined in [1].

Let $p(n)$ be the least prime factor of the positive integer n . Clearly we have

$$\sum_{i=2}^n \log p(i) \leq \sum_{i=2}^n \log P(i).$$

In this note we prove the following result

$$\sum_{i=2}^n \log p(i) = o(n \log n).$$

Let $\beta_{p_h}(n)$ be the number of numbers that do not exceed n such that their least prime factor is p_h . We shall need the following theorems.

Theorem 1.1. *The following asymptotic formula holds*

$$\beta_{p_h}(n) = \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h} n + O(1).$$

Proof. See [3]. □

Note that

$$\sum_{2 \leq p_h \leq n} \beta_{p_h}(n) = n - 1. \tag{1.1}$$

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Theorem 1.2. *We have*

$$\sum_{h=1}^{\infty} \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h} = \frac{1}{2} + \left(1 - \frac{1}{2} \right) \frac{1}{3} + \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \frac{1}{5} \\ + \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \left(1 - \frac{1}{5} \right) \frac{1}{7} + \cdots = 1.$$

Proof. See [3]. □

Theorem 1.3. *We have the following asymptotic formula*

$$\prod_{p \leq x} \left(1 - \frac{1}{p} \right) \sim \frac{e^{-\gamma}}{\log x},$$

where γ is the Euler's constant.

Proof. See [2, Chapter XXII]. □

Theorem 1.4. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{i \rightarrow \infty} \frac{a_i}{b_i} = 1$. If $\sum_{i=1}^{\infty} b_i$ is divergent then we have the following limit*

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} = 1.$$

Therefore $\sum_{i=1}^{\infty} a_i$ is also divergent.

Proof. See [4, page 332]. □

Now, we can to prove the following two theorems.

Theorem 1.5. *The series*

$$\sum_{h=1}^{\infty} \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{\log p_h}{p_h} = \frac{\log 2}{2} + \left(1 - \frac{1}{2} \right) \frac{\log 3}{3} + \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \frac{\log 5}{5} \\ + \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \left(1 - \frac{1}{5} \right) \frac{\log 7}{7} + \cdots .$$

diverges.

Proof. We have (see Theorem 1.3)

$$\left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{\log p_h}{p_h} = \prod_{p \leq p_{h-1}} \left(1 - \frac{1}{p} \right) \frac{\log p_h}{p_h} \sim \frac{\log p_h}{p_h} \frac{e^{-\gamma}}{\log p_{h-1}} \sim \frac{e^{-\gamma}}{p_h}.$$

Note that $\log p_h \sim \log p_{h-1}$ due to the well-known equation $\log p_h \sim \log h$. Now, it is well-known that the series $\sum_{h=1}^{\infty} \frac{1}{p_h}$ diverges. Therefore this last fact, the former equation and Theorem 1.4 prove Theorem 1.5. □

Theorem 1.6. *Let us consider a strictly increasing sequence $f(n)$ such that $f(n) \rightarrow \infty$ and $\frac{f(n)}{n} \rightarrow 0$. We have the following asymptotic formula*

$$\sum_{2 \leq p_h \leq f(n)} \beta_{p_h}(n) \sim n. \tag{1.2}$$

Proof. We have (see equation(1.1))

$$\sum_{2 \leq p_h \leq f(n)} \beta_{p_h}(n) \leq n. \quad (1.3)$$

Let $\epsilon > 0$. We can choose m such that (see Theorem 1.2)

$$\sum_{h=1}^m \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h} \geq 1 - \epsilon. \quad (1.4)$$

Therefore we have (see Theorem 1.1 and (1.4))

$$\begin{aligned} \sum_{2 \leq p_h \leq f(n)} \beta_{p_h}(n) &\geq \sum_{2 \leq p_h \leq p_m} \beta_{p_h}(n) = n \sum_{h=1}^m \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h} + O(1) \\ &\geq n(1 - \epsilon) + O(1) \geq n(1 - 2\epsilon). \end{aligned} \quad (1.5)$$

Equations (1.3) and (1.5) give

$$n(1 - 2\epsilon) \leq \sum_{2 \leq p_h \leq f(n)} \beta_{p_h}(n) \leq n.$$

That is

$$(1 - 2\epsilon) \leq \frac{\sum_{2 \leq p_h \leq f(n)} \beta_{p_h}(n)}{n} \leq 1. \quad (1.6)$$

Equation (1.2) is an immediate consequence of equation (1.6) as ϵ is arbitrarily small. $\square$

2. MAIN RESULTS

Theorem 2.1. *We have*

$$\sum_{i=2}^n \log p(i) = ng(n), \quad (2.1)$$

where $g(n) \rightarrow \infty$ and $g(n) = o(\log n)$.

Proof. Let us put

$$\sum_{i=2}^n \log p(i) = ng(n). \quad (2.2)$$

We can choose M such that (see Theorem 1.5)

$$\sum_{h=1}^m \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{\log p_h}{p_h} \geq M. \quad (2.3)$$

Therefore (see Theorem 1.1 and (2.3))

$$\begin{aligned} \sum_{i=2}^n \log p(i) &= \sum_{2 \leq p_h \leq n} \log p_h \beta_{p_h}(n) \geq \sum_{2 \leq p_h \leq p_m} \log p_h \beta_{p_h}(n) \\ &= n \left(\sum_{h=1}^m \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{\log p_h}{p_h} \right) + O(1) \geq nM + O(1) \geq n \frac{M}{2}. \end{aligned}$$

That is (see (2.2))

$$\frac{\sum_{i=2}^n \log p(i)}{n} = g(n) \geq \frac{M}{2}.$$

Now, M is arbitrarily large, consequently

$$g(n) \rightarrow \infty.$$

On the other hand, we have (see Theorem 1.6 with $f(n) = \log n$ and (1.1))

$$\begin{aligned} \sum_{i=2}^n \log p(i) &= \sum_{2 \leq p_h \leq n} \log p_h \beta_{p_h}(n) = \sum_{2 \leq p_h \leq \log n} \log p_h \beta_{p_h}(n) \\ &+ \sum_{\log n < p_h \leq n} \log p_h \beta_{p_h}(n) \leq \log \log n \sum_{2 \leq p_h \leq \log n} \beta_{p_h}(n) \\ &+ \log n \sum_{\log n < p_h \leq n} \beta_{p_h}(n) = \log \log n h(n) n + \log n o(n), \end{aligned}$$

where $h(n) \rightarrow 1$.

Therefore

$$0 \leq \frac{\sum_{i=2}^n \log p(i)}{n \log n} \leq \frac{\log \log n h(n) n + \log n o(n)}{n \log n} \rightarrow 0.$$

That is (see (2.2))

$$\frac{\sum_{i=2}^n \log p(i)}{n \log n} = \frac{ng(n)}{n \log n} = \frac{g(n)}{\log n} \rightarrow 0.$$

That is

$$g(n) = o(\log n).$$

□

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