

GLOBAL ATTRACTIVITY AND EXISTENCE OF WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTION FOR GHNNs WITH DELAYS AND VARIABLE COEFFICIENTS

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ABSTRACT. In this paper, we shall explain a new result concerning weighted pseudo almost automorphic solutions of Hopfield neural network with time-varying delay and variable coefficients. Precisely, we shall prove existence and the global attractivity of the unique weighted pseudo almost automorphic solution of the considered model. Then, by employing suitable analytic techniques, global attractivity of the unique weighted pseudo almost automorphic solution is established. Lastly, an example is provided to demonstrate the validity of the results proposed.

1. INTRODUCTION

In the past two decades, neural networks has been received considerable attention, and there have been extensive research results presented about the stability analysis of neural network and its applications (See [2], [14], [20], [21]). In particular, the stability research related to Hopfield neural networks have been extensively studied and developed in recent years since it has been widely used to model many of the phenomena arising in areas such as signal processing, pattern recognition, static image processing, associative memory. We refer the reader to ([2], [16], [17], [19], [22]) and the references cited therein.

As is well known, both in biological and man-made neural networks, delays are inevitable, due to various reasons. For instance, time delays can be caused by the finite switching speed of amplifier circuits in neural networks. Time delays in the neural networks make the dynamic behaviors become more complex, and may destabilize the stable equilibria and admit oscillations, bifurcation and chaos. Thus, it is very important to study the dynamics of neural networks delay.

In 1955, Bochner suggested another generalization of the concept of almost periodicity that to say, almost automorphy. In fact, In the beginning of sixties, Bochner introduced the concept of almost automorphic functions in his papers [4] and [5] in relation to some aspects of differential geometry. This concept became a generalization of almost periodicity which is one of the most attractive

Date: Received: Jul 22, 2013; Accepted: Aug 12, 2013.

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2010 *Mathematics Subject Classification.* Primary 92B20, 43A60; Secondary 44B20.

Key words and phrases. Hopfield neural network, Weighted pseudo almost automorphic, Global attractivity.

topics in the qualitative theory of differential equations. Recently, the concept of almost automorphic functions has widely been used in the investigation of the existence of almost automorphic solutions of various kinds of evolution equations by N'Guérékata and others (see for example [10] and [14]). Notice that some fundamental properties of almost periodic functions are not verified by the almost automorphic functions, as example the property of uniform continuity.

The new concept of pseudo-almost automorphy generalizes the one of pseudo-almost periodicity, in fact, a pseudo-almost automorphic function is the sum of an almost automorphic function and of an ergodic perturbation. These functions were introduced recently by Liang, Xiao and Zhang in [25] and [18]. Consequently, the research for the almost automorphic and pseudo almost periodic solutions for dynamic systems are more complicated. Recently, Blot et. al [3] have proposed an extension of pseudo-almost automorphic functions called weighted pseudo-almost automorphic functions. The existence and uniqueness of the pseudo-almost automorphic and almost periodic solutions of differential equations have been investigated by many authors recently ([3], [10], [25], [26]).

Several important results have been established to guarantee the existence, uniqueness, and qualitative properties of stability of the equilibrium point of neural networks. These results are mainly based on the M-matrix theory, topological theory, fixed point method and Lyapunov functional approach. Hence in ([6], [7]) by using analysis techniques, the authors have obtained several sufficient criteria for ascertaining the stability and exponential stability of the equilibrium point or periodic solution for the following model with constant coefficients

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau_j)) + I_i$$

where I_i are constants and τ_j are nonnegative constants. It is well known that delays arise frequently in practical applications and it is difficult to measure them precisely. Most researches on neural networks have been restricted to simple cases of constant delays and few papers consider variable delay case. Although, in most situations, delays are variable and neural network usually has a spatial nature due to the presence of various parallel pathways the entire history affects the present, so it is necessary to model them by introducing time-varying delay, and such delay terms, more suitable to practical neural nets. Further, time delays in the neural networks make the dynamic behaviors more complex, and may destabilize the stable equilibria and admit almost-periodic oscillation, pseudo almost-periodic motion, bifurcation, and chaos (see for example [1], [23] and [15]).

Motivated by the above discussion and in order to given an extension of the papers [6] and [7], we are concerned with the following GHNNs with time-varying coefficients and variable delay

$$\begin{cases} \dot{x}_i(t) = -d_i(t) x_i(t) + \sum_{j=1}^n a_{ij}(t) f(x_j(t)) + \sum_{j=1}^n b_{ij}(t) g(x_j(t - \tau_j(t))) + I_i(t) \\ x_i(t) = \psi_i(t), -\tau \leq t \leq 0, 1 \leq i \leq n. \end{cases} \quad (1.1)$$

The main purpose of this paper is to study the existence and attractivity of the pseudo-almost automorphic solution for the model (1.1).

The structure of this paper is outlined as follows: In Section 2, we will introduce some necessary notations, definitions and fundamental properties of the space $PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$ which will be used in the paper. In Section 3, based on different methods and analysis techniques and provides several sufficient conditions ensuring the existence and uniqueness of the weighted pseudo-almost automorphic solution for the considered system. Section 4 is devoted to the global attractivity of the weighted pseudo-almost automorphic solution of (1.1). At last, an illustrative example is given. It should be mentioned that the main results include Theorems 4.4, 4.5 and 5.1.

2. PRELIMINARIES: THE FUNCTIONS SPACES

Let $BC(\mathbb{R}, \mathbb{R}^n)$ denote the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space where $\|\cdot\|_\infty$ denotes the sup norm

$$\|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\|.$$

The following definition is due first to Bochner [5]:

Definition 2.1. A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$g(t) := \lim_{n \rightarrow \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} g(t - s_n) = f(t).$$

Denote by $AA(\mathbb{R}, \mathbb{R}^n)$ the collection of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{R}^n$.

Remark 2.2. By the pointwise convergence, the function g is just measurable and not necessarily continuous. If the convergence in both limits is uniform, then f is almost periodic. Besides, the concept of almost automorphy is then larger than almost periodicity. In fact, a classical example of an almost automorphic function which is not almost periodic, as it is not uniformly continuous, is the function defined by

$$t \mapsto \cos \left(\frac{1}{2 + \sin t + \sin \sqrt{2}t} \right), \quad t \in \mathbb{R}.$$

Recently, a new generalization of Bohr almost periodic functions was introduced by Diagana [11]. This new concept is called weighted pseudo-almost periodicity and implements in a natural fashion the notion of pseudo-almost periodicity introduced in the literature by Zhang [29]. To construct those new spaces, the main idea consists of enlarging the so-called ergodic component, utilized in Zhang's definition of pseudo-almost periodicity, with the help of a weighted measure $d\mu(x) = \varrho(x) dx$, where $\varrho : \mathbb{R} \rightarrow]0, +\infty[$ is a locally integrable function

over $\mathbb{R}$, which is commonly called weight. In other words weighted pseudo-almost periodic functions are good generalizations of the Zhang's pseudo-almost periodic functions. Roughly speaking, let $\mathbb{U}$ denote the collection of all functions (weights) $\varrho : \mathbb{R} \rightarrow]0, +\infty[$ which are locally integrable over $\mathbb{R}$ such $\varrho(x) > 0$ for almost each $x \in \mathbb{R}$. For $\varrho \in \mathbb{U}$ and for $r > 0$, we set

$$m(r, \varrho) := \int_{-r}^r \varrho(x) dx.$$

Throughout this section, the set of weights $\mathbb{U}_\infty$ stands for

$$\mathbb{U}_\infty := \left\{ \varrho \in \mathbb{U}, \lim_{r \rightarrow \infty} m(r, \varrho) = +\infty \right\}.$$

Obviously, $\mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions. To introduce the weighted pseudo-almost automorphic functions, we need to define the "weighted ergodic" space $PAP_0(\mathbb{R}, \mathbb{R}^k, \varrho)$. Let $\varrho \in \mathbb{U}_\infty$. Define

$$PAP_0(\mathbb{R}, \mathbb{R}^k, \varrho) = \left\{ f \in BC(\mathbb{R}, \mathbb{R}^k) / \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \|f(t)\| \varrho(t) dt = 0 \right\}.$$

Clearly, the spaces $PAP_0(\mathbb{R}, \mathbb{R}^k, \varrho)$ are richer than $PAP_0(\mathbb{R}, \mathbb{R}^k)$ and give rise to an enlarged space of weighted pseudo-almost periodic functions. In the same way, we define $PAP_0(\mathbb{R} \times \mathbb{R}^k, \mathbb{R}^k, \varrho)$ as the collection of jointly continuous functions $f : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $f(\cdot, y)$ is bounded for each $y \in Y$ and

$$\lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \|f(t, y)\| \varrho(t) dt = 0$$

uniformly in $y \in Y$.

Definition 2.3. Let $\varrho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{R}^k)$ is called weighted pseudo-almost automorphic if it can be expressed as

$$f = g + \varphi$$

where $g \in AA(\mathbb{R}, \mathbb{R}^k)$ and $\varphi \in PAP_0(\mathbb{R}, \mathbb{R}^k, \varrho)$. The collection of such functions will be denoted by $PAA(\mathbb{R}, \mathbb{R}^k, \varrho)$.

Remark 2.4. The weighted pseudo-almost periodic functions will then appear as perturbations of almost automorphic functions by elements of $PAP_0(\mathbb{R}, \mathbb{R}^k, \varrho)$. Besides, the functions h and φ in above definition are respectively called the almost-automorphic component and the ergodic perturbation of the weighted pseudo-almost automorphic function f . Besides the decomposition given in definition 2.3 is unique.

Remark 2.5. The space $PAA(\mathbb{R}, \mathbb{R}^k, \varrho)$ is a closed subspace of $(B(\mathbb{R}, \mathbb{R}^k), \|\cdot\|_\infty)$. This yields $PAA(\mathbb{R}, \mathbb{R}^k, \varrho)$ is a Banach space [11]. Further, we have the following hierarchy

$$AP(\mathbb{R}, \mathbb{R}^k) \subset AA(\mathbb{R}, \mathbb{R}^k) \subset PAA(\mathbb{R}, \mathbb{R}^k, \varrho).$$

Clearly

$$PAA(\mathbb{R}, \mathbb{R}^k, 1) = PAA(\mathbb{R}, \mathbb{R}^k).$$

Example 2.6. Let $\varrho_1(x) = e^x$ for each $x \in \mathbb{R}$. Clearly $\varrho_1(\cdot) \in \mathbb{U}_\infty$ and the function

$$f(t) = \cos\left(\frac{1}{2 + \sin t + \sin \sqrt{2}t}\right) + e^{-t}$$

belongs to $PAA(\mathbb{R}, \mathbb{R}, \varrho_1)$ since

$$\lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T e^t e^{-t} dt = 0.$$

3. PROBLEM FORMULATION

Let us consider the following Hopfield Neural network with time-varying delay and variable coefficients

$$\begin{cases} \dot{x}_i(t) &= -d_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(t - \tau_j(t))) + I_i(t) \\ x_i(t) &= \psi_i(t), -\tau \leq t \leq 0, 1 \leq i \leq n. \end{cases} \quad (1.1)$$

where n denotes the total number of units in the GHNNs, $x_i(t)$ corresponds to the state of the i -th unit at time t ; $d_i(\cdot) > 0$ represents the neuron firing rate, $f(x_j(t))$ and $g(x_j(t - \tau_j))$ denote the outputs of the j -th unit at time t and $(t - \tau_j)$ respectively; $a_{ij}(\cdot)$ and $b_{ij}(\cdot)$ denote the connection weights between the j -th unit and the i -th unit with which the i -th unit at time t and $(t - \tau_j(t))$ respectively. The function $I_i(t)$ is an external input on the i -th unit at time t . $\tau_j(t)$ denotes the transmission delay along the axon of the j -th unit and $0 \leq \tau_j(t)$.

Let us list some assumptions which will be used in this paper.

(H₁) For all $1 \leq i \leq n$, the functions $d_i(\cdot) > 0$.

(H₂) For all $1 \leq i, j \leq n$, the functions $d_i(\cdot)$, $a_{ij}(\cdot)$, $b_{ij}(\cdot)$ and $I_i(\cdot)$ are weighted pseudo-almost automorphic.

(H₃) The functions $f_j(\cdot)$ and $g_j(\cdot)$ are weighted pseudo-almost automorphic and satisfy the Lipschitz condition, i.e., there are constants $L_j^{f_j} > 0$, $L_j^{g_j} > 0$ such that for all $x, y \in \mathbb{R}$, and for all $1 \leq j \leq n$, one has

$$|f_j(x) - f_j(y)| \leq L_j^{f_j} |x - y|, \quad |g_j(x) - g_j(y)| \leq L_j^{g_j} |x - y|.$$

(H₄) Denote

$$\begin{aligned} a_{ij}^+ &= \max_{t \in \mathbb{R}} a_{ij}(t), \quad b_{ij}^+ = \max_{t \in \mathbb{R}} b_{ij}(t), \\ \tilde{d}_i &= \min_{t \in \mathbb{R}} \tilde{d}_i(t), \quad \tilde{d} = \min_{1 \leq i \leq n} \tilde{d}_i \end{aligned}$$

and

$$r = \max_{1 \leq i \leq n} \left[\frac{\sum_{j=1}^n L_j^{f_j} a_{ij}^+ + \sum_{j=1}^n L_j^{g_j} b_{ij}^+}{\widetilde{d}} \right] < 1.$$

4. EXISTENCE AND UNIQUENESS OF WEIGHTED PSEUDO-ALMOST AUTOMORPHIC SOLUTION

In this section, we establish some results for the existence, uniqueness of the weighted pseudo-almost automorphic solution of (1.1). Let us fix $\varrho \in \mathbb{U}_\infty$. First, following along the same lines as in the proof of [1] it follows that

Lemma 4.1. *If $\varphi, \psi \in PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$, then $\varphi \times \psi \in PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$.*

Lemma 4.2. [25] *$(PAA(\mathbb{R}, \mathbb{R}^n, \varrho), \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is the supremum norm.*

Reasoning in a similar way to Lemma 1 in [8] we obtain

Lemma 4.3. *If $f(\cdot) \in PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$ then $f(\cdot - h) \in PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$ where h is a fixed constant.*

Theorem 4.4. *Suppose that assumptions $(H_1), (H_2)$ and (H_3) hold. Define the nonlinear operator Γ . Then Γ maps $PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$ into itself.*

Proof: First of all, let us check that Γ is well defined. Indeed, by lemma 4.3, for all $x_j(\cdot) \in PAA(\mathbb{R}, \mathbb{R}, \varrho)$ the function $x_j(\cdot - h) \in PAA(\mathbb{R}, \mathbb{R}, \varrho)$ since $PAA(\mathbb{R}, \mathbb{R}, \varrho)$ is a translation invariant subspace of $BC(\mathbb{R}, \mathbb{R})$. Further, by the composition theorem of pseudo-almost automorphic functions (see for example [10]) $f_j(x_j(\cdot))$ and $g_j(x_j(\cdot - \tau_j(\cdot)))$ are in $PAA(\mathbb{R}, \mathbb{R}, \varrho)$. So, by lemma 4.1 and lemma 4.2, for all $1 \leq i \leq n$ the function

$$\chi_i : s \mapsto \sum_{j=1}^n a_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^n b_{ij}(s) g_j(x_j(s - \tau_j(s)))$$

belongs to $PAA(\mathbb{R}, \mathbb{R}, \varrho)$. This ensures the existence of two functions $\alpha_i \in AA(\mathbb{R}, \mathbb{R})$ and $\varphi_i \in PAA_0(\mathbb{R}, \mathbb{R}, \varrho)$ such that for all $1 \leq i \leq n$

$$\chi_i = \alpha_i + \varphi_i.$$

So, one can write

$$\begin{aligned} (\Gamma \chi_i)(t) &= \int_{-\infty}^t \exp \left(- \int_s^t d_i(\xi) d\xi \right) \chi_i(s) ds \\ &= \int_{-\infty}^t \exp \left(- \int_s^t d_i(\xi) d\xi \right) \alpha_i(s) ds + \int_{-\infty}^t \exp \left(- \int_s^t d_i(\xi) d\xi \right) \varphi_i(s) ds \\ &= (\Gamma \alpha_i)(t) + (\Gamma \varphi_i)(t). \end{aligned}$$

Let (s'_n) be a sequence of real numbers. By (H_3) we can extract a subsequence (s_n) of (s'_n) such that for all $t, s \in \mathbb{R}$ and for all $1 \leq i \leq n$

$$\lim_{n \rightarrow +\infty} d_i(t + s_n) = d_i^1(t) \quad \lim_{n \rightarrow +\infty} d_i^1(t - s_n) = d_i(t)$$

and

$$\lim_{n \rightarrow +\infty} \alpha_i(t + s_n) = \alpha_i^1(t) \quad \lim_{n \rightarrow +\infty} \alpha_i^1(t - s_n) = \alpha_i(t).$$

Pose

$$(\Gamma^1 \alpha_i)(t) := \int_{-\infty}^t \exp \left(- \int_s^t d_i^1(\xi) d\xi \right) \alpha_i^1(s) ds.$$

Then it follows that

$$\begin{aligned} (\Gamma \alpha_i)(t + s_n) - (\Gamma^1 \alpha_i)(t) &= \int_{-\infty}^{t+s_n} e^{-\int_s^{t+s_n} d_i(\xi) d\xi} \alpha_i(s) ds - \int_{-\infty}^t e^{-\int_s^t d_i^1(\xi) d\xi} \alpha_i^1(s) ds \\ &= \int_{-\infty}^{t+s_n} e^{-\int_{s-s_n}^t d_i(\sigma+s_n) d\sigma} \alpha_i(s) ds - \int_{-\infty}^t e^{-\int_s^t d_i^1(\xi) d\xi} \alpha_i^1(s) ds \\ &= \int_{-\infty}^t e^{-\int_u^t d_i(\sigma+s_n) d\sigma} \alpha_i(s + s_n) ds - \int_{-\infty}^t e^{-\int_s^t d_i^1(\xi) d\xi} \alpha_i^1(s) ds \\ &= \int_{-\infty}^t e^{-\int_u^t d_i(\sigma+s_n) d\sigma} \alpha_i(s + s_n) ds - \int_{-\infty}^t e^{-\int_u^t d_i(\sigma+s_n) d\sigma} \alpha_i^1(u) du \\ &\quad + \int_{-\infty}^t e^{-\int_u^t d_i(\sigma+s_n) d\sigma} \alpha_i^1(u) du - \int_{-\infty}^t e^{-\int_s^t d_i^1(\xi) d\xi} \alpha_i^1(s) ds \\ &= \int_{-\infty}^t e^{-\int_u^t d_i(\sigma+s_n) d\sigma} (\alpha_i(s + s_n) - \alpha_i^1(s)) ds \\ &\quad + \int_{-\infty}^t \left(e^{-\int_u^t d_i(\sigma+s_n) d\sigma} - e^{-\int_s^t d_i^1(\xi) d\xi} \right) \alpha_i^1(s) ds. \end{aligned}$$

So there exists $\theta \in]0, 1[$ such that

$$\begin{aligned}
|(\Gamma\alpha_i)(t + s_n) - (\Gamma^1\alpha_i)(t)| &\leq |\alpha_i^1|_\infty \int_{-\infty}^t \left(e^{-\int_s^t d_i(\sigma+s_n)d\sigma} - e^{-\int_s^t d_i^1(\xi)d\xi} \right) ds \\
&+ \int_{-\infty}^t e^{-\int_s^t d_i(\sigma+s_n)d\sigma} |\alpha_i(s + s_n) - \alpha_i^1(s)| ds \\
&\leq |\alpha_i^1|_\infty \int_{-\infty}^t \left\{ e^{-\left[\int_s^t d_i(\sigma+s_n)d\sigma + \theta \left(\int_s^t d_i^1(\xi)d\xi - \int_s^t d_i(\sigma+s_n)d\sigma \right) \right]} \right. \\
&\times \left. \int_s^t |d_i^1(\sigma) - d_i(\sigma + s_n)| d\sigma \right\} ds \\
&+ \int_{-\infty}^t e^{-(t-s)\underline{d}_i} |\alpha_i(s + s_n) - \alpha_i^1(s)| ds \\
&\leq |\alpha_i^1|_\infty \int_{-\infty}^t \left\{ e^{-\int_s^t d_i(\sigma+s_n)d\sigma} e^{-\theta \left(\int_s^t |d_i^1(\sigma) - d_i(\sigma+s_n)| d\sigma \right)} \right. \\
&\times \left. \int_s^t |d_i^1(\sigma) - d_i(\sigma + s_n)| d\sigma \right\} ds + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{d}_i} ds \\
&\leq |\alpha_i^1|_\infty \int_{-\infty}^t \left[e^{-(t-s)\underline{d}_i} e^{-\theta \left(\int_s^t |d_i^1(\sigma) - d_i(\sigma+s_n)| d\sigma \right)} \int_s^t |d_i^1(\sigma) - d_i(\sigma + s_n)| d\sigma \right] ds \\
&+ \int_{-\infty}^t |\alpha_i(s + s_n) - \alpha_i^1(s)| e^{-(t-s)\underline{d}_i} ds \\
&\leq |\alpha_i^1|_\infty \int_{-\infty}^t \left[e^{-(t-s)\underline{d}_i} \int_s^t |d_i^1(\sigma) - d_i(\sigma + s_n)| d\sigma \right] ds \\
&+ \int_{-\infty}^t |\alpha_i(s + s_n) - \alpha_i^1(s)| e^{-(t-s)\underline{d}_i} ds \\
&\leq \int_{-\infty}^t \Phi_n(t, s) ds + \int_{-\infty}^t \Psi_n(t, s) ds,
\end{aligned}$$

where

$$\Phi_n(t, s) = \frac{e^{-(t-s)d_i}}{|\alpha_i^1|_\infty} \int_s^t |d_i^1(\sigma) - d_i(\sigma + s_n)| d\sigma$$

and

$$\Psi_n(t, s) = |\alpha_i(s + s_n) - \alpha_i^1(s)| e^{-(t-s)d_i}.$$

By the Lebesgue dominated convergence theorem we obtain immediately that

$$\lim_{n \rightarrow +\infty} (\Gamma \alpha_i)(t + s_n) = (\Gamma^1 \alpha_i)(t) \text{ for all } t \in \mathbb{R}.$$

Reasoning in a similar way to first step we can show easily that

$$\lim_{n \rightarrow +\infty} (\Gamma^1 \alpha_i)(t - s_n) = (\Gamma \alpha_i)(t) \text{ for all } t \in \mathbb{R}.$$

At this stage we can thus assert that, for all $1 \leq i \leq n$, the function $(\Gamma \alpha_i)$ is almost automorphic. Now, let us show that for all $1 \leq i \leq n$ the function $(\Gamma \varphi_i)$ belongs to $PAP_0(\mathbb{R}, \mathbb{R}, \varrho)$. One can see without difficulty that:

$$\lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left| \int_{-\infty}^t \exp \left(- \int_s^t d_i(\xi) d\xi \right) \varphi_i(s) ds \right| \varrho(t) dt \leq J + K,$$

where

$$J = \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-T}^t e^{-\tilde{d}_i(t-s)} |\varphi_i(s)| ds \right) \varrho(t) dt \text{ and}$$

$$K = \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-\infty}^{-T} e^{-\tilde{d}_i(t-s)} |\varphi_i(s)| ds \right) \varrho(t) dt.$$

Clearly, in order to show that for all $1 \leq i \leq n$, $(\Gamma\varphi_i)$ belongs to $PAP_0(\mathbb{R}, \mathbb{R}, \varrho)$ it will suffice to prove that $J = K = 0$. In fact, one has

$$\begin{aligned}
\frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-T}^t |e^{-(t-s)d_i} \varphi_i(s)| ds \right) dt &= \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-T}^t e^{-\tilde{d}_i(t-s)} |\varphi_i(s)| ds \right) \varrho(t) dt \\
&\leq \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_0^{+\infty} e^{-\tilde{d}_i \xi} |\varphi_i(t-\xi)| d\xi \right) \varrho(t) dt \\
&= \int_0^{+\infty} e^{-\tilde{d}_i \xi} \left(\frac{1}{m(T, \varrho)} \int_{-T}^T |\varphi_i(t-\xi)| \varrho(t) dt \right) d\xi \\
&\leq \int_0^{+\infty} e^{-\tilde{d}_i \xi} \left(\frac{1}{m(T, \varrho)} \int_{-T-\xi}^{T-\xi} |\varphi_i(u)| \varrho(u+\xi) du \right) d\xi \\
&\leq \int_0^{+\infty} e^{-\tilde{d}_i \xi} \left(\frac{1}{m(T, \varrho)} \int_{-T-\xi}^{T+\xi} |\varphi_i(u)| \varrho(u+\xi) du \right) d\xi \\
&\leq \sup_{\xi \in \mathbb{R}} \left(\frac{m(T+\xi, \varrho)}{m(T, \varrho)} \right) \sup_{\xi \in \mathbb{R}} \left(\frac{\varrho(u+\xi)}{\varrho(u)} \right) \\
&\quad \times \int_0^{+\infty} e^{-\tilde{d}_i \xi} \left(\frac{1}{m(T+\xi, \varrho)} \int_{-T-\xi}^{T+\xi} |\varphi_i(u)| \varrho(u) du \right) d\xi.
\end{aligned}$$

Since the function $\varphi_i(\cdot) \in PAA_0(\mathbb{R}, \mathbb{R}, \varrho)$ then the function $\phi_{i,T}$ defined by

$$\phi_{i,T}(\xi) = \sup_{\xi \in \mathbb{R}} \left(\frac{m(T+\xi, \varrho)}{m(T, \varrho)} \right) \sup_{\xi \in \mathbb{R}} \left(\frac{\varrho(u+\xi)}{\varrho(u)} \right) \left(\frac{1}{m(T+\xi, \varrho)} \int_{-T-\xi}^{T+\xi} |\varphi_i(u)| \varrho(u) du \right)$$

is bounded and satisfy $\lim_{T \rightarrow +\infty} \phi_{i,T}(\xi) = 0$. Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$J = \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-T}^t e^{-\tilde{d}_i(t-s)} |\varphi_i(s)| ds \right) \varrho(t) dt = 0.$$

On the other hand, notice that $|\varphi_i|_\infty = \sup_{t \in \mathbb{R}} |\varphi_i(t)| < \infty$ then

$$\begin{aligned}
K &= \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-\infty}^{-T} \left| e^{-\tilde{d}_i(t-s)} \varphi_i(s) \right| ds \right) \varrho(t) dt \\
&= \lim_{T \rightarrow +\infty} \frac{1}{m(T, \varrho)} \int_{-T}^T \left(\int_{-\infty}^{-T} e^{-\tilde{d}_i(t-s)} |\varphi_i(s)| ds \right) \varrho(t) dt \\
&\leq \lim_{T \rightarrow +\infty} \frac{\sup_{t \in \mathbb{R}} |\varphi_i(t)|}{m(T, \varrho)} \int_{-T}^T \left(\int_{t+T}^{+\infty} e^{-\tilde{d}_i \xi} d\xi \right) \varrho(t) dt \\
&\leq \lim_{T \rightarrow +\infty} \frac{\sup_{t \in \mathbb{R}} |\varphi_i(t)|}{m(T, \varrho)} \int_{-T}^T \left(\int_{2T}^{+\infty} e^{-\tilde{d}_i \xi} d\xi \right) \varrho(t) dt \\
&= \lim_{T \rightarrow +\infty} \frac{\sup_{t \in \mathbb{R}} |\varphi_i(t)|}{\tilde{d}} e^{-2\tilde{d}_i T} = 0.
\end{aligned}$$

This obviously implies that $(\Gamma \varphi_i)$ belongs to $PAP_0(\mathbb{R}, \mathbb{R}, \varrho)$ and consequently, $(\Gamma \chi_i)$ belongs to $PAA(\mathbb{R}, \mathbb{R}, \varrho)$ for all $1 \leq i \leq n$. This ends the proof of Theorem 4.4. Let us give now the first main result of this paper

Theorem 4.5. *Suppose that assumptions $(H_1) - (H_4)$ hold. Then the GHNNs (1.1) has a unique weighted pseudo-almost automorphic solution in the region*

$$\mathcal{B} = \left\{ \psi \in PAA(\mathbb{R}, \mathbb{R}^n, \varrho), \|\psi - \varphi_0\| \leq \frac{r \|I\|_\infty}{\tilde{d}(1-r)} \right\},$$

where

$$\varphi_0(t) = \begin{pmatrix} \int_{-\infty}^t \exp\left(-\int_s^t d_1(\xi) d\xi\right) I_1(s) ds \\ \vdots \\ \int_{-\infty}^t \exp\left(-\int_s^t d_n(\xi) d\xi\right) I_n(s) ds \end{pmatrix}$$

Proof. Obviously, $\mathcal{B}$ is a closed convex subset of $PAA(\mathbb{R}, \mathbb{R}^n, \varrho)$ and one has

$$\begin{aligned}
\|\varphi_0(t)\| &= \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{-\int_s^t d_i(\xi) d\xi} I_i(s) ds \right\| \\
&\leq \|I\|_\infty \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-(t-s)\tilde{d}_i} ds \\
&= \frac{\|I\|_\infty}{\tilde{d}}.
\end{aligned}$$

Afterwards, we see without difficulty that for any $\varphi \in \mathcal{B}$, the following estimates

$$\|\varphi\| \leq \frac{\|I\|_\infty}{\tilde{d}(1-r)}.$$

Let us prove that the operator Γ is a self-mapping from $\mathcal{B}$ to $\mathcal{B}$. In fact, for any $\varphi \in \mathcal{B}$, we have

$$\begin{aligned}
\|(\Gamma\varphi)(t) - \varphi_0(t)\| &\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left[\frac{\sum_{j=1}^n L_j^{f_j} |a_{ij}(t)| + \sum_{j=1}^n L_j^{g_j} |b_{ij}(t)|}{\tilde{d}} \right] \|\varphi\|_\infty \\
&\leq \frac{r \|I\|_\infty}{\tilde{d}(1-r)},
\end{aligned}$$

which implies that $(\Gamma\varphi) \in \mathcal{B}$. Next, we prove the mapping Γ is a contraction mapping of $\mathcal{B}$. In view of (H_3) , for any $\varphi, \psi \in \mathcal{B}$, we have

$$\begin{aligned}
\|(\Gamma\varphi)(t) - (\Gamma\psi)(t)\| &= \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{-\int_s^t d_i(\xi) d\xi} \left\{ \sum_{j=1}^n a_{ij}(s) f_j(\varphi_j(s)) \right. \right. \\
&\quad + \sum_{j=1}^n b_{ij}(s) g_j(\varphi_j(s - \tau_j(s))) \\
&\quad + \sum_{j=1}^n b_{ij}(s) g_j(\varphi_j(s - \tau_j(s))) \\
&\quad \left. \left. - \sum_{j=1}^n a_{ij}(s) f_j(\psi_j(s)) + \sum_{j=1}^n b_{ij}(s) g_j(\psi_j(s - \tau_j(s))) \right\} \right\| \\
&\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t d_i(\xi) d\xi} \left\{ \sum_{j=1}^n |a_{ij}(s)| |f_j(\varphi_j(s)) - f_j(\psi_j(s))| \right. \\
&\quad \left. + \sum_{j=1}^n |b_{ij}(s)| |g_j(\varphi_j(s - \tau_j(s))) - g_j(\psi_j(s - \tau_j(s)))| \right\} \\
&\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t d_i(\xi) d\xi} \left\{ \sum_{j=1}^n |a_{ij}(s)| L_f^j |\varphi_j(s) - \psi_j(s)| \right. \\
&\quad \left. + \sum_{j=1}^n |b_{ij}(s)| L_g^j |\varphi_j(s - \tau_j(s)) - \psi_j(s - \tau_j(s))| \right\} ds \\
&\leq \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left[\frac{\sum_{j=1}^n L_f^j |a_{ij}(t)| + \sum_{j=1}^n L_g^j |b_{ij}(t)|}{\widetilde{d}} \right] \|\varphi - \psi\| \\
&\leq r \|\varphi - \psi\|,
\end{aligned}$$

which proves that Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $x^* \in \mathcal{B}$ that is $\Gamma(x^*) = x^*$. Hence, x^* is the unique weighted pseudo-almost automorphic solution of (1.1) in $\mathcal{B}$.

5. THE GLOBAL ATTRACTIVITY OF THE PAA SOLUTION

Let $x^*(\cdot) = (x_1^*(\cdot), \dots, x_n^*(\cdot))^T$ be the weighted pseudo-almost automorphic solution of Theorem 4.4 and $x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))^T$ be an arbitrary solution of (1.1). So, one has

$$\dot{x}_i^*(t) = -d_i(t) x_i^*(t) + \sum_{j=1}^n [a_{ij}(t) f(x_j^*(t)) + b_{ij}(t) g(x_j^*(t - \tau_j(t)))] + I_i(t)$$

and

$$\dot{x}_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^n [a_{ij}(t)f(x_j(t)) + b_{ij}(t)g_j(x_j(t - \tau_j(t)))] + I_i(t).$$

Let us pose for all $1 \leq i \leq n$, $z_i(\cdot) = x_i(\cdot) - x_i^*(\cdot)$. Consequently, we obtain

$$\begin{cases} \dot{z}_i(t) = -d_i(t)z_i(t) + \sum_{j=1}^n a_{ij}(t)F_j(z_j(t)) + \sum_{j=1}^n b_{ij}(t)G_j(x_j(t - \tau_j(t))) \\ z_i(t) = \theta_i(t), -r \leq t \leq 0, 1 \leq i \leq n, \end{cases} \quad (1.2)$$

where for all $1 \leq i, j \leq n$,

$$F_j(z_j(\cdot)) = f_j(x_j(\cdot)) - f_j(x_j^*(\cdot)), G_j(z_j(\cdot)) = g_j(x_j(\cdot)) - g_j(x_j^*(\cdot))$$

and

$$\theta_i(\cdot) = \psi_i(\cdot) - x_i^*(\cdot).$$

Clearly, the weighted pseudo-almost automorphic solution $x^*(\cdot)$ of system (1.1) is global attractive if and only if the equilibrium point O of system (1.2) is globally attractive. It is obviously sufficient to study the global attractivity of the equilibrium point O for system (1.2).

Theorem 5.1. *Suppose that assumptions $(H_1) - (H_4)$ hold, then the equilibrium point O of the nonlinear system (1.2) is global attractive.*

Proof. First, let us prove that the solutions of system (1.2) are uniformly bounded. In other words, there exists $M > 0$ such that for all $t \geq 0$ one has

$$\|z(t)\| \leq M.$$

In fact, there exists a large number $M > 0$, such that

$$\|\theta\|_\infty < M.$$

Let us take a number κ such that, $\kappa > 1$. We shall prove that for all $t \geq 0$, $\|z(t)\| \leq \kappa M$. Let us assume, by contradiction, that $z(\cdot)$ is not bounded. In that case one finds at least one number $t' > 0$, such that

$$\begin{cases} \|z(t')\| = \kappa M \\ \|z(t)\| \leq \kappa M, \quad 0 \leq t \leq t'. \end{cases}$$

Using the conditions (H_3) , (H_4) and the equation (1.2), we obtain

$$\begin{aligned}
\|z(t')\| &\leq \max_{1 \leq i \leq n} \left\{ |\theta_i(0)| e^{-\tilde{d}_i t'} + \int_0^{t'} e^{-\tilde{d}(t'-s)} \|z(s)\| \times \right. \\
&\quad \left. \left(\sum_{j=1}^n a_{ij}^+ + L_j^f |z_j(s)| + b_{ij}^+ L_j^g |z_j(s - \tau_j(s))| \right) ds \right\} \\
&\leq \max_{1 \leq i \leq n} \left\{ \kappa M e^{-\tilde{d}_i t'} + \int_0^{t'} e^{-\tilde{d}(t'-s)} \kappa M \left(\sum_{j=1}^n a_{ij}^+ + L_j^f + b_{ij}^+ L_j^g \right) ds \right\} \\
&\leq \max_{1 \leq i \leq n} \left\{ e^{-\tilde{d}_i t'} + \int_0^{t'} e^{-\tilde{d}_i(t'-s)} \left(\sum_{j=1}^n a_{ij}^+ + L_j^f + b_{ij}^+ L_j^g \right) ds \right\} \kappa M \\
&\leq \max_{1 \leq i \leq n} \left\{ e^{-\tilde{d}_i t'} + \left(\frac{\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g}{\tilde{d}_i} \right) (1 - e^{-\tilde{d}_i t'}) \right\} \kappa M \\
&< \kappa M,
\end{aligned}$$

which gives a contradiction. Hence, we have by now proved that for all $t \geq 0$, $\|z(t)\| \leq \kappa M$. Let us take $\kappa \rightarrow 1$, then for all $t \geq 0$, $\|z(t)\| \leq M$. Thus, there is a constant $\lambda \geq 0$, such that

$$\limsup_{t \rightarrow +\infty} \|z(t)\| = \lambda.$$

It follows that

$$\forall \varepsilon > 0, \exists t_2 < 0, \forall t, (t \geq t_2 \implies \|z(t)\| \leq (1 + \varepsilon) \lambda)$$

and

$$\begin{aligned}
\dot{z}_i(t) + d_i(t) z_i(t) &= \sum_{j=1}^n a_{ij}(t) F_j(z_j(t)) + \sum_{j=1}^n b_{ij}(t) G_j(x_j(t - \tau_j(t))) \\
&\leq \sum_{j=1}^n |a_{ij}(t)| |F_j(z_j(t))| + \sum_{j=1}^n |b_{ij}(t)| |G_j(x_j(t - \tau_j(t)))| \\
&\leq \sum_{j=1}^n |a_{ij}(t)| L_j^f |x_j(t) - x_j^*(t)| + \\
&\quad + \sum_{j=1}^n |b_{ij}(t)| L_j^g |x_j(t - \tau_j(t)) - x_j^*(t - \tau_j(t))| \\
&\leq \max_{1 \leq i \leq n} \left(\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g \right) \|z(t)\| \\
&\leq \max_{1 \leq i \leq n} \left(\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g \right) (1 + \varepsilon) \lambda.
\end{aligned}$$

So, through the integration, we obtain the inequality

$$\begin{aligned}
|z_i(t)| &\leq |\theta_i(0)| e^{-\int_0^t d_i(u) du} + \left\{ \max_{1 \leq i \leq n} \left(\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g \right) (1 + \varepsilon) \lambda \right\} \int_0^t e^{-\int_s^t d_i(u) du} ds \\
&\leq \|\theta\| e^{-\underline{d}_i t} + \left\{ \max_{1 \leq i \leq n} \left(\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g \right) (1 + \varepsilon) \lambda \right\} \int_0^t e^{-\underline{d}_i(t-s)} ds \\
&\leq \|\theta\| e^{-\underline{d}_i t} + \left\{ \frac{\max_{1 \leq i \leq n} \left(\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g \right)}{\underline{d}_i} \right\} (1 + \varepsilon) \lambda (1 - e^{-\underline{d}_i t}) \\
&\leq \|\theta\| e^{-\underline{d}_i t} + \left\{ \max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g}{\underline{d}_i} \right) \right\} (1 + \varepsilon) \lambda (1 - e^{-\underline{d}_i t})
\end{aligned}$$

Hence,

$$\|z(t)\| \leq \max_{1 \leq i \leq n} \left[\|\theta\| e^{-\underline{d}_i t} + \left\{ \max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^n a_{ij}^+ L_j^f + b_{ij}^+ L_j^g}{\underline{d}_i} \right) \right\} (1 + \varepsilon) \lambda (1 - e^{-\underline{d}_i t}) \right]$$

In particular,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \|z(t)\| &\leq \limsup_{t \rightarrow +\infty} \max_{1 \leq i \leq n} \left[\|\theta\| e^{-\tilde{d}_i t} \right. \\ &\quad \left. + \left\{ \max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^n a_{ij}^+ L_j^{f_j} + b_{ij}^+ L_j^{g_j}}{\tilde{d}_i} \right) \right\} (1 + \varepsilon) \lambda (1 - e^{-\tilde{d}_i t}) \right] \\ &\leq r (1 + \varepsilon) \beta \end{aligned}$$

Passing to limit when $\varepsilon \rightarrow 0$, we obtain

$$\lambda (1 - r) \leq 0$$

By condition (H_4) , we obtain $\lambda = 0$ which imply that

$$\lim_{t \rightarrow +\infty} \|z(t)\| = \lim_{t \rightarrow +\infty} \|x_i(t) - x_i^*(t)\| = 0,$$

and consequently the proof of this theorem is completed.

Remark 5.2. Our results and the method used in the proof of Theorem 4.4 are essentially new since we don't use the exponential dichotomy. Moreover, it should be pointed out that in the proof of the global attractivity the pseudo-almost automorphy is without importance. So, similar result can be obtained for the class of almost periodic or pseudo-almost periodic functions. Notice that Theorem 5.1 is same to theorem 4 in [30], but our approach to get the result is different. Besides, our model is more general than [24] since in Ref. [24], the authors studied the almost periodicity and the model considered is with constant coefficients and delay. Notice that, some CNNs and HNNs with and without constant delays can be seen as a special case of our model. Finally, to our best knowledge, there is no published paper considering the pseudo-almost automorphic solutions for Hopfield neural networks with impulses and varying-time delay.

6. AN EXAMPLE

In order to illustrate some feature of our main results, in this section, we will apply our main results to some special three-dimensional systems and demonstrate the efficiencies of our criteria. Let us consider the following Hopfield neural network

$$\dot{x}_i(t) = -d_i(t) x_i(t) + \sum_{j=1}^3 a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^3 b_{ij}(t) g_j(x_j(t - \tau_j(t)))$$

where

$$\begin{pmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{pmatrix} = \begin{pmatrix} 3 + \cos^2 \pi t \\ 7 + 2 \cos \sqrt{2}t \\ 5 + 2 \sin \sqrt{3}t \end{pmatrix} \implies \tilde{d} = 2$$

for all $x \in \mathbb{R}$, for all $t \in \mathbb{R}, \forall 1 \leq j \leq 3$

$$f_j(t) = g_j(t) = \sin t \implies L_j^1 = L_j^g = 1$$

$$\begin{aligned}
\text{and for } \varrho(t) = e^t \text{ let } (a_{ij}(t))_{1 \leq i, j \leq 3} &= \begin{pmatrix} \frac{\cos t}{5} & \frac{\cos\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)+0,5e^{-t}}{2} & \frac{\sin\left(\frac{1}{1+\sin t+\sin \sqrt{5}t}\right)}{5} \\ \frac{\cos \sqrt{3}t}{10} & \frac{3 \cos\left(\frac{1}{2+\sin t+\sin \sqrt{3}t}\right)+2e^{-t}}{10} & \frac{\cos\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)}{5} \\ \frac{\sin \sqrt{2}t}{5} & \frac{2 \sin\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)+e^{-t}}{10} & \frac{\sin\left(\frac{1}{2+\sin t+\sin \sqrt{3}t}\right)}{5} \end{pmatrix}, \\
(b_{ij}(t))_{1 \leq i, j \leq 3} &= \begin{pmatrix} \frac{\sin \sqrt{2}t}{10} & \frac{2 \cos\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)+e^{-t}}{10} & \frac{\sin\left(\frac{1}{1+\sin t+\sin \sqrt{5}t}\right)}{10} \\ \frac{\cos \sqrt{5}t}{5} & \frac{\cos\left(\frac{1}{2+\sin t+\sin \sqrt{3}t}\right)+2e^{-t}}{10} & \frac{\cos\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)}{5} \\ \frac{\sin \sqrt{3}t}{5} & \frac{\sin\left(\frac{1}{2+\sin t+\sin \sqrt{2}t}\right)+e^{-t}}{10} & \frac{\sin\left(\frac{1}{2+\sin t+\sin \sqrt{3}t}\right)}{10} \end{pmatrix} \text{ then} \\
r &= \max_{1 \leq i \leq n} \sup_{t \in \mathbb{R}} \left[\frac{\sum_{j=1}^3 L_j^{f_j} |a_{ij}(t)| + \sum_{j=1}^n L_j^{g_j} |b_{ij}(t)|}{\tilde{d}} \right] \\
&= \max \frac{\sum_{j=1}^3 a_{ij}^+ + b_{ij}^+}{\tilde{d}} \\
&= \max \sum_{j=1}^3 \left(\frac{1.8}{2}, \frac{1.5}{2}, \frac{1.2}{2} \right) < 1
\end{aligned}$$

Therefore, all conditions of Theorem 4.4 are satisfied, then the delayed Hopfield neural networks (1.1) has a unique weighted pseudo-almost periodic solution in the region

$$\mathcal{B} = B(\varphi_0, r) = \left\{ x \in PAP^1(\mathbb{R}, \mathbb{R}^n, \varrho), \|\varphi - \varphi_0\| \leq \frac{0, 9 \times 3 + 0}{2(1 - 0, 9)} = 0, 135 \right\}.$$

7. CONCLUSION

In this paper, some novel sufficient conditions are presented ensuring the existence and uniqueness of the weighted pseudo-almost automorphic solution for the Hopfield neural networks with varying-time coefficients and delays. We claim that many results in the literature dealing with periodic or almost periodic solutions of Hopfield neural networks are special cases of the results in this paper since the concept of weighted pseudo-almost-automorphy generalizes the one of the pseudo-almost-periodicity. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained results.

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