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ANTI-INVARIANT RIEMANNIAN SUBMERSIONS FROM A LOCALLY RIEMANNIAN PRODUCT MANIFOLD TO ANY RIEMANNIAN MANIFOLD

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Abstract. In this paper, we define and study anti-invariant Riemannian submersions from a locally Riemannian product manifold onto a Riemannian manifold. We investigate the geometric properties of distributions which are involved from the definition of an anti-invariant Riemannian submersion in locally Riemannian product manifolds.

1. Introduction and preliminaries

Immersions and submersions, which are special tools in differential geometry, also play a fundamental role in Riemannian geometry, especially when the involved manifolds carry an additional structure (such as contact, Hermitian quaternion and product structure).

In particularly, Riemannian submersions(which we always assume to have connected fibers) are fundamentally important in several areas of Riemannian geometry. For instance, it is a classical and important problem in Riemannian geometry to construct Riemannian manifolds with positive or non-negative sectional curvature. While there are a few methods, the most abundant source of examples comes via submersions from compact Lie groups. Riemannian submersions between Riemannian manifolds are important geometric structures.

The study of submersion between two Riemannian manifolds was initiated by B. O'Neill and A. Gray[1, 2]. This theory was developed very much in the last thirty five years. A submersion gives rise to two distributions called the vertical and horizontal distributions of which the vertical distribution is always integrable[6].

In [1], A. Gray used the metric g of a Riemannian manifold M and added to the already known condition imposed on the field P the requirement of symmetric, which has the form of the relation $g(P.,P.) = g(.,.)$, similar to the condition of invariance of a Kaehler metric of an almost complex manifold.

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In [7], S. Kobayashi considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kaehler manifold in terms of distributions.

Let (M, g_M) and (B, g_B) be Riemannian manifolds of dimension m and b, respectively. A differentiable map $\pi : M \longrightarrow B$ is said to be submersion if it has maximal rank at any point of M. The implicit function theorem states that the fibre over any $q \in B$, $\pi^{-1}(q)$ is a closed $(m-b)$-dimensional submanifold of M. The submanifolds $\pi^{-1}(q)$,$q \in B$, are called fibres. A vector field on M is called vertical if it is always tangent to fibres and horizontal if it is always orthogonal to fibres. Putting $\mathcal{V}_p = Kerd\pi_p$, for any $p \in M$, we obtain an integrable distribution $\mathcal{V}$ which corresponds to the foliation of M determined by the fibres of π, since each $\mathcal{V}_p$ coincides with the tangent space $\pi^{-1}(q)$ at $p \in M$, $\pi(p) = q$, that is, $\mathcal{V}_p = T_{\pi^{-1}(q)}p$[5].

Each $\mathcal{V}$ is called the vertical space at p, $\mathcal{V}$ is the vertical distribution, the sections of $\mathcal{V}$ are so-called vertical vector fields and determine a Lie-subalgebra $\Gamma(\mathcal{V})$ of $\Gamma(TM)$.

Let $\mathcal{H}$ be the complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g_M. So at any $p \in M$, one has the orthogonal decomposition such that

$$T_M(p) = \mathcal{V}_p \oplus \mathcal{H}_p, \tag{1.1}$$

where $\mathcal{H}_p$ is called the horizontal distribution space at p. The sections of the horizontal distribution $\mathcal{H}$ are the horizontal vector fields. They set up a subspace $\Gamma(\mathcal{H})$ of $\Gamma(TM)$.

Now, we denote by $\hbar$ and ν the projection operators of $\Gamma(TM)$ onto $\mathcal{H}$ and $\mathcal{V}$, respectively, then we have

$$\hbar^2 = \hbar, \quad \nu^2 = \nu, \quad and \quad \hbar\nu = \nu\hbar = 0. \tag{1.2}$$

Putting $P = \hbar - \nu$, by virtue of (1.2), the tensor field P satisfy the identity $P^2 = I$, where I denotes the identity map of $\Gamma(TM)$.

For a submersion $\pi : M \longrightarrow B$, we denote by $d\pi$(or sometimes π_*) the tangent map of π. A vector field on M is called basic if X is horizontal and π-related to a vector field Y on B, that is, $d\pi(X_p) = Y_{\pi(p)}$ for all $p \in M$[9].

Definition 1.1. Let (M, g_M) and (B, g_B) two be Riemannian manifolds. A submersion $\pi : M \longrightarrow B$ is called a Riemannian submersion if at each point $p \in M$, $d\pi$ preserves the length of the horizontal vectors, that is,

$$d\pi_p : \mathcal{H}_p \longrightarrow T_B\pi(p)$$

is a linear isometry[5].

We state the following Proposition for later use[6].

Proposition 1.2. *Let $\pi : (M, g_M) \longrightarrow (B, g_B)$ be Riemannian submersion and denote by ∇ and ∇' the Levi-Civita connections of M and B, respectively. If X and Y are basic vector fields, π-related to X' and Y', one has*
1.) $g_M(X, Y) = g_B(X', Y')o\pi$,
2.) $\hbar[X, Y]$ is the basic vector field π-related to $[X', Y']$,

3.) $\hbar(\nabla_X Y)$ *is the basic vector field* π*-related to* ${\nabla'}_{X'} Y'$*,*
4.) $[X, V]$ *is a vertical vector field for any vertical vector field* V*.*

A Riemannian submersion $\pi : (M, g_M) \longrightarrow (B, g_B)$ determines two (1,2)-tensor fields such as $\mathcal{T}$ and $\mathcal{A}$ on M. They are called the fundamental tensor fields or the invariants of Riemannian submersion π and are defined by means of the vertical and horizontal projections

$$\hbar : \Gamma(TM) \longrightarrow \Gamma(\mathcal{H}), \quad \nu : \Gamma(TM) \longrightarrow \Gamma(\mathcal{V}), \tag{1.3}$$

according to the formulas

$$\mathcal{T}(E, F) = \mathcal{T}_E F = \hbar \nabla_{\nu E} \nu F + \nu \nabla_{\nu E} \hbar F \tag{1.4}$$

and

$$\mathcal{A}(E, F) = \mathcal{A}_E F = \nu \nabla_{\hbar E} \hbar F + \hbar \nabla_{\hbar E} \nu F \tag{1.5}$$

for any $E, F \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of the differentiable vector fields on M. It is easy to prove that $\mathcal{A}$ and $\mathcal{T}$ are horizontal and vertical tensor fields, respectively, that is, $\mathcal{A}_E = \mathcal{A}_{\hbar E}$ and $\mathcal{T}_E = \mathcal{T}_{\nu E}$. The geometry of Riemannian submersions is characterized by $\mathcal{A}$ and $\mathcal{T}$ tensor fields. $\mathcal{A}_E$ and $\mathcal{T}_E$ are skew-symmetric tensor fields on M reversing the horizontal and vertical subspaces, that is,

$$\mathcal{T}_E(\mathcal{V}_p) \subseteq \mathcal{H}_p, \quad \mathcal{T}_E(\mathcal{H}_p) \subseteq \mathcal{V}_p$$

and

$$\mathcal{A}_E(\mathcal{V}_p) \subseteq \mathcal{H}_p, \quad \mathcal{A}_E(\mathcal{H}_p) \subseteq \mathcal{V}_p$$

Furthermore, from [3], we state that the tensor fields $\mathcal{A}$ and $\mathcal{T}$ satisfy

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\nu[X, Y], \quad \forall X, Y \in \Gamma(\mathcal{H}), \tag{1.6}$$

and

$$\mathcal{T}_U V = \mathcal{T}_V U, \quad \forall U, V \in \Gamma(\mathcal{V}). \tag{1.7}$$

The mappings between Riemannian manifolds satisfying these two conditions were studied by T. Nagano in [9]. In particular, he derived the fundamental equations analogues to Weingarten's formulas for submanifolds in Riemannian geometry. O'Neill further studied such mappings in [2] and called such mappings Riemannian submersions.

Geometrical features of the fibres will be distinguished by $\hbar$ and $\mathcal{V}$. For the covariant derivative, for example, $\nabla^{\mathcal{V}}_V W = \mathcal{V} \nabla_V W$ representation will be used in this paper, where W, V are vertical vector fields and ∇ is the Levi-Civita connection on M.

Lemma 1.3. *Let X, Y be horizontal vector fields and V, W vertical vector fields. Then one has*

$$\begin{aligned}\nabla_V W &= \mathcal{T}_V W + \nabla_V^{\mathcal{V}} W \\ \nabla_V X &= \nabla_V^{\hbar} X + \mathcal{T}_V X \\ \nabla_X V &= \mathcal{A}_X V + \nabla_X^{\mathcal{V}} V \\ \nabla_X Y &= \nabla_X^{\hbar} Y + \mathcal{A}_X Y.\end{aligned}$$

Furthermore, if X is basic, $\nabla_V^{\hbar} X = \mathcal{A}_X V$[7].

Definition 1.4. Let $\pi : (M, g_M) \longrightarrow (B, g_B)$ be a map between Riemannian manifolds. The differential π_* of π can be seen a section of the bundle $(TM, \pi^{-1}TB) \longrightarrow M$, where $\pi^{-1}(TB)$ is the pullback bundle and it has fibres $(\pi^{-1}(TB))_p = T_B\pi(p)$ for each $p \in M$. $Hom(TM, \pi^{-1}(TB))$ has a connection ∇ induced from the Levi-Civita connection ∇' and pullback connection.

The second fundamental form of π is the map

$$\nabla\pi_* : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TB)$$

defined by

$$(\nabla\pi_*)(X, Y) = \nabla_X^2 \pi_* Y - \pi_*(\nabla'_X Y), \tag{1.8}$$

where ∇^2 also denotes the pullback connection of ∇^2 along π. It is well known that the second fundamental form is symmetric[6].

Let (M, g, F) be an almost Riemannian product manifold. This means that Riemannian manifold M admits a tensor field F of type $(1,1)$ such that

$$g(X, Y) = g(FX, FY), F^2 = I, F \neq \mp I, \tag{1.9}$$

for all $X, Y \in \Gamma(TM)$. We denote the Levi-Civita connection on M by ∇. If F is parallel with respect ∇, that is, $\nabla F = 0$, the almost Riemannian product manifold is called locally Riemannian product manifold.

2. Anti-Invariant Riemannian Submersions in a Locally Riemannian Product Manifold

Definition 2.1. Let (M, g_M, F) be a locally Riemannian product manifold and (B, g_B) be a Riemannian manifold. We suppose that there exists a Riemannian submersion $\pi : M \longrightarrow B$ such that $Ker\pi_*$ is anti-invariant with respect to F, that is, $F(Ker\pi_*) \subseteq (Ker\pi_*)^{\perp}$, then we call π an anti-invariant Riemannian submersion.

Now, we will give an example in order to guarantee the existence of ant-invariant Riemannian submersions in locally Riemannian product manifolds and demonstrate that the method presented in this paper is effective.

Example 2.2. Let $\pi : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be a map defined by

$$\pi(x_1, x_2, x_3, x_4, x_5) = (\frac{x_1 - x_4}{\sqrt{2}}, x_3, \frac{x_2 + x_5}{\sqrt{2}}).$$

Then by direct calculations, we have

$$Kerd\pi = S_p\{V = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, W = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5}\}$$

and

$$(Kerd\pi)^{\perp} = S_p\{X = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4}, Y = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, Z = \frac{\partial}{\partial x_3}\}.$$

Since $rank(\pi) = 3$, the map π is a submersion. Furthermore, we conclude that the derivative map $d\pi$ preserves lengths of the horizontal vectors with respect to the standard metric tensors of $\mathbb{R}^5$ and $\mathbb{R}^3$, that is, π is a Riemannian submersion. On the other hand, one of the Riemannian product structure of $\mathbb{R}^5$ is defined by

$$F : \mathbb{R}^5 \longrightarrow \mathbb{R}^5, \quad F(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, -x_4, -x_5)$$

We can easily see that $FV = X$, $FW = Y$ which imply that $F(Kerd\pi) \subset (Kerd\pi)^{\perp}$. So π is an anti-invariant Riemannian submersion.

Here we state that If $\pi : M \longrightarrow B$ is an anti-invariant Riemannian submersion from a locally Riemannian product manifold (M, g, F) to a Riemannian manifold (B, g_B), then from Definition 2.1 we can derive $F(Kerd\pi)^{\perp} \cap Kerd\pi \neq 0$. Therefore, we denote the orthogonal complementary distribution of $FKerd\pi$ in $(Kerd\pi)^{\perp}$ by μ. Hence we have direct sum

$$(Kerd\pi)^{\perp} = F(Kerd\pi) \oplus \mu, \tag{2.1}$$

which is equivalent to

$$F(Kerd\pi)^{\perp} = Kerd\pi \oplus F(\mu). \tag{2.2}$$

We can easily to see that μ is an invariant subbundle of $(Kerd\pi)^{\perp}$ with respect to F and $\dim(\mu) = 2b - m$.
since π is a Riemannian submersion and consider (2.1), we can write

$$TB = d\pi(FKerd\pi) \oplus d\pi(\mu). \tag{2.3}$$

So for any $X \in (Kerd\pi)^{\perp}$, FX can be written follow as

$$FX = tX + \omega X, \tag{2.4}$$

where tX and ωX belong to the distributions $\Gamma(Kerd\pi)$ and $\Gamma(\mu)$, respectively.

Lemma 2.3. *$\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be an anti-invariant Riemannian submersion. Then for all $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$ we have*

$$\mathcal{A}_X FV = t\mathcal{A}_X V = -\mathcal{A}_{FV} X, \tag{2.5}$$

$$(\nabla_V t)X = -\mathcal{T}_V \omega X, \quad (\nabla_V \omega)X = F\mathcal{T}_V X - \mathcal{T}_V tX, \tag{2.6}$$

and

$$(\nabla_Y \omega)X = F\mathcal{A}_Y X - \mathcal{A}_Y tX \quad and \quad (\nabla_Y t)X = \mathcal{A}_{\omega X} Y. \tag{2.7}$$

Proof.
Since Riemannian product structure F is integrable and consider Lemma 1.3 we have

$$\begin{aligned}\nabla_X FV &= F\nabla_X V\\ \nabla^{\hbar}_X FV + \mathcal{A}_X FV &= t\mathcal{A}_X V + \omega\mathcal{A}_X V + F\nabla^{\mathcal{V}}_X V,\end{aligned}$$

whereof the vertical components give (2.5). On the other hand, for the same reason, we reach

$$\begin{aligned}\nabla_V FX &= F\nabla_V X\\ \mathcal{T}_V tX + \nabla^{\mathcal{V}}_V tX + \nabla^{\hbar}_V \omega X + \mathcal{T}_V \omega X &= t\nabla^{\hbar}_V X + \omega\nabla^{\hbar}_V X + F\mathcal{T}_V X.\end{aligned}$$

The horizontal and vertical components of the last equation, respectively, we get

$$\nabla^{\hbar}_V \omega X - \omega\nabla^{\hbar}_V X = F\mathcal{T}_V X - \mathcal{T}_V tX,$$

that is,

$$(\nabla_V \omega)X = F\mathcal{T}_V X - \mathcal{T}_V tX$$

and

$$\nabla^{\mathcal{V}}_V tX - t\nabla^{\hbar}_V X = -\mathcal{T}_V \omega X,$$

$$(\nabla_V t)X = -\mathcal{T}_V \omega X,$$

which gives (2.6). Again making use of (2.5) and consider Lemma 1.3, we have

$$\begin{aligned}\nabla_Y FX &= F\nabla_Y X\\ \mathcal{A}_Y tX + \nabla^{\mathcal{V}}_Y tX + \nabla^{\hbar}_Y \omega X + \mathcal{A}_Y \omega X &= t\nabla^{\hbar}_Y X + \omega\nabla^{\hbar}_Y X + F\mathcal{A}_Y X.\end{aligned}$$

From the horizontal and vertical components of this equation, respectively, we obtain

$$\nabla^{\hbar}_Y \omega X - \omega(\nabla^{\hbar}_Y X) = F\mathcal{A}_Y X - \mathcal{A}_t X, \quad \nabla^{\mathcal{V}}_Y tX - t\nabla^{\hbar}_Y X = -\mathcal{A}_Y \omega X,$$

that is,

$$(\nabla_Y \omega)X = F\mathcal{A}_Y X - \mathcal{A}_Y tX \quad \textit{and} \quad (\nabla_Y t)X = -\mathcal{A}_Y \omega X = \mathcal{A}_{\omega X} Y,$$

where the covariant derivatives of t and ω are defined by

$$(\nabla_E t)F = \nabla^{\hbar}_E tF - t(\nabla_E F)$$

and

$$(\nabla_E \omega)F = \nabla^{\hbar}_E \omega F - \omega(\nabla^{\hbar}_E F),$$

for any $E, F \in \Gamma(TM)$. Hence the proof is complete. □

In the rest of this paper, for the shake of brevity, we denote the horizontal and vertical distributions of the anti-invariant Riemannian submersion $\pi : M \longrightarrow B$ by $\mathcal{H}$ and $\mathcal{V}$, respectively, and π_* stand for the differential of π.

We note that an anti-invariant Riemannian submersion is called a lagrangian anti-invariant Riemannian submersion if $F(\mathcal{V}_p) = \mathcal{H}_p$ for each $p \in M$.

Theorem 2.4. *Let $\pi : (M, g_B, F) \longrightarrow (B, g_B)$ be an anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*
1.) The horizontal distribution $\mathcal{H}$ is integrable

$$\begin{aligned} 2.) g_M(\mathcal{A}_X tY, FV) - g_M(\mathcal{A}_Y tX, FV) &= g_B((\nabla_Y \omega)X, FV) - g_B((\nabla_X \omega)Y, FV); \\ 3.) g_M(\mathcal{A}_{FV} X, tY) - g_M(\mathcal{A}_{FV} Y, tX) &= g_B((\nabla \pi_*)(X, V), \pi_* F\omega Y) \\ &- g_B((\nabla \pi_*)(Y, V), \pi_* F\omega X), \end{aligned}$$

for any $X, Y \in \mathcal{H}$ and $V \in \mathcal{V}$

Proof. By using (1.9) and (2.3), we have

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ &= g_M(\nabla_X FY, FV) - g_M(\nabla_Y FX, FV) \\ &= g_M(\nabla_X tY, FV) - g_M(\nabla_Y tX, FV) + g_M(\nabla_X \omega Y, FV) \\ &- g_M(\nabla_Y \omega X, FV) \\ &= g_M(\mathcal{A}_X tY, FV) - g_M(\mathcal{A}_Y tX, FV) + g(\nabla^{\hbar}_X \omega Y, FV) \\ &- g_M(\nabla^{\hbar}_Y \omega X, FV) \\ &= g_M(\mathcal{A}_X tY, FV) - g_M(\mathcal{A}_Y tX, FV) + g_M((\nabla_X \omega)Y, FV) \\ &- g_M((\nabla_Y \omega)X, FV) \end{aligned}$$

which implies that 1.) $\Leftrightarrow$ 2.) On the other hand, for any $V \in \Gamma(\mathcal{V})$, differentiating covariant the $\pi_* V = 0$ in the direction $X \in \Gamma(\mathcal{H})$, also making use of (1.8) and consider Lemma 1.3, we get

$$(\nabla \pi_*)(X, V) = \pi_*(\nabla_X V) = \pi_*(\mathcal{A}_X V). \tag{2.8}$$

To show equivalence of 2.) to 3.), we also use (2.7)

$$\begin{aligned} g_M(\mathcal{A}_X tY, FV) &- g_M(\mathcal{A}_Y tX, FV) + g_M(\nabla_X \omega Y, FV) - g_M(\nabla_Y \omega X, FV) \\ &= g_M(\mathcal{A}_X tY, FV) - g_M(\mathcal{A}_Y tX, FV) - g_M(\nabla_X FV, \omega Y) \\ &+ g_M(\nabla_Y FV, \omega X) \\ &= g_M(\mathcal{A}_{FV} X, tY) - g_M(\mathcal{A}_{FV} Y, tX) - g_M(\mathcal{A}_X V, F\omega Y) \\ &+ g_M(\mathcal{A}_Y V, F\omega X). \end{aligned}$$

Riemannian submersion of the fact that where π and (2.8) are used, we reach

$$\begin{aligned} g_M(\mathcal{A}_{FV} X, tY) - g_M(\mathcal{A}_{FV} Y, tX) &= g_B(\pi_*(\mathcal{A}_X V), \pi_* F\omega Y) \\ &- g_B(\pi_*(\mathcal{A}_Y V), \pi_* F\omega X) \\ &= g_B((\nabla \pi_*)(X, V), \pi_* F\omega Y) \\ &- g_B((\nabla \pi_*)(Y, V), \pi_* F\omega X) \end{aligned}$$

which implies 2.) $\Leftrightarrow$ 3.) □

Since $\omega = 0$ in the Lagrangian anti-invariant Riemannian submersions, the following corollary can be given.

Corollary 2.5. *Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*

1.) The horizontal distribution $\mathcal{H}$ is integrable,
2.) $\mathcal{A}_X FY = \mathcal{A}_Y FX$,
3.) $\mathcal{A}_{FV} FY = F\mathcal{A}_{FV} Y$, for any $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$.

Theorem 2.6. *Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be an anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*
1.) The horizontal distribution $\mathcal{H}$ is total geodesic on M;
2.) $g_M(\mathcal{A}_{FV} X, tY) + g_M((\nabla_X \omega)Y, FV) = 0$;
3.) $t\mathcal{A}_X tY + \mathcal{A}_X F\omega Y = 0$;
for any $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$.

Proof. For any $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\nabla_X FY, FV) = g_M(\nabla_X tY, FV) + g_M(\nabla_X \omega Y, FV) \\ &= g_M(\mathcal{A}_X tY, FV) + g_M(\nabla^{\hbar}_X \omega Y, FV) \\ &= g_M(\mathcal{A}_X tY, FV) + g_M((\nabla_X \omega)Y + \omega(\nabla_X Y), FV) \\ &= g_M(\mathcal{A}_{FV} X, tY) + g_M((\nabla_X \omega)Y, FV) \end{aligned}$$

which implies that 1.) $\Leftrightarrow$ 2.). On the other hand, by direct calculations, we have

$$\begin{aligned} g_M(\mathcal{A}_{FV} X, tY) + g_M((\nabla_X \omega)Y, FV) &= g_M(\mathcal{A}_{FV} X, tY) + g_M(\nabla_X \omega Y, FV) \\ &= g_M(t\mathcal{A}_X tY, V) + g_M(\nabla_X F\omega Y, V) \\ &= g_M(t\mathcal{A}_X tY + \mathcal{A}_X F\omega Y, V), \end{aligned}$$

which implies 2.) $\Leftrightarrow$ 3.) This proves our assertion. □

Corollary 2.7. *Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*
1.) The horizontal distribution $\mathcal{H}$ is totally geodesic on M,
2.) $\mathcal{A}_{FV} X = 0$,
3.) $\mathcal{A}_X V = 0$,
for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$.

Theorem 2.8. *Let $\pi : (M, g_B, F) \longrightarrow (B, g_B)$ be an anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*
1.) The vertical distribution $\mathcal{V}$ is total geodesic on M,
2.) $g_M(\mathcal{T}_U tX, FV) = g_B((\nabla \pi_*)(FV, X), \pi_* \omega X)$,
3.) $\mathcal{T}_V tX + \mathcal{A}_{\omega X} V$ lies in the invariant subbundle $\Gamma(\mu)$,
for any $X \in \Gamma(\mathcal{H})$ and $U, V \in \Gamma(\mathcal{V})$.

Proof. By using (1.9), (2.4) and consider Lemma 1.3, we arrive at

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\nabla_U FV, FX) = g_M(\nabla_U FV, tX) + g_M(\nabla_U FV, \omega X) \\ &= g_M(\mathcal{T}_U FV, tX) + g_M(\nabla_{FV} U + [FV, U], \omega X). \end{aligned}$$

Since $[FV, U] \in \Gamma(\mathcal{V})$ and taking into account that π is a Riemannian submersion, we obtain

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{T}_U FV, tX) + g_M(\nabla_{FV} U, \omega X) \\ &= g_M(\mathcal{T}_U FV, tX) + g_M(\mathcal{A}_{FV} U, \omega X) \\ &= g_M(\mathcal{T}_U FV, tX) + g_B(\pi_*(\mathcal{A}_{FV} U), \pi_* \omega X). \end{aligned}$$

Also by using (2.8), we conclude

$$g_M(\nabla_U V, X) \;=\; g_M(\mathcal{T}_U FV, tX) + g_B((\nabla\pi_*)(U, FV), \pi_*\omega X),$$

which implies 1) $\Leftrightarrow$ 2). Let us suppose that 2.) is satisfied. Since $\mathcal{T}_U$ is a skew-symmetric on M, by using (1.6) and (1.7), we have

$$\begin{aligned}
g_M(\mathcal{T}_U tX, FV) &+ g_B((\nabla\pi_*)(FV, U), \pi_*\omega X) = -g_M(\mathcal{T}_U tX, FV)\\
&+ g_B(\pi_*(\nabla_{FV} U), \pi_*\omega X)\\
&= -g_M(\mathcal{T}_U tX, FV) + g_M(\nabla_{FV} U, \omega X) = -g_M(\mathcal{T}_U tX, FV)\\
&+ g_M(\mathcal{A}_{FV} U, \omega X)\\
&= -g_M(\mathcal{T}_U tX, FV) - g_M(\mathcal{A}_{FV}\omega X, U)\\
&= -g_M(\mathcal{T}_U tX, FV) + g_M(\mathcal{A}_{\omega X} FV, U)\\
&= -g_M(\mathcal{T}_U tX, FV) - g_M(\mathcal{A}_{\omega X} U, FV)\\
&= g_M(\mathcal{T}_U tX + \mathcal{A}_{\omega X} U, FV),
\end{aligned}$$

which is equivalent to 3.) □

Corollary 2.9. *Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion. Then the following assertions are equivalent to each other;*
1.) The vertical distribution $\mathcal{V}$ is totally geodesic on M,
2.) $\mathcal{T}_V FX = 0$,
3.) $\mathcal{T}_V X = 0$, for any $V \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$.

The Riemannian submersion map π is called totally geodesic map if the map π_* is parallel with respect to ∇, i.e., $\nabla\pi_* = 0$.

Theorem 2.10. *Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion. Then π defines a totally geodesic map if and only if*

$$\mathcal{T}_U FV = 0, \quad \forall U, V \in \Gamma(\mathcal{V}) \tag{2.9}$$

and

$$\mathcal{A}_{FV} X = 0, \quad X \in \Gamma(\mathcal{H}). \tag{2.10}$$

Proof. It is well known that $(\nabla\pi_*)(X, Y) = 0$, for any $X, Y \in \Gamma(\mathcal{H})$, in the Riemannian submersions. So we have to show that

$$(\nabla\pi_*)(X, U) = (\nabla\pi_*)(V, U) = 0$$

which is equivalent to (2.9) and (2.10). Thus, for any $X \in \Gamma(\mathcal{H})$ and $U, V \in \Gamma(\mathcal{V})$, from (1.8), (1.9) and consider Lemma 1.3 we have

$$\begin{aligned}
(\nabla\pi_*)(U, V) &= -\pi_*(\nabla_U V) = -\pi_*(F\nabla_U FV) = -\pi_*(F\nabla^{\hbar}_U FV + F\mathcal{T}_U FV)\\
&= -\pi_*(F\mathcal{T}_U FV)
\end{aligned}$$

and

$$\begin{aligned}
(\nabla\pi_*)(U, X) &= -\pi_*(\nabla_X U) = -\pi_*(F\nabla_X FU) = -\pi_*(F\nabla^{\hbar}_X FU + F\mathcal{A}_X FU)\\
&= -\pi_*(F\mathcal{A}_X FU) = \pi_*(F\mathcal{A}_{FU} X).
\end{aligned}$$

Taking into account of F being a regularly map and π being a lagrangian anti-invariant Riemannian submersion, we get desired results. $\square$

Theorem 2.11. *Let $f : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion. Then $M = M_{\mathcal{H}} \times_f M_{\mathcal{V}}$ is a twisted product if and only if the tensors $\mathcal{T}_U$ and $\mathcal{A}_X$ satisfy the conditions*

$$\mathcal{T}_V FX = g_M(\mathcal{T}_V V, X)\|V\|^{-2} FV \tag{2.11}$$

and

$$\mathcal{A}_X FY = 0, \tag{2.12}$$

for any $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$.

Proof. We note that the twisted product $M = M_{\mathcal{H}} \times_f M_{\mathcal{V}}$ is characterized by the fact that $M_{\mathcal{H}}$ and $M_{\mathcal{V}}$ being totally geodesic and totally umbilical submanifolds of M, respectively, (for the more detail, we refer to [8]).

If $M = M_{\mathcal{H}} \times_f M_{\mathcal{V}}$ is a twisted product, we have

$$g_M(\nabla_U V, X) \;=\; g_M(\nabla_U FV, FX) = g_M(\mathcal{T}_U FV, FX) = -g_M(\mathcal{T}_U FX, FV),$$

that is,

$$X(\ln f) g_M(U, V) = g_M(\mathcal{T}_U FX, FV).$$

This implies that

$$\mathcal{T}_U FX = X(\ln f) FU. \tag{2.13}$$

By taking inner product both sides of (2.13) by FU and $\mathcal{T}_U$ is a skew-symmetric, we have

$$\begin{aligned} X(\ln f) g(FU, FU) &= g_M(\mathcal{T}_U FX, FU) = -g_M(\mathcal{T}_U FU, FX) = -g_M(\nabla_U FU, FX) \\ &= -g_M(\nabla_U U, X) = -g_M(\mathcal{T}_U U, X). \end{aligned}$$

Also, combining with (2.13), we reach

$$\mathcal{T}_U FX = -g_M(\mathcal{T}_U U, X)\|U\|^{-2} FU.$$

So (2.11) is satisfied.

On the other hand, since $M_{\mathcal{H}}$ is a totally geodesic in M, f is a function on M and $\mathcal{A}_X FY \in \Gamma(\mathcal{H})$, for any $X, Y \in \Gamma(\mathcal{H})$ we conclude

$$g_M(\mathcal{A}_X FY, X) = g_M(\nabla_X FY, X) = g_M(\nabla_X Y, FX) = 0,$$

which implies (2.12). $\square$

Theorem 2.12. *Let (M, g_M, F) be a locally Riemannian product manifold and (B, g_B) be a Riemannian manifold. Then there is no lagrangian anti-invariant Riemannian submersion from M onto B such that M is proper locally twisted product manifold which are form $M_{\mathcal{V}} \times_f M_{\mathcal{H}}$*

Proof. Let $\pi : (M, g_M, F) \longrightarrow (B, g_B)$ be a lagrangian anti-invariant Riemannian submersion and $\bar{M} = M_{\mathcal{V}} \times_f M_{\mathcal{H}}$ be a locally twisted product of $M_{\mathcal{V}}$ and $M_{\mathcal{H}}$. σ denote the second fundamental form of $M_{\mathcal{H}}$ in M. Then we have

$$g(\nabla_X Y, V) = g(\sigma(X, Y), V),$$

for any $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$. Taking into account of $M_{\mathcal{H}}$ being totally umbilical submanifold of M, we get

$$g(\nabla_X Y, V) = g(h, V)g(X, Y), \tag{2.14}$$

where h denote the mean curvature of $M_{\mathcal{H}}$ in M. Furthermore, since ∇ is a Levi-Civita connection, $\mathcal{H}$ and $\mathcal{V}$ are orthogonal distributions on M, we have

$$g(\nabla_X Y, V) \;=\; -g(\nabla_X V, Y) = -g(\nabla_X FV, FY) = -g(\mathcal{A}_X FV, FY). \tag{2.15}$$

So combining (2.14) and (2.15) , we reach

$$\mathcal{A}_X FV = -g(V, h)FX. \tag{2.16}$$

Taking inner product both sides of (2.16) by FX, we arrive

$$\begin{aligned}
g_M(\mathcal{A}_X FV, FX) &= -g_M(V, h)g_M(FX, FX)\\
g_M(\nabla_X FV, FX) &= -g_M(V, h)g_M(X, X)\\
g_M(\nabla_X V, X) &= -g_M(V, h)g_M(X, X)\\
g_M(\mathcal{A}_X V, X) &= -g_M(V, h)g_M(X, X)\\
g_M(\mathcal{A}_X X, V) &= g_M(V, h)g_M(X, X) = 0.
\end{aligned}$$

$\mathcal{A}_{\mathcal{H}}$ is skew-symmetric, we conclude that $h = 0$, that is, $M_{\mathcal{H}}$ is totally geodesic submanifold. This completes the proof. □

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