

A NOTE ON SOME RECENT BEST PROXIMITY POINT RESULTS FOR NON-SELF MAPPINGS

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ABSTRACT. In this note, we show how an easy lemma, proved recently by Abkar and Gabeleh, can be applied to obtain Geraghty-type results for best proximity points. In particular, we obtain improvements of some recent best proximity point results and show by an example that these improvements are proper.

1. INTRODUCTION

Let $\mathcal{B}$ denote the class of real functions $\beta : [0, +\infty) \rightarrow [0, 1)$ satisfying the condition

$$\beta(t_n) \rightarrow 1 \text{ implies } t_n \rightarrow 0. \quad (1.1)$$

An example of a function in $\mathcal{B}$ may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$. In an attempt to generalize the Banach contraction principle, M. Geraghty proved in 1973 the following

Theorem 1.1. [5] *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a self-map. Suppose that there exists $\beta \in \mathcal{B}$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to z when $n \rightarrow \infty$.

On the other hand, if we consider a non-self mapping $T : A \rightarrow B$, where A, B are disjoint subsets of a metric space (X, d) , it is natural to ask about the existence of a point $x^* \in A$ such that $d(x^*, Tx^*)$ is minimal, possibly $d(x^*, Tx^*) = d(A, B)$. Such point x^* is usually called a best proximity point of the mapping T and there is a vast literature on this subject, since the positive answer has many applications, especially in Optimization Theory.

Very recently, Abkar and Gabeleh have proved in [2] an easy lemma which can be used (and has already been used, e.g., in [2, 6]) for obtaining best proximity point results. In this paper, we show how this lemma can be applied to obtain Geraghty-type results for best proximity points. In particular, we obtain an

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improvement of the main result from the paper [7] by E. Karapınar, which we show not to be a generalization of the results of Caballero et al. [3].

2. PRELIMINARIES

Let A and B be nonempty subsets of a metric space (X, d) . We adopt the following notations and definitions.

$$\begin{aligned} d(A, B) &:= \inf\{d(x, y) : (x, y) \in A \times B\}, \\ A_0 &:= \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in A\}. \end{aligned}$$

The notion of P -property was introduced by Sankar Raj [10] as follows.

Definition 2.1. [10] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P -property if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

For example, it was shown in [1, 10] that the pair (A, B) satisfies the P -property in the following cases:

- (1) A, B are nonempty, closed and convex subsets of a Hilbert space X ;
- (2) A, B are nonempty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $d(A, B) = 0$;
- (3) A, B are nonempty, bounded, closed and convex subsets of a uniformly convex Banach space X .

Definition 2.2. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) and let $T : A \rightarrow B$. A point $x^* \in A$ is said to be a best proximity point of T if

$$d(x^*, Tx^*) = d(A, B).$$

The following result was proved by Caballero et al. in [3].

Theorem 2.3. [3, Theorem 3.1] *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and having the P -property. Let a map $T : A \rightarrow B$ be such that $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \tag{2.1}$$

for some $\beta \in \mathcal{B}$ and all $x, y \in A$. Then there exists a unique best proximity point $x^ \in A$ of the mapping T .*

Recently, Abkar and Gabeleh [2] proved the following result.

Lemma 2.4. [2, Lemma 2.2] *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and having the P -property. Then there is a bijective isometry $g : A_0 \rightarrow B_0$ such that $d(x, gx) = d(A, B)$ for $x \in A_0$.*

This simple result appears to be very useful in obtaining best proximity point results for non-self mappings (see, e.g., [2, 6]). In particular, using this lemma, the main result of V. Sankar Ray [10, Theorem 3.1] can be easily deduced from a well known fixed point result for weak contractions.

3. MAIN RESULTS

Consider the following properties of functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$:

- (a) ψ is nondecreasing;
- (b) ψ is subadditive, i.e., $\psi(s + t) \leq \psi(s) + \psi(t)$ for $s, t \in [0, +\infty)$;
- (c) ψ is continuous;
- (d) $\psi^{-1}(\{0\}) = \{0\}$.

In an attempt to generalize Theorem 2.3, E. Karapınar proved in [7] the following

Theorem 3.1. [7, Theorem 2.1] *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and having the P -property. Let a map $T : A \rightarrow B$ be such that $T(A_0) \subseteq B_0$ and*

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)), \quad (3.1)$$

for all $x, y \in A$, some $\beta \in \mathcal{B}$ and some $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the conditions (a), (b), (c) and (d). Then there exists a unique best proximity point $x^ \in A$ of the mapping T .*

However, it is easy to see that this is not a generalization of Theorem 2.3. Namely, under the given assumptions on the function ψ , the mapping $D : X \times X \rightarrow [0, +\infty)$ given by $D = \psi \circ d$, i.e., $D(x, y) = \psi(d(x, y))$, is again a metric on X , and the condition (3.1) can be rewritten as

$$D(Tx, Ty) \leq \beta(D(x, y))D(x, y),$$

which is simply the condition (2.1) of Theorem 2.3.

We will prove now a variant of this assertion, omitting the assumption (b) for the function ψ , so that it cannot be directly reduced to Theorem 2.3. For this, we need the following

Lemma 3.2. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a self-map. Suppose that there exist $\beta \in \mathcal{B}$ and a function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying conditions (a), (c) and (d) and such that*

$$\psi(d(Tx, Ty)) \leq \beta(d(x, y))\psi(d(x, y)), \quad (3.2)$$

for all $x, y \in X$. Then T has a unique fixed point $x^ \in X$ and, for each $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^* when $n \rightarrow \infty$.*

Proof. Let $x_0 \in X$ be arbitrary and let $x_n = T^n x_0$ be the corresponding Picard sequence. As usual, if $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then the proof is finished, hence assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. We will prove:

- (1) $d(x_{n+1}, x_n) \rightarrow 0$ when $n \rightarrow \infty$;
- (2) $\{x_n\}$ is a Cauchy sequence in (X, d) .

(1) By (3.2) we have that

$$\begin{aligned}\psi(d(x_{n+1}, x_n)) &= \psi(d(Tx_n, Tx_{n-1})) \leq \beta(d(x_n, x_{n-1}))\psi(d(x_n, x_{n-1})) \\ &< \psi(d(x_n, x_{n-1})).\end{aligned}$$

Using the property (a) of function ψ we get that the sequence $\{d(x_n, x_{n-1})\}$ is decreasing, implying that $d(x_n, x_{n-1}) \rightarrow d^* \geq 0$ when $n \rightarrow \infty$.

Suppose that $d^* > 0$. Using the properties (c) and (d) of function ψ , we get that $\psi(d(x_{n+1}, x_n)) \rightarrow \psi(d^*) > 0$ when $n \rightarrow \infty$. It follows from

$$0 < \frac{\psi(d(x_{n+1}, x_n))}{\psi(d(x_n, x_{n-1}))} \leq \beta(d(x_n, x_{n-1})) < 1$$

that $\beta(d(x_n, x_{n-1})) \rightarrow 1$ when $n \rightarrow \infty$, and since $\beta \in \mathcal{B}$, we get that $d(x_n, x_{n-1}) \rightarrow 0$ when $n \rightarrow \infty$, which is a contradiction. Hence, (1) is proved.

(2) Suppose that $\{x_n\}$ is not a Cauchy sequence. Applying the standard procedure (see, e.g., the proof of [9, Lemma 2.1]), we conclude that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $d(x_{n_k+1}, x_{m_k}) \rightarrow \varepsilon$ and $d(x_{n_k}, x_{m_k-1}) \rightarrow \varepsilon$ when $k \rightarrow \infty$. Applying (3.2), we get that

$$0 < \frac{\psi(d(x_{n_k+1}, x_{m_k}))}{\psi(d(x_{n_k}, x_{m_k-1}))} \leq \beta(d(x_{n_k}, x_{m_k-1})) < 1.$$

Passing to the limit when $n \rightarrow \infty$, since $\beta \in \mathcal{B}$, we get that $d(x_{n_k}, x_{m_k-1}) \rightarrow 0$, a contradiction. Thus, (2) is proved.

Now, since (X, d) is a complete metric space, there is an $x^* \in X$ such that $x_n \rightarrow x^*$ when $n \rightarrow \infty$. Since T is continuous (which follows from (3.2)), we get that also $x_{n+1} = Tx_n \rightarrow Tx^*$, hence $Tx^* = x^*$. The uniqueness of the fixed point follows easily from (3.2). $\square$

Remark 3.3. For $\psi(t) = t$, Lemma 3.2 reduces to Geraghty's Theorem 1.1.

Taking $\beta(t) = c$, $0 < c < 1$ for $t \geq 0$, we obtain the following

Corollary 3.4. [8] *Let (X, d) be a complete metric spaces and $T : X \rightarrow X$ a self-map such that, for some function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the properties (a), (c) and (d),*

$$\psi(d(Tx, Ty)) \leq c\psi(d(x, y)),$$

for all $x, y \in X$. Then T has a unique fixed point $x^ \in X$ and each Picard sequence $\{T^n x\}$ converges to x^* .*

Now, we are in position to prove the main result of this section.

Theorem 3.5. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and having the P-property. Let a map $T : A \rightarrow B$ be such that $T(A_0) \subseteq B_0$ and, for some $\beta \in \mathcal{B}$ and some $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the conditions (a), (c) and (d), the inequality (3.2) holds for all $x, y \in A$. Then there exists a unique best proximity point $x^* \in A$ of the mapping T . Further, the sequence $\{x_n\}$ in A_0 , satisfying*

$$d(x_{n+1}, Tx_n) = d(A, B), \quad n \in \mathbb{N}, \tag{3.3}$$

converges to x^ .*

Proof. Denote the restriction of T to A_0 again by T . Since $T(A_0) \subseteq B_0$, we have that $T : A_0 \rightarrow B_0$. By Lemma 2.4, there is a bijective isometry $g : A_0 \rightarrow B_0$, such that $d(x, gx) = d(A, B)$ for all $x \in A_0$. Consider the mapping

$$S := g^{-1} \circ T : A_0 \rightarrow A_0.$$

By [2, Lemma 2.1], (A_0, d) is a complete metric space. We will check that S satisfies the condition (3.2). Indeed,

$$\begin{aligned} \psi(d(Sx, Sy)) &= \psi(d(g^{-1}Tx, g^{-1}Ty)) = \psi(d(Tx, Ty)) \\ &\leq \beta(d(x, y))\psi(d(x, y)), \end{aligned}$$

for all $x, y \in X$, since g^{-1} is an isometry.

By Lemma 3.2, we conclude that there exists $x^* \in A_0$ such that $Sx^* = x^*$. We will prove that x^* is a best proximity point of the mapping T . We have that $0 = d(x^*, Sx^*) = d(x^*, g^{-1}Tx^*)$, hence $g^{-1}Tx^* = x^*$ and so $Tx^* = gx^*$, since g is a bijection. Since $d(x^*, gx^*) = d(A, B)$, we get that $d(x^*, Tx^*) = d(x^*, gx^*) = d(A, B)$, which proves that x^* is a best proximity point of T .

Let $\{x_n\}$ be a sequence in A_0 satisfying (3.3). Since $d(x_{n+1}, gx_{n+1}) = d(A, B)$, $n \in \mathbb{N}$, by the P -property of the pair (A, B) , we get that $gx_{n+1} = Tx_n$, $n \in \mathbb{N}$. Hence, $x_{n+1} = g^{-1}Tx_n = Sx_n$, $n \in \mathbb{N}$, and it follows that $x_n \rightarrow x^*$, $n \rightarrow \infty$, which finishes the proof. $\square$

We present an example (adapted from [4, Example 3.4]), showing possible usage of our result, and that this generalization of Theorem 2.3 is proper.

Example 3.6. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and let $A = \{(x, 0) : 0 \leq x \leq 1\}$, $B = \{(x, 1) : 0 \leq x \leq 1\}$. It is clear that $d(A, B) = 1$, $A_0 = A$, $B_0 = B$ and that (A, B) satisfies the P -property.

Consider the mapping $T : A \rightarrow B$ given by $T(x, 0) = (\frac{1}{5}x, 1)$, $0 \leq x \leq 1$ and the function $\beta \in \mathcal{B}$ given by $\beta(t) = e^{-t}/(t+1)$, for $t > 0$, $\beta(0) \in (0, 1)$. If we take $\mathbf{x} = (0, 0)$, $\mathbf{y} = (1, 0)$, then

$$d(T\mathbf{x}, T\mathbf{y}) = d((0, 1), (\frac{1}{5}, 1)) = \frac{1}{5} > \frac{1}{2e} = \frac{e^{-1}}{1+1} \cdot 1 = \beta(d(\mathbf{x}, \mathbf{y}))d(\mathbf{x}, \mathbf{y}),$$

which means that the condition (2.3) of Theorem 2.3 is not satisfied.

Take now the function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ given by $\psi(t) = t^2$, which satisfies conditions (a), (c) and (d), but which is not subadditive. Then, for arbitrary $\mathbf{x} = (x, 0)$, $\mathbf{y} = (y, 0) \in A$,

$$\begin{aligned} \psi(d(T\mathbf{x}, T\mathbf{y})) &= \frac{1}{25}|x - y|^2 < \frac{e^{-|x-y|}}{|x-y|+1} \cdot |x-y|^2 \\ &= \beta(d(\mathbf{x}, \mathbf{y}))\psi(d(\mathbf{x}, \mathbf{y})), \end{aligned}$$

since

$$\frac{1}{25} < \frac{e^{-t}}{t+1}$$

for $0 \leq t \leq 1$ (the last inequality follows because the function β is decreasing and $\beta(1) = 1/2e > 1/25$).

All the conditions of Theorem 3.5 are satisfied, and T has a (unique) best proximity point $\mathbf{x} = (0, 0)$.

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