

A REMARK ON AN ELLIPTIC EQUATION IN AN UNBOUNDED DOMAIN

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ABSTRACT. We give in this work some remarks on the resolution of an elliptic equation with homogeneous Dirichlet boundary conditions posed on the unbounded domain $\Sigma = (0, +\infty) \times (0, 1)$. Our aim is to use the famous Dore-Venni theorem, in UMD Banach spaces, to prove the existence, uniqueness and maximal regularity of the strict solution.

1. INTRODUCTION

In this paper, we study the resolution of the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) + \frac{\partial^2 u}{\partial x^2}(t, x) = \delta u(t, x) - f(t, x), & (t, x) \in \Sigma, \\ u(0, x) = 0, & x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0, \end{cases} \quad (1.1)$$

where $\Sigma = (0, +\infty) \times (0, 1)$ and $\delta = \delta_L + \delta_A$ for some fixed reals $\delta_L > 0$ and $\delta_A > 0$.

This problem can be written in the following operational form, considered in the Banach space E

$$u''(t) + Au(t) - \delta_L u(t) = g(t), \quad t \in (0, +\infty), \quad u(0) = 0. \quad (1.2)$$

Here $g(t) := -f(t)$, $f \in L^p(0, +\infty; E)$ with $E = L^q(0, 1)$, $p, q \in]1, +\infty[$ and A is the closed linear operator with dense domain $D(A)$ in E , defined by

$$\begin{cases} (Av)(x) = v''(x) - \delta_A v(x) = (Pv)(x) - \delta_A v(x), & v \in D(A), \\ D(A) = D(P) = W^{2,q}(0, 1) \cap W_0^{1,q}(0, 1). \end{cases}$$

Note that E is an UMD complex Banach space (Unconditional Martingale Difference property). It means that for some $p \in (1, \infty)$ (then for all $p \in (1, \infty)$), the Hilbert transform on $L^p(\mathbb{R}; E)$ is bounded. These spaces are also said to be of class $\mathcal{HT}$. In concrete applications, any space constructed on L^p is UMD if $p \in (1, \infty)$. See [1].

We say that u is a strict solution of (1.2) if

$$u \in L^p(0, +\infty; D(A)) \cap W^{2,p}(0, +\infty; E),$$

and satisfies (1.2).

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Equation (1.2) has been extensively studied by many authors, in various mathematical settings (homogeneous or nonhomogeneous boundary conditions, variable operators) but, in most cases, on a bounded domain. See for instance [2, 6]. However, we point out the fact that other important results have been proved when the domain is unbounded, one can see for example [5, 9].

In [5], the authors have studied the abstract equation

$$u''(x) + 2Bu'(x) + Au(x) = f(x), \quad x \in (0, R), \quad (0 < R \leq +\infty),$$

where A, B are two closed linear operators with domains $D(A), D(B)$ and satisfying the assumption

$$\sup_{\lambda \geq 0} \left\| \lambda (\lambda + B^2 - A)^{-1} \right\| < +\infty.$$

We can find in Prüss [9] a study of the problem

$$-u''(y) + F^2 u(y) = g(y), \quad y > 0, \quad u(0) = \varphi,$$

this author essentially use a variant of the remarkable Dore-Venni result [4], obtained by Prüss and Sohr [10]. In this extended version, the operators are not necessarily invertible.

In [4], G. Dore and A. Venni have considered the sum $A + B$ of closed linear operators A and B such that $0 \in \rho(A) \cap \rho(B)$ and

$$(DV) \begin{cases} i) \rho(A) \supset]-\infty, 0] \text{ and } \exists K_A > 0 : \\ \quad \forall \lambda \geq 0, \|(A + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_A/(1 + \lambda), \\ ii) \rho(B) \supset]-\infty, 0] \text{ and } \exists K_B > 0 : \\ \quad \forall \lambda \geq 0, \|(B + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_B/(1 + \lambda). \end{cases}$$

$\rho(A)$ and $\rho(B)$ are the resolvent sets of A and B . Their celebrated theorem, used by several authors, is restricted to invertible operators. Note that, one can remove this too strong assumption. See, for example [10], where (DV) is replaced by the weaker

$$\begin{cases} (-\infty, 0) \subset \rho(A) \cap \rho(B) \\ \max \{ \|(A + \lambda I)^{-1}\|, \|(B + \lambda I)^{-1}\| \} \leq K/\lambda \text{ for all } \lambda > 0, \end{cases}$$

for some $K < \infty$. Here A and B are not necessarily invertible. See also [2], for the Da Prato-Grisvard theorem in which the spaces need not to be UMD. In particular, their theory can be used in the framework of continuous or Hölder continuous functions.

In the present study ($\delta = \delta_L + \delta_A \neq 0$), our approach is fundamentally based on the direct use of the sum theory developped by Dore and Venni [4], in UMD Banach spaces. So, the operators we consider in the sum must be invertible. In our case, this condition is fulfilled, since one has $\delta_A > 0$ and $\delta_L > 0$.

Throughout this work, the resolvent of A must verify the condition

$$\exists K > 0 : \forall \lambda \geq 0, \|(A - \lambda I)^{-1}\|_{\mathcal{L}(E)} \leq K/(1 + \lambda),$$

which is, in many cases, too strong.

Our main result regarding this study is the following

Theorem 1.1. *For all functions $f \in L^p(0, +\infty; L^q(0, 1))$ where $p, q \in]1, \infty[$, Problem (1.1) admits a unique strict solution u satisfying*

$$\begin{aligned} u \in & L^p(0, \infty; W^{2,q}(0, 1) \cap W_0^{1,q}(0, 1)) \\ & \cap W^{2,p}(0, \infty; L^q(0, 1)) \cap W_0^{1,p}(0, \infty; L^q(0, 1)). \end{aligned}$$

2. PRELIMINARIES

For a closed linear operator A in a complex Banach space X , $D(A)$, $N(A)$, $R(A)$, $\rho(A)$, $\sigma(A)$ denote the domain, the kernel, the range, the resolvent set and the spectrum of A respectively. A is called sectorial if $D(A)$ and $R(A)$ are dense in X , $N(A) = \{0\}$, $(-\infty, 0) \subset \rho(A)$ and

$$\|t(tI + A)^{-1}\| \leq C \text{ for all } t > 0,$$

for some $C < \infty$. The class of sectorial operators in X will be denoted by $S(X)$.

Let $\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$; the spectral angle ϕ_A of $A \in S(X)$ is defined by

$$\phi_A = \inf \left\{ \phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty \right\}.$$

It is known that $\phi_A \in [0, \pi)$. See [11, 3].

A sectorial operator A admits bounded imaginary powers if $A^{is} \in \mathcal{L}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $\|A^{is}\| \leq C$ for $|s| \leq 1$. The class of such A is denoted by $\mathcal{BIP}(X)$.

In this case $\{A^{is} : s \in \mathbb{R}\}$ forms a strongly continuous group of bounded linear operators in X . His type

$$\theta_A = \overline{\lim}_{|s| \rightarrow \infty} \frac{1}{|s|} \log \|A^{is}\|$$

will be called the power angle of A . We have $\phi_A \leq \theta_A$ and for any $\varphi_A > \theta_A$ there is M_A such that

$$\forall s \in \mathbb{R}, \quad \|A^{is}\| \leq M_A e^{\varphi_A |s|},$$

see for example [11, 3, 8]. The most interesting case, in applications, is when $\theta_A \in [0, \pi)$.

Now and following [10], a sectorial operator $A \in S(X)$ belongs to the class $\mathcal{BIP}(X, \theta)$ for some $\theta \in [0, \pi)$ if A satisfies the condition: for all $s \in \mathbb{R}$, $A^{is} \in \mathcal{L}(X)$ and there is some K satisfying

$$\|A^{is}\| \leq K e^{\theta |s|}, \quad s \in \mathbb{R},$$

so that $A \in \bigcup_{\theta \in [0, \pi)} \mathcal{BIP}(X, \theta)$ if and only if $A \in \mathcal{BIP}(X)$ and $\theta_A \in [0, \pi)$.

Equivalently, $A \in \mathcal{BIP}(X)$ if $A^{is} \in \mathcal{L}(X)$, $s \in \mathbb{R}$, and there are constants θ and K such that $\|A^{is}\| \leq K e^{\theta |s|}$, $s \in \mathbb{R}$. Moreover,

$$\theta_A = \inf \{ \theta \geq 0 : \exists C, \quad \|A^{is}\| \leq C e^{\theta |s|}, \quad s \in \mathbb{R} \},$$

see [8].

3. ON THE SUM OF OPERATORS

In this section X stands for an UMD Banach space and L, M are two closed linear operators with dense domains $D(L), D(M)$ in X . Denote by S their sum

$$\begin{cases} Su = Lu + Mu \\ u \in D(L) \cap D(M), \end{cases}$$

and assume that

$$(DV1) \begin{cases} i) \rho(L) \supset]-\infty, 0] \text{ and } \exists K_L > 0 : \\ \quad \forall \lambda \geq 0, \|(L + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_L/(1 + \lambda), \\ ii) \rho(M) \supset]-\infty, 0] \text{ and } \exists K_M > 0 : \\ \quad \forall \lambda \geq 0, \|(M + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_M/(1 + \lambda). \end{cases}$$

$$(DV2) \begin{cases} \forall \lambda \in \rho(-L), \forall \mu \in \rho(-M) \\ (L + \lambda I)^{-1}(M + \mu I)^{-1} - (M + \mu I)^{-1}(L + \lambda I)^{-1} = 0. \end{cases}$$

$$(DV3) \begin{cases} L, M \in \mathcal{BIP}(X) \text{ with} \\ \theta_L + \theta_M < \pi, \end{cases}$$

where $\rho(L)$ and $\rho(M)$ are the resolvent sets of L and M .

Then the main result given in [4] is

Theorem 3.1. *Under the assumptions (DV1), (DV2) and (DV3) the operator S is closed and $S^{-1} \in \mathcal{L}(X)$.*

This result provides a powerful tool in PDE. It has particularly solved, for example, the question of L^p -regularity of the Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & 0 \leq t \leq T, \\ u(0) = 0, \end{cases}$$

for $1 < p < \infty$ and $f \in L^p(0, T; E)$. See [4].

4. APPLICATION OF THE SUMS

Let $X = L^p(0, \infty; E)$ and let us define in this Banach space the two following operators

$$\begin{cases} (Lu)(t) = -u''(t) + \delta_L u(t) = (Bu)(t) + \delta_L u(t) \text{ for almost all } t \in (0, \infty), \\ D(L) = D(B) = W^{2,p}(0, \infty; E) \cap W_0^{1,p}(0, \infty; E), \end{cases}$$

and

$$\begin{cases} (Mu)(t) = -A(u(t)), \quad u \in D(M) \text{ for almost all } t \in (0, \infty), \\ D(M) = L^p(0, \infty; D(A)). \end{cases}$$

M is said to be the L^p -realization of $(-A)$; it has the same properties in X as $(-A)$ in E .

Then problem (1.2) can be written, in X , in the form of the following sum

$$Lu + Mu = f.$$

We are now in position to apply the Dore-Venni Theorem 3.1. In particular, thanks to the optimal results regarding the operators A, M and L which will be stated in the Propositions 4.1, 4.2 and 4.3, we will show that the resolvents commute and that the strong parabolicity condition $\theta_L + \theta_M < \pi$ in (DV3) holds.

Proposition 4.1. *The linear closed operator A verifies*

$$\begin{cases} \mathbb{R}_+ \subset \rho(A) \text{ and } \exists K > 0 : \\ \forall \lambda \geq 0, \|(A - \lambda I)^{-1}\|_{\mathcal{L}(E)} \leq K/(1 + \lambda), \end{cases} \quad (4.1)$$

and

$$\begin{cases} \forall s \in \mathbb{R}, (-A)^{is} \in \mathcal{L}(E) \text{ and} \\ \forall \varphi \in]0, \frac{\pi}{2}[, \exists M'' > 0, \forall s \in \mathbb{R}, \|(-A)^{is}\| \leq M'' e^{\varphi|s|}. \end{cases} \quad (4.2)$$

Proof. By Lemma 8.1 on p. 372-373 in [2], the operator $P : D(P) \rightarrow E$ is densely defined in E such that

$$\begin{cases}]0, +\infty[\subset \rho(P) \text{ and } \exists K > 0 : \\ \forall \lambda > 0, \|(P - \lambda I)^{-1}\|_{\mathcal{L}(E)} \leq K/\lambda. \end{cases}$$

We also have

$$\begin{cases} \text{for all } s \in \mathbb{R}, (-P)^{is} \in \mathcal{L}(E) \text{ and for each } 0 < \varphi < \frac{\pi}{2}, \\ \text{there is } M' > 0 \text{ such that for all } s \in \mathbb{R}, \|(-P)^{is}\| \leq M' e^{\varphi|s|}. \end{cases}$$

For more details, see [12], Theorem 3.5 on p. 98 and [11], Theorem C on p. 166-167. Then we deduce

$$\forall \varphi \in]0, \frac{\pi}{2}[, (-P) \in \mathcal{BIP}(E, \varphi).$$

On the other hand, one has for all $\lambda \geq 0$,

$$\|(A - \lambda I)^{-1}\|_{\mathcal{L}(E)} = \|(P - (\delta_A + \lambda) I)^{-1}\|_{\mathcal{L}(E)} \leq K/(\delta_A + \lambda),$$

and then we get

$$\|(A - \lambda I)^{-1}\|_{\mathcal{L}(E)} \leq \begin{cases} \frac{K}{1 + \lambda} \text{ if } \delta_A \geq 1, \\ \frac{K(1/\delta_A)}{1 + \lambda} \text{ if } 0 < \delta_A < 1. \end{cases}$$

Let $\varphi \in]0, \frac{\pi}{2}[$, since $(-A) = (-P) + \delta_A I$ and $(-P) \in \mathcal{BIP}(E, \varphi)$ one can apply Theorem 3 on p. 437 in [10]. Therefore, it follows that

$$\forall \varphi \in]0, \frac{\pi}{2}[, (-A) \in \mathcal{BIP}(E, \varphi).$$

□

Proposition 4.2. *M is linear closed with dense domain and satisfies*

$$\begin{cases} \rho(M) \supset \mathbb{R}_- \text{ and } \exists K_M > 0 : \\ \forall \lambda \geq 0, \|(M + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_M/(1 + \lambda), \end{cases} \quad (4.3)$$

moreover

$$\begin{cases} \forall s \in \mathbb{R}, M^{is} \in \mathcal{L}(X) \text{ and} \\ \forall \varphi_M \in]0, \frac{\pi}{2}[, \exists K > 0, \forall s \in \mathbb{R}, \|M^{is}\| \leq K e^{\varphi_M|s|}. \end{cases} \quad (4.4)$$

Proof. Let $u \in X$, $h \in X$ and $\lambda \geq 0$. It is easy to see that

$$v(t) := ((M + \lambda I)^{-1} h)(t) = (-A + \lambda I)^{-1} h(t) \text{ for almost all } t \in (0, \infty),$$

and that for all $s \in \mathbb{R}$,

$$(M^{is}u)(t) = (-A)^{is}u(t) \text{ for almost all } t \in (0, \infty).$$

Let us consider the functions

$$u_n(t) := n((M + nI)^{-1}u)(t) = -n(A - nI)^{-1}u(t).$$

They belong to $D(M)$ and are bounded a.e. by the function $K\|u(t)\|_E$. In addition, they converge to $u(t)$ a.e. in E . Thus by Lebesgue's theorem $u_n \rightarrow u$, in X . Hence M is densely defined.

We also have for almost all $t \in (0, \infty)$

$$v(t) \in D(A) \text{ and } \|v(t)\| \leq \frac{K}{1 + \lambda} \|h(t)\|,$$

thus

$$\|v\| = \|(M + \lambda I)^{-1} h\| \leq \frac{K_M}{1 + \lambda} \|h\|.$$

Furthermore, one has

$$\|M^{is}u\| = \left(\int_0^\infty \|(-A)^{is}u(t)\|^p dt \right)^{1/p} \leq \|(-A)^{is}\| \|u\|.$$

Therefore, M^{is} is bounded and $\|M^{is}\| \leq \|(-A)^{is}\|$. Consequently, estimates (4.3) and (4.4) are satisfied for the operator M . $\square$

Proposition 4.3. *L is linear closed with dense domain and verifies*

$$\begin{cases} \rho(L) \supset \mathbb{R}_- \text{ and } \exists K_L > 0 : \\ \forall \lambda \geq 0, \|(L + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq K_L/(1 + \lambda), \end{cases} \quad (4.5)$$

and

$$\begin{cases} \forall s \in \mathbb{R}, L^{is} \in \mathcal{L}(X) \text{ and} \\ \forall \varphi_B \in]\theta_B, \pi[, \exists K_B > 0, \forall s \in \mathbb{R}, \|L^{is}\| \leq K_B e^{\varphi_B |s|}. \end{cases} \quad (4.6)$$

Proof. We have

$$\begin{cases} (Lu)(t) = -u''(t) + \delta_L u(t) = (Bu)(t) + \delta_L u(t) \text{ for almost all } t \in (0, \infty), \\ D(L) = D(B) = W^{2,p}(0, \infty; E) \cap W_0^{1,p}(0, \infty; E). \end{cases}$$

Now, to study $\rho(L)$, we solve the spectral equation

$$Lv + \lambda v = h.$$

Let $\lambda \geq 0$, then the problem

$$\begin{cases} -v''(t) + (\lambda + \delta_L)v(t) = h(t), \\ v(0) = 0, \end{cases}$$

admits the solution

$$\begin{aligned} v(t) &= ((L + \lambda I)^{-1}h)(t) \\ &= \int_0^{+\infty} k(t, s)h(s)ds, \end{aligned}$$

where

$$k(t, s) = \begin{cases} \frac{e^{-\sqrt{\lambda+\delta_L}t} \sinh \sqrt{\lambda+\delta_L}s}{\sqrt{\lambda+\delta_L}} & \text{if } 0 \leq s \leq t, \\ \frac{e^{-\sqrt{\lambda+\delta_L}s} \sinh \sqrt{\lambda+\delta_L}t}{\sqrt{\lambda+\delta_L}} & \text{if } s \geq t. \end{cases}$$

Moreover, we have

$$\begin{aligned} \int_0^{+\infty} |k(t, s)| ds &\leq \frac{e^{-\sqrt{\lambda+\delta_L}t}}{\sqrt{\lambda+\delta_L}} \int_0^t \cosh \sqrt{\lambda+\delta_L}s ds \\ &\quad + \frac{\cosh \sqrt{\lambda+\delta_L}t}{\sqrt{\lambda+\delta_L}} \int_t^{+\infty} e^{-\sqrt{\lambda+\delta_L}s} ds \\ &\leq \frac{e^{-\sqrt{\lambda+\delta_L}t}}{\lambda+\delta_L} \sinh \sqrt{\lambda+\delta_L}t + \frac{\cosh \sqrt{\lambda+\delta_L}t}{\lambda+\delta_L} e^{-\sqrt{\lambda+\delta_L}t} \\ &\leq \frac{1}{\lambda+\delta_L}, \end{aligned}$$

and

$$\begin{aligned} \int_0^{+\infty} |k(t, s)| dt &\leq \frac{e^{-\sqrt{\lambda+\delta_L}s}}{\sqrt{\lambda+\delta_L}} \int_0^s \cosh \sqrt{\lambda+\delta_L}t dt \\ &\quad + \frac{\cosh \sqrt{\lambda+\delta_L}s}{\sqrt{\lambda+\delta_L}} \int_s^{+\infty} e^{-\sqrt{\lambda+\delta_L}t} dt \\ &\leq \frac{e^{-\sqrt{\lambda+\delta_L}s}}{\lambda+\delta_L} \sinh \sqrt{\lambda+\delta_L}s + \frac{\cosh \sqrt{\lambda+\delta_L}s}{\lambda+\delta_L} e^{-\sqrt{\lambda+\delta_L}s} \\ &\leq \frac{1}{\lambda+\delta_L}, \end{aligned}$$

then

$$\begin{cases} \sup_t \int_0^{+\infty} |k(t, s)| ds \leq \frac{1}{\lambda+\delta_L}, \\ \sup_s \int_0^{+\infty} |k(t, s)| dt \leq \frac{1}{\lambda+\delta_L}. \end{cases}$$

This allows us to obtain

$$\|(L + \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \begin{cases} \frac{1}{1+\lambda} & \text{if } \delta_L \geq 1, \\ \frac{(1/\delta_L)}{1+\lambda} & \text{if } 0 < \delta_L < 1. \end{cases}$$

Here, we use Theorem 5.6 on p. 342, in [7], also known as Schur's lemma or Schur's test about boundedness of integral operators on L^p spaces. Notice that these estimates are related to the case $X = L^p(0, \infty; E)$. We also mention that, one can find on p. 372-373 in Da Prato-Grisvard [2], similar technique obtained with a different kernel, in $X = L^p(0, 1; E)$ where E is a Banach space.

For property (4.6), note that the operator B has been used with a variant of the Dore-Venni theorem, by J. Prüss [9], to study an abstract second order problem with Dirichlet condition.

So, for any $\varphi_B > \theta_B$ there is M_B such that

$$\forall s \in \mathbb{R}, \quad \|B^{is}\| \leq M_B e^{\varphi_B |s|},$$

where θ_B is the power angle of B . Thus, we obtain

$$\forall \varphi_B \in]\theta_B, \pi[, \quad B \in \mathcal{BIP}(X, \varphi_B).$$

Now, since we have $L = B + \delta_L I$ and $\delta_L > 0$, we can apply Theorem 3 on p. 437 in [10]. We deduce that

$$\forall \varphi_B \in]\theta_B, \pi[, \quad L \in \mathcal{BIP}(X, \varphi_B).$$

□

Let φ_B be such (but fixed) element and take $\varphi_M = \frac{\pi - \varphi_B}{2}$. Then $\varphi_M \in]0, \frac{\pi}{2}[$ and $\varphi_M + \varphi_B \in]\varphi_B, \pi[$. It follows that

$$\theta_L + \theta_M \leq \varphi_B + \varphi_M < \pi.$$

On the other hand, one has (for $h \in X$) for almost all $t \in (0, \infty)$

$$\begin{aligned} ((L + \lambda I)^{-1}(M + \mu I)^{-1}h)(t) &= \int_0^{+\infty} k(t, s) ((M + \mu I)^{-1}h)(s) ds \\ &= \int_0^{+\infty} k(t, s) (-A + \mu I)^{-1}h(s) ds \\ &= (-A + \mu I)^{-1} \int_0^{+\infty} k(t, s) h(s) ds \\ &= (-A + \mu I)^{-1} ((L + \lambda I)^{-1}h)(t) \\ &= ((M + \mu I)^{-1}(L + \lambda I)^{-1}h)(t). \end{aligned}$$

Hence, M and L verify the assumption (DV2). Therefore, Dore-Venni Theorem 3.1 leads to the following

Theorem 4.4. *For all functions $f \in L^p(0, \infty; E)$ where $p \in]1, \infty[$, Problem (1.2) admits a unique strict solution u satisfying*

$$u \in L^p(0, \infty; D(A)) \cap W^{2,p}(0, \infty; E) \cap W_0^{1,p}(0, \infty; E).$$

Finally, Theorem 4.4 leads to Theorem 1.1.

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