ON A HALF-DISCRETE REVERSE HILBERT-TYPE INEQUALITY

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ABSTRACT. By using the weight functions and the techniques of real analysis, a half-discrete reverse Hilbert-type inequality with the best constant factor is given. A best extension with a parameter $\lambda$ and the equivalent forms are also considered.

1. INTRODUCTION

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L^p(0, \infty), g(\geq 0) \in L^q(0, \infty), ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0, ||g||_q > 0$, we have the following Hardy-Hilbert’s integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}||f||_p||g||_q,$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we still have the following discrete Hardy-Hilbert’s inequality with the same constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q.$$

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [10], [11], [12]). Also we have the following Mulholland’s inequality with the same constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], [13]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=2}^\infty m^{p-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^\infty n^{q-1} b_n^q \right\}^{\frac{1}{q}}.$$

In 1998, by introducing an independent parameter $\lambda \in (0, 1], Yang$ [14] gave an extension of (1.1) (for $p = q = 2$). By refining the results of [14], Yang [15] gave some extensions that are the best possible of (1.1) and (1.2) as follows: If $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda, k_\lambda(t, 1) t^{\lambda_1-1}dt \in R_+, \phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}$,
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\[ f(\geq 0) \in L_{p,\phi}(0, \infty) = \{ f|||f|||_{p,\phi} := \left\{ \int_0^\infty \phi(x)|f(x)|^pdx \right\}^{\frac{1}{p}} < \infty \}, g(\geq 0) \in L_{q,\psi}(0, \infty), ||f||_{p,\phi}, ||g||_{q,\psi} > 0, \] then we have

\[
\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi},
\]

(1.4)

where the constant factor \(k(\lambda_1)\) is the best possible. Moreover if \(k_\lambda(x, y)\) is also finite and \(k_\lambda(x, y)x^{\lambda_1-1}(x, y)^{\lambda_2-1}\) is decreasing for \(x > 0 (y > 0)\), then for \(a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a|||_{p,\phi} := \left\{ \sum_{n=1}^\infty \phi(n)|a_n|^p \right\}^{\frac{1}{p}} < \infty \}, b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0, \) we have

\[
\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m, n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi},
\]

(1.5)

where the constant factor \(k(\lambda_1)\) is the best possible. For \(\lambda = 1, k_1(x, y) = \frac{1}{x+y}, \)

\(\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}, (1.4) \) reduces to (1.1), and (1.5) reduces to (1.2). Some other results including the reverse Hilbert-type inequalities are provided by [16]-[9].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the the constant factors in the inequalities are the best possible. And Yang [18] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [19] gave a half-discrete Hilbert’s inequality and [20] gave the following half-discrete reverse Hilbert-type inequality with the best constant factor 4:

\[
\int_0^\infty f(x) \sum_{n=1}^\infty \min\{x, n\}a_n dx
\]

\[
> 4 \left\{ \int_0^\infty (1 - \theta_1(x))x^{\frac{3p}{2}-1}f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{3q}{2}-1}a_n^q \right\}^{\frac{1}{q}}.
\]

(1.6)

In this paper, by using the weight functions and the techniques of real analysis, a half-discrete reverse Hilbert-type inequality similar to (1.6) with the best constant factor is given as follows \((\theta_1(x) \in (0, 1)):\)

\[
\int_1^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{\ln e^x a_n} dx
\]

\[
> \pi \left\{ \int_1^\infty (1 - \theta_1(x))x^{p-1}(\ln x)^{\frac{p}{2}-1}f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{q}{2}-1}a_n^q \right\}^{\frac{1}{q}}.
\]

(1.7)

A best extension of (1.7) with a parameter \(\lambda\) and the equivalent forms are considered.
2. SOME LEMMAS

Lemma 2.1. If $0 < \lambda \leq 2$, setting weight functions $\omega(n)$ and $\varpi(x)$ as follows:

\[
\omega(n) : = n^{\frac{1}{2}} \int_{1}^{\infty} \frac{1}{x(\ln ex^n)^{\lambda}} (\ln x)^{\frac{3}{2} - 1} dx, n \in \mathbb{N},
\]

\[
\varpi(x) : = (\ln x)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^{\lambda}} n^{\frac{3}{2} - 1}, x \in (1, \infty),
\]

then we have

\[
B^{\frac{\lambda}{2}, \frac{\lambda}{2}}(1 - \theta_\lambda(x)) < \varpi(x) < \omega(n) = B^{\frac{\lambda}{2}, \frac{\lambda}{2}},
\]

where, $\theta_\lambda(x) := \frac{1}{B^{\frac{\lambda}{2}, \frac{\lambda}{2}}} \int_{0}^{\ln x} \frac{t^{(\lambda/2) - 1}}{(1 + t)^{\lambda}} dt$ and

\[
\theta_\lambda(x) = O((\ln x)^{\frac{\lambda}{2}}) \in (0, 1)(x \in (1, \infty)).
\]

Proof. Setting $t = n \ln x$ in (2.1), by calculation, we have

\[
\omega(n) = \int_{0}^{\infty} \frac{1}{(1 + t)^{\lambda}} t^{\frac{1}{2} - 1} dt = B^{\frac{\lambda}{2}, \frac{\lambda}{2}}.
\]

Since for fixed $x > 1$,

\[
h(x, y) := \frac{1}{(\ln ex^y)^{\lambda}} y^{\frac{1}{2} - 1} = \frac{1}{(1 + y \ln x)^{\lambda}} y^{1 - \lambda/2}
\]

is strictly decreasing for $y \in (0, \infty)$, then we find

\[
\varpi(x) < (\ln x)^{\frac{1}{2}} \int_{0}^{\infty} \frac{1}{(1 + y \ln x)^{\lambda}} y^{\frac{1}{2} - 1} dy
\]

\[
= \int_{0}^{\infty} \frac{1}{(1 + t)^{\lambda}} t^{\frac{1}{2} - 1} dt = B^{\frac{\lambda}{2}, \frac{\lambda}{2}},
\]

\[
\varpi(x) > (\ln x)^{\frac{1}{2}} \int_{1}^{\infty} \frac{1}{(1 + y \ln x)^{\lambda}} y^{\frac{1}{2} - 1} dy
\]

\[
= \int_{\ln x}^{\infty} \frac{t^{(\lambda/2) - 1}}{(1 + t)^{\lambda}} dt = B^{\frac{\lambda}{2}, \frac{\lambda}{2}}(1 - \theta_\lambda(x)) > 0,
\]

\[
0 < \theta_\lambda(x) = \frac{1}{B^{\frac{\lambda}{2}, \frac{\lambda}{2}}} \int_{0}^{\ln x} \frac{t^{(\lambda/2) - 1}}{(1 + t)^{\lambda}} dt
\]

\[
< \frac{1}{B^{\frac{\lambda}{2}, \frac{\lambda}{2}}} \int_{0}^{\ln x} t^{\frac{1}{2} - 1} dt = \frac{2}{\lambda B^{\frac{\lambda}{2}, \frac{\lambda}{2}}} (\ln x)^{\frac{\lambda}{2}},
\]

and then (2.3) is valid.

Lemma 2.2. Let the assumptions of Lemma 2.1 be fulfilled and additionally, $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \in \mathbb{N}, f(x)$ is a non-negative measurable function
in $(1, \infty)$. Then we have the following inequalities (Note: in this paper, if $a_n = 0$, then we take $a_n^q = 0(q < 0)$):

\[
J = \left\{ \sum_{n=1}^{\infty} n^{\frac{\lambda}{2} - 1} \left[ \int_1^{\infty} \frac{f(x)}{(\ln e x)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} 
\geq [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^{\frac{1}{2}} \left\{ \int_1^{\infty} \frac{\varpi(x) x^{p-1} (\ln x)^{p(1 - \frac{\lambda}{2})-1} f_p(x) dx}{} \right\}^{\frac{1}{p}}, \quad (2.4)
\]

\[
L_1 = \left\{ \int_1^{\infty} \frac{(\ln x)^{(q\lambda/2) - 1}}{x(x^{q-1})} \left[ \sum_{n=1}^{\infty} \frac{a_n}{(\ln e x)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} 
\geq \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \sum_{n=1}^{\infty} n^{q(1 - \frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}}. \quad (2.5)
\]

**Proof.** (i) By the reverse Hölder’s inequality with weight (cf. [7]) and (2.3), it follows that

\[
\left[ \int_1^{\infty} \frac{f(x) dx}{(\ln e x)^{\lambda}} \right]^p = \left\{ \int_1^{\infty} \frac{1}{(\ln e x)^{\lambda}} \left[ \frac{(\ln x)^{(1 - \lambda/2)q}}{n^{(1 - \lambda/2)q/p}} \frac{1}{x^{1/q}} f(x) \right] \right\}^p 
\geq \int_1^{\infty} \frac{1}{(\ln e x)^{\lambda}} \frac{x^{p-1} (\ln x)^{(1 - \lambda/2)(p-1)}}{n^{1 - \lambda/2}} f_p(x) dx 
\times \left\{ \int_1^{\infty} \frac{1}{(\ln e x)^{\lambda}} \frac{n^{(1 - \lambda/2)(q-1)}}{x(\ln x)^{1 - \lambda/2}} dx \right\}^{p-1} 
= \left\{ \omega(n) n^{(1 - \frac{\lambda}{2})-1} \right\}^{p-1} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1 - \lambda/2)(p-1)}}{(\ln e x)^{\lambda} n^{1 - \lambda/2}} f_p(x) dx 
= \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right\}^{p-1} n^{1 - \frac{\lambda}{2}} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1 - \lambda/2)(p-1)}}{(\ln e x)^{\lambda} n^{1 - \lambda/2}} f_p(x) dx.
\]

Then by Lebesgue term by term integration theorem (cf. [8]), we have

\[
J \geq \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1 - \lambda/2)(p-1)}}{(\ln e x)^{\lambda} n^{1 - \lambda/2}} f_p(x) dx \right\}^{\frac{1}{p}} 
= \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right\}^{\frac{1}{2}} \left\{ \int_1^{\infty} \sum_{n=1}^{\infty} \frac{x^{p-1} (\ln x)^{(1 - \lambda/2)(p-1)}}{(\ln e x)^{\lambda} n^{1 - \lambda/2}} f_p(x) dx \right\}^{\frac{1}{p}} 
= \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right\}^{\frac{1}{2}} \left\{ \int_1^{\infty} \varpi(x) x^{p-1} (\ln x)^{p(1 - \frac{\lambda}{2})-1} f_p(x) dx \right\}^{\frac{1}{p}},
\]
and (2.4) follows. Still by the reverse Hölder’s inequality with weight, for \( q < 0 \), we have
\[
\left[ \sum_{n=1}^{\infty} \frac{a_n}{(\ln e x^n)^{\lambda}} \right]^q = \left[ \sum_{n=1}^{\infty} \frac{1}{(\ln e x^n)^{\lambda}} \left[ \frac{(\ln x)^{(1-\lambda)/q}}{n^{(1-\lambda)/p}} x^{1/q} \right]^q \right] \times \left[ \frac{n^{(1-\lambda)/p}}{(\ln x)^{(1-\lambda)/q}} x^{1/q} a_n \right]^{q-1} \\
\leq \left\{ \sum_{n=1}^{\infty} \frac{1}{(\ln e x^n)^{\lambda}} \right\}^{q-1} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\lambda)/q}} \frac{(\ln x)^{(1-\lambda)/2}}{x} \frac{1}{n^{(1-\lambda)/2}} a_n^q.
\]

Then by Lebesgue term by term integration theorem, we have
\[
L_1 \geq \left\{ \int_1^{\infty} \left[ \sum_{n=1}^{\infty} \frac{1}{(\ln e x^n)^{\lambda}} \frac{(\ln x)^{(1-\lambda)/2}}{x} n^{(1-\lambda)/2} a_n^q \right] dx \right\}^{1/q} \\
= \left\{ \sum_{n=1}^{\infty} \left[ n^{\lambda} \int_1^{\infty} \frac{(\ln x)^{(1-\lambda)/2}}{x (\ln e x^n)^{\lambda}} dx \right] n^{q(1-\lambda)/2} a_n^q \right\}^{1/q} \\
= \left\{ \sum_{n=1}^{\infty} \omega(n) n^{q(1-\lambda)/2} a_n^q \right\}^{1/q},
\]
and then in view of (2.3), inequality (2.5) follows. \( \square \)

3. Main results

In the following, for \( 0 < p < 1, q < 0 \), we still use the normal expressions of \( ||f||_{p,\Phi} \) and \( ||a||_{q,\Psi} \). Setting \( \Phi(x) := (1 - \theta_{\lambda}(x)) x^{p-1} (\ln x)^{p(1-\frac{\lambda}{2})-1} \) \((x \in (1, \infty))\), \( \Psi(n) := n^{q(1-\frac{\lambda}{2})-1} \) \((n \in \mathbb{N})\), we have \( [\Phi(x)]^{1-q} = \frac{(\ln x)^{(p\lambda/2)-1}}{x^{(1-\theta_{\lambda}(x))^{q-1}}} \) and

**Theorem 3.1.** If \( 0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq 2, f(x), a_n \geq 0, 0 < ||f||_{p,\Phi} < \infty \) and \( 0 < ||a||_{q,\Psi} < \infty \), then we have the following equivalent inequalities:
\[
I := \sum_{n=1}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{\ln e x^n} \lambda \ dx \\
= \int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{\ln e x^n} \lambda \ dx > B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi} ||a||_{q,\Psi}, \quad (3.1)
\]
\[ J = \left\{ \sum_{n=1}^{\infty} n^{\frac{q-1}{p}} \left[ \int_1^{\infty} \frac{f(x)dx}{\ln(x^n)^\lambda} \right]^p \right\}^{\frac{1}{p}} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\Phi}, \quad (3.2) \]

\[ L = \left\{ \int_1^{\infty} \frac{(\ln x)^{\frac{q-1}{p}}}{x(1-\theta_\lambda(x))^{q-1}} \left[ \sum_{n=1}^{\infty} \frac{a_n}{\ln(x^n)^\lambda} \right]^q \right\}^{\frac{1}{q}} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||a||_{q,\Psi}, \quad (3.3) \]

where the same constant factor \( B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \) in the above inequalities is the best possible.

**Proof.** By Lebesgue term by term integration theorem, there are two expressions for \( I \) in (3.1). In view of (2.4), for \( \varpi(x) > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1-\theta_\lambda(x)) \), we have (3.2). By the reverse H"older’s inequality, we have

\[ I = \sum_{n=1}^{\infty} \left[ \Psi\left(\frac{n}{x}\right) \int_1^{\infty} \frac{1}{\ln(x^n)^\lambda} f(x)dx \right] \left[ \Psi\left(\frac{n}{x}\right) a_n \right] \geq J ||a||_{q,\Psi}. \quad (3.4) \]

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

\[ a_n := \left[ \Psi(n) \right]^{1-p} \left[ \int_1^{\infty} \frac{1}{\ln(x^n)^\lambda} f(x)dx \right]^{p-1}, \quad n \in \mathbb{N}, \]

then \( J^{q-1} = ||a||_{q,\Psi} \). By (2.4), we find \( J > 0 \). If \( J = \infty \), then (3.2) is naturally valid; if \( J < \infty \), then by (3.1), we have

\[ ||a||_{q,\Psi} = J^p = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\Phi} ||a||_{q,\Psi}, \]

\[ ||a||_{q,\Psi} = J > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\Phi}, \]

and we have (3.2), which is equivalent to (3.1).

In view of (2.5), for \([\varpi(x)]^{1-q} > [B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1-\theta_\lambda(x))]^{1-q} \), we have (3.3). By the reverse Hölder’s inequality, we find that

\[ I = \int_1^{\infty} \left[ \Phi\left(\frac{1}{x}\right) f(x) \right] \left[ \Phi\left(\frac{1}{x}\right) \sum_{n=1}^{\infty} \frac{a_n}{\ln(x^n)^\lambda} \right] dx \geq ||f||_{p,\Phi} L. \quad (3.5) \]

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

\[ f(x) := \left[ \Phi(x) \right]^{1-q} \left[ \sum_{n=1}^{\infty} \frac{1}{\ln(x^n)^\lambda} a_n \right]^{q-1}, \quad x \in (1, \infty), \]

then \( L^{q-1} = ||f||_{p,\Phi} \). By (2.5), we find \( L > 0 \). If \( L = \infty \), then (3.3) is naturally valid; if \( L < \infty \), then by (3.1), we have

\[ ||f||_{p,\Phi} = L^q = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\Phi} ||a||_{q,\Psi}, \]

\[ ||f||_{p,\Phi}^p = L > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||a||_{q,\Psi}, \]
and we have (3.3) which is equivalent to (3.1). Hence inequalities (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting \( f(x) = \frac{1}{x} (\ln x)^{\frac{1}{2} + \frac{\varepsilon}{p} - 1}, x \in (1, e); f(x) = 0, x \in [e, \infty), \)
and \( \overline{a}_n = n^{\frac{1}{2} - \frac{\varepsilon}{p} - 1}, n \in \mathbb{N}, \) if there exists a positive number \( k \geq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \), such that (3.1) is valid as we replace \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) by \( k \), then in particular, it follows that

\[
\overline{I} := \sum_{n=1}^{\infty} \int_{1}^{\infty} \frac{1}{(\ln x^n)\lambda} \overline{a}_n f(x) dx > k \|f\|_{p, \Phi} \|\overline{a}\|_{\Psi},
\]

\[
= k \left\{ \int_{1}^{e} \frac{1 - O((\ln x)^{\frac{1}{2}})}{x (\ln x)^{-\varepsilon + 1}} dx \right\} \left\{ 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\varepsilon + 1}} \right\}^{\frac{1}{q}}
\]

\[
> k \left( \frac{1}{\varepsilon} - O(1) \right)^{\frac{1}{p}} \left\{ 1 + \int_{1}^{\infty} \frac{1}{x^{\varepsilon + 1}} dx \right\}^{\frac{1}{q}}
\]

\[
= k \left( 1 - \varepsilon O(1) \right)^{\frac{1}{p}} (\varepsilon + 1)^{\frac{1}{q}}, \tag{3.6}
\]

\[
\overline{I} = \sum_{n=1}^{\infty} n^{\frac{1}{2} - \frac{\varepsilon}{p} - 1} \int_{1}^{\infty} \frac{1}{x (\ln x)^{\lambda} (\ln x)^{\frac{1}{2} + \frac{\varepsilon}{p} - 1}} dx
\]

\[
t = n \ln x \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon + 1}} \int_{0}^{n} \frac{1}{(t + 1)^{\lambda} t^{\frac{1}{2} + \frac{\varepsilon}{p} - 1}} dt
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon + 1}} \int_{0}^{\infty} \frac{1}{(t + 1)^{\lambda} t^{\frac{1}{2} + \frac{\varepsilon}{p} - 1}} dt
\]

\[
= B \left( \frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) \left( 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\varepsilon + 1}} \right)
\]

\[
< B \left( \frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) \left( 1 + \int_{1}^{\infty} \frac{1}{y^{\varepsilon + 1}} dy \right)
\]

\[
= \frac{1}{\varepsilon} B \left( \frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) (1 + \varepsilon). \tag{3.7}
\]

Hence by (3.6) and (3.7), it follows that

\[
B \left( \frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) (1 + \varepsilon) > k \{ 1 - \varepsilon O(1) \}^{\frac{1}{p}} \{ \varepsilon + 1 \}^{\frac{1}{q}}, \tag{3.8}
\]

and \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \geq k(\varepsilon \to 0^+) \). Hence \( k = B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) is the best value of (3.1).

We confirm that the constant factor \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) in (3.2) ((3.3)) is the best possible. Otherwise we can came to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. \( \square \)
Remark 3.2. For \( \lambda = 1 \) in (3.1), (3.2) and (3.3), we have (1.7) and the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} \left[ \int_{1}^{\infty} \frac{f(x)dx}{\ln ex^n} \right]^p > \pi^p \int_{1}^{\infty} (1 - \theta_1(x))x^{p-1}(\ln x)^{\frac{q}{2}-1}f^p(x)dx, \tag{3.9}
\]

\[
\int_{1}^{\infty} \frac{(\ln x)^{\frac{q}{2}-1}}{x(1 - \theta_1(x))^{q-1}} \left[ \sum_{n=1}^{\infty} \frac{a_n}{\ln ex^n} \right]^q \, dx < \pi^q \sum_{n=1}^{\infty} n^{\frac{q}{2}-1}a_n^q. \tag{3.10}
\]

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