

ON A HALF-DISCRETE REVERSE HILBERT-TYPE INEQUALITY

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ABSTRACT. By using the weight functions and the techniques of real analysis, a half-discrete reverse Hilbert-type inequality with the best constant factor is given. A best extension with a parameter λ and the equivalent forms are also considered.

1. INTRODUCTION

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L^p(0, \infty), g(\geq 0) \in L^q(0, \infty), \|f\|_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0, \|g\|_q > 0$, we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \quad (1.1)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, \|a\|_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, \|b\|_q > 0$, then we still have the following discrete Hardy-Hilbert's inequality with the same constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \quad (1.2)$$

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [10], [11], [12]). Also we have the following Mulholland's inequality with the same constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], [13]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{m=2}^\infty m^{p-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^\infty n^{q-1} b_n^q \right\}^{\frac{1}{q}}. \quad (1.3)$$

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [14] gave an extension of (1.1) (for $p = q = 2$). By refining the results of [14], Yang [15] gave some extensions that are the best possible of (1.1) and (1.2) as follows: If $\lambda, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda, k(\lambda_1) = \int_0^\infty k_\lambda(t, 1)t^{\lambda_1-1}dt \in R_+, \phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}$,

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$f(\geq 0) \in L_{p,\phi}(0, \infty) = \{f \mid \|f\|_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\}$, $g(\geq 0) \in L_{q,\psi}(0, \infty)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad (1.4)$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if $k_\lambda(x, y)$ is also finite and $k_\lambda(x, y)x^{\lambda_1-1}(k_\lambda(x, y)y^{\lambda_2-1})$ is decreasing for $x > 0(y > 0)$, then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a \mid \|a\|_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m, n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \quad (1.5)$$

where the constant factor $k(\lambda_1)$ is the best possible. For $\lambda = 1$, $k_1(x, y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (1.4) reduces to (1.1), and (1.5) reduces to (1.2). Some other results including the reverse Hilbert-type inequalities are provided by [16]-[9].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the constant factors in the inequalities are the best possible. And Yang [18] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [19] gave a half-discrete Hilbert's inequality and [20] gave the following half-discrete reverse Hilbert-type inequality with the best constant factor 4:

$$\begin{aligned} & \int_0^\infty f(x) \sum_{n=1}^\infty \min\{x, n\} a_n dx \\ & > 4 \left\{ \int_0^\infty (1 - \theta_1(x)) x^{\frac{3p}{2}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{3q}{2}-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (1.6)$$

In this paper, by using the weight functions and the techniques of real analysis, a half-discrete reverse Hilbert-type inequality similar to (1.6) with the best constant factor is given as follows ($\theta_1(x) \in (0, 1)$):

$$\begin{aligned} & \int_1^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{\ln ex^n} dx \\ & > \pi \left\{ \int_1^\infty (1 - \theta_1(x)) x^{p-1} (\ln x)^{\frac{p}{2}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{q}{2}-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (1.7)$$

A best extension of (1.7) with a parameter λ and the equivalent forms are considered.

2. SOME LEMMAS

Lemma 2.1. *If $0 < \lambda \leq 2$, setting weight functions $\omega(n)$ and $\varpi(x)$ as follows:*

$$\omega(n) : = n^{\frac{\lambda}{2}} \int_1^\infty \frac{1}{x(\ln ex^n)^\lambda} (\ln x)^{\frac{\lambda}{2}-1} dx, n \in \mathbf{N}, \quad (2.1)$$

$$\varpi(x) : = (\ln x)^{\frac{\lambda}{2}} \sum_{n=1}^\infty \frac{1}{(\ln ex^n)^\lambda} n^{\frac{\lambda}{2}-1}, x \in (1, \infty), \quad (2.2)$$

then we have

$$B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1 - \theta_\lambda(x)) < \varpi(x) < \omega(n) = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right), \quad (2.3)$$

where, $\theta_\lambda(x) := \frac{1}{B(\frac{\lambda}{2}, \frac{\lambda}{2})} \int_0^{\ln x} \frac{t^{(\lambda/2)-1}}{(1+t)^\lambda} dt$ and

$$\theta_\lambda(x) = O((\ln x)^{\frac{\lambda}{2}}) \in (0, 1) (x \in (1, \infty)).$$

Proof. Setting $t = n \ln x$ in (2.1), by calculation, we have

$$\omega(n) = \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{\lambda}{2}-1} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).$$

Since for fixed $x > 1$,

$$h(x, y) := \frac{1}{(\ln ex^y)^\lambda} y^{\frac{\lambda}{2}-1} = \frac{1}{(1 + y \ln x)^\lambda y^{1-\lambda/2}}$$

is strictly decreasing for $y \in (0, \infty)$, then we find

$$\varpi(x) < (\ln x)^{\frac{\lambda}{2}} \int_0^\infty \frac{1}{(1 + y \ln x)^\lambda} y^{\frac{\lambda}{2}-1} dy$$

$$\stackrel{t=y \ln x}{=} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{\lambda}{2}-1} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right),$$

$$\varpi(x) > (\ln x)^{\frac{\lambda}{2}} \int_1^\infty \frac{1}{(1 + y \ln x)^\lambda} y^{\frac{\lambda}{2}-1} dy$$

$$\stackrel{t=y \ln x}{=} \int_{\ln x}^\infty \frac{t^{(\lambda/2)-1}}{(1+t)^\lambda} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1 - \theta_\lambda(x)) > 0,$$

$$\begin{aligned} 0 &< \theta_\lambda(x) = \frac{1}{B(\frac{\lambda}{2}, \frac{\lambda}{2})} \int_0^{\ln x} \frac{t^{(\lambda/2)-1}}{(1+t)^\lambda} dt \\ &< \frac{1}{B(\frac{\lambda}{2}, \frac{\lambda}{2})} \int_0^{\ln x} t^{\frac{\lambda}{2}-1} dt = \frac{2}{\lambda B(\frac{\lambda}{2}, \frac{\lambda}{2})} (\ln x)^{\frac{\lambda}{2}}, \end{aligned}$$

and then (2.3) is valid. $\square$

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $n \in \mathbf{N}$, $f(x)$ is a non-negative measurable function*

in $(1, \infty)$. Then we have the following inequalities (Note: in this paper, if $a_n = 0$, then we take $a_n^q = 0$ ($q < 0$)).):

$$\begin{aligned} J &: = \left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\int_1^{\infty} \frac{f(x)}{(\ln ex^n)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} \\ &\geq [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^{\frac{1}{q}} \left\{ \int_1^{\infty} \varpi(x) x^{p-1} (\ln x)^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \quad (2.4)$$

$$\begin{aligned} L_1 &: = \left\{ \int_1^{\infty} \frac{(\ln x)^{(q\lambda/2)-1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{(\ln ex^n)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} \\ &\geq \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (2.5)$$

Proof. (i) By the reverse Hölder's inequality with weight (cf. [7]) and (2.3), it follows that

$$\begin{aligned} \left[\int_1^{\infty} \frac{f(x) dx}{(\ln ex^n)^\lambda} \right]^p &= \left\{ \int_1^{\infty} \frac{1}{(\ln ex^n)^\lambda} \left[\frac{(\ln x)^{(1-\lambda/2)/q}}{n^{(1-\lambda/2)/p}} x^{1/q} f(x) \right] \right. \\ &\quad \times \left. \left[\frac{n^{(1-\lambda/2)/p}}{(\ln x)^{(1-\lambda/2)/q}} \frac{1}{x^{1/q}} \right] dx \right\}^p \\ &\geq \int_1^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{n^{1-\lambda/2}} f^p(x) dx \\ &\quad \times \left\{ \int_1^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{n^{(1-\lambda/2)(q-1)}}{x (\ln x)^{1-\lambda/2}} dx \right\}^{p-1} \\ &= \left\{ \omega(n) n^{q(1-\frac{\lambda}{2})-1} \right\}^{p-1} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{(\ln ex^n)^\lambda n^{1-\lambda/2}} f^p(x) dx \\ &= \left[B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^{p-1} n^{1-\frac{p\lambda}{2}} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{(\ln ex^n)^\lambda n^{1-\lambda/2}} f^p(x) dx. \end{aligned}$$

Then by Lebesgue term by term integration theorem (cf. [8]), we have

$$\begin{aligned} J &\geq \left[B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^{\frac{1}{q}} \left\{ \sum_{n=1}^{\infty} \int_1^{\infty} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{(\ln ex^n)^\lambda n^{1-\lambda/2}} f^p(x) dx \right\}^{\frac{1}{p}} \\ &= \left[B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^{\frac{1}{q}} \left\{ \int_1^{\infty} \sum_{n=1}^{\infty} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{(\ln ex^n)^\lambda n^{1-\lambda/2}} f^p(x) dx \right\}^{\frac{1}{p}} \\ &= \left[B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^{\frac{1}{q}} \left\{ \int_1^{\infty} \varpi(x) x^{p-1} (\ln x)^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned}$$

and (2.4) follows. Still by the reverse Hölder's inequality with weight, for $q < 0$, we have

$$\begin{aligned}
\left[\sum_{n=1}^{\infty} \frac{a_n}{(\ln ex^n)^\lambda} \right]^q &= \left\{ \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} \left[\frac{(\ln x)^{(1-\lambda/2)/q}}{n^{(1-\lambda/2)/p}} x^{1/q} \right] \right. \\
&\quad \times \left. \left[\frac{n^{(1-\lambda/2)/p}}{(\ln x)^{(1-\lambda/2)/q}} \frac{1}{x^{1/q}} a_n \right] \right\}^q \\
&\leq \left\{ \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{x^{p-1} (\ln x)^{(1-\lambda/2)(p-1)}}{n^{1-\lambda/2}} \right\}^{q-1} \\
&\quad \times \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{n^{(1-\lambda/2)(q-1)}}{x (\ln x)^{1-\lambda/2}} a_n^q \\
&= \frac{x[\varpi(x)]^{q-1}}{(\ln x)^{(q\lambda/2)-1}} \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{(\ln x)^{\lambda/2-1}}{x} n^{(q-1)(1-\frac{\lambda}{2})} a_n^q.
\end{aligned}$$

Then by Lebesgue term by term integration theorem, we have

$$\begin{aligned}
L_1 &\geq \left\{ \int_1^\infty \sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} \frac{(\ln x)^{\lambda/2-1}}{x} n^{(q-1)(1-\frac{\lambda}{2})} a_n^q dx \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{n=1}^{\infty} \left[n^{\frac{\lambda}{2}} \int_1^\infty \frac{(\ln x)^{\lambda/2-1}}{x (\ln ex^n)^\lambda} dx \right] n^{q(1-\frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{n=1}^{\infty} \omega(n) n^{q(1-\frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}},
\end{aligned}$$

and then in view of (2.3), inequality (2.5) follows. $\square$

3. MAIN RESULTS

In the following, for $0 < p < 1, q < 0$, we still use the normal expressions of $\|f\|_{p,\Phi}$ and $\|a\|_{q,\Psi}$. Setting $\Phi(x) := (1 - \theta_\lambda(x))x^{p-1}(\ln x)^{p(1-\frac{\lambda}{2})-1}$ ($x \in (1, \infty)$), $\Psi(n) := n^{q(1-\frac{\lambda}{2})-1}$ ($n \in \mathbf{N}$), we have $[\Phi(x)]^{1-q} = \frac{(\ln x)^{(q\lambda/2)-1}}{x(1-\theta_\lambda(x))^{q-1}}$, $[\Psi(n)]^{1-p} = n^{\frac{p\lambda}{2}-1}$ and

Theorem 3.1. *If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq 2, f(x), a_n \geq 0, 0 < \|f\|_{p,\Phi} < \infty$ and $0 < \|a\|_{q,\Psi} < \infty$, then we have the following equivalent inequalities:*

$$\begin{aligned}
I &: = \sum_{n=1}^{\infty} a_n \int_1^\infty \frac{f(x)}{(\ln ex^n)^\lambda} dx \\
&= \int_1^\infty f(x) \sum_{n=1}^{\infty} \frac{a_n}{(\ln ex^n)^\lambda} dx > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \tag{3.1}
\end{aligned}$$

$$J = \left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\int_1^{\infty} \frac{f(x)dx}{(\ln ex^n)^\lambda} \right]^p \right\}^{\frac{1}{p}} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi}, \quad (3.2)$$

$$\begin{aligned} L &: = \left\{ \int_1^{\infty} \frac{(\ln x)^{\frac{q\lambda}{2}-1}}{x(1-\theta_\lambda(x))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{(\ln ex^n)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} \\ &> B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|a\|_{q,\Psi}, \end{aligned} \quad (3.3)$$

where the same constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in the above inequalities is the best possible.

Proof. By Lebesgue term by term integration theorem, there are two expressions for I in (3.1). In view of (2.4), for $\varpi(x) > B(\frac{\lambda}{2}, \frac{\lambda}{2})(1-\theta_\lambda(x))$, we have (3.2). By the reverse Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_1^{\infty} \frac{1}{(\ln ex^n)^\lambda} f(x)dx \right] [\Psi^{\frac{1}{q}}(n)a_n] \geq J \|a\|_{q,\Psi}. \quad (3.4)$$

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_1^{\infty} \frac{1}{(\ln ex^n)^\lambda} f(x)dx \right]^{p-1}, \quad n \in \mathbf{N},$$

then $J^{p-1} = \|a\|_{q,\Psi}$. By (2.4), we find $J > 0$. If $J = \infty$, then (3.2) is naturally valid; if $J < \infty$, then by (3.1), we have

$$\|a\|_{q,\Psi}^q = J^p = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi},$$

$$\|a\|_{q,\Psi}^{q-1} = J > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi},$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.5), for $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2}, \frac{\lambda}{2})(1-\theta_\lambda(x))]^{1-q}$, we have (3.3). By the reverse Hölder's inequality, we find that

$$I = \int_1^{\infty} [\Phi^{\frac{1}{p}}(x)f(x)] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=1}^{\infty} \frac{a_n}{(\ln ex^n)^\lambda} \right] dx \geq \|f\|_{p,\Phi} L. \quad (3.5)$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := [\Phi(x)]^{1-q} \left[\sum_{n=1}^{\infty} \frac{1}{(\ln ex^n)^\lambda} a_n \right]^{q-1}, \quad x \in (1, \infty),$$

then $L^{q-1} = \|f\|_{p,\Phi}$. By (2.5), we find $L > 0$. If $L = \infty$, then (3.3) is naturally valid; if $L < \infty$, then by (3.1), we have

$$\|f\|_{p,\Phi}^p = L^q = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi},$$

$$\|f\|_{p,\Phi}^{p-1} = L > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|a\|_{q,\Psi},$$

and we have (3.3) which is equivalent to (3.1). Hence inequalities (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\tilde{f}(x) = \frac{1}{x}(\ln x)^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1}$, $x \in (1, e)$; $\tilde{f}(x) = 0$, $x \in [e, \infty)$, and $\tilde{a}_n = n^{\frac{\lambda}{2} - \frac{\varepsilon}{q} - 1}$, $n \in \mathbf{N}$, if there exists a positive number $k(\geq B(\frac{\lambda}{2}, \frac{\lambda}{2}))$, such that (3.1) is valid as we replace $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ by k , then in particular, it follows that

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty} \int_1^{\infty} \frac{1}{(\ln ex^n)^{\lambda}} \tilde{a}_n \tilde{f}(x) dx > k \|\tilde{f}\|_{p, \Phi} \|\tilde{a}\|_{q, \Psi} \\
&= k \left\{ \int_1^e \frac{(1 - O((\ln x)^{\frac{\lambda}{2}})) dx}{x (\ln x)^{-\varepsilon+1}} \right\}^{\frac{1}{p}} \left\{ 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\varepsilon+1}} \right\}^{\frac{1}{q}} \\
&> k \left(\frac{1}{\varepsilon} - O(1) \right)^{\frac{1}{p}} \left\{ 1 + \int_1^{\infty} \frac{1}{x^{\varepsilon+1}} dx \right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon} \{1 - \varepsilon O(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}}, \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty} n^{\frac{\lambda}{2} - \frac{\varepsilon}{q} - 1} \int_1^e \frac{1}{x (\ln ex^n)^{\lambda}} (\ln x)^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dx \\
&\stackrel{t=n \ln x}{=} \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon+1}} \int_0^n \frac{1}{(t+1)^{\lambda}} t^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dt \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^{\varepsilon+1}} \int_0^{\infty} \frac{1}{(t+1)^{\lambda}} t^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dt \\
&= B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p}\right) \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{\varepsilon+1}}\right) \\
&< B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p}\right) \left(1 + \int_1^{\infty} \frac{1}{y^{\varepsilon+1}} dy\right) \\
&= \frac{1}{\varepsilon} B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p}\right) (1 + \varepsilon). \tag{3.7}
\end{aligned}$$

Hence by (3.6) and (3.7), it follows that

$$B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p}\right) (1 + \varepsilon) > k \{1 - \varepsilon O(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}}, \tag{3.8}$$

and $B(\frac{\lambda}{2}, \frac{\lambda}{2}) \geq k(\varepsilon \rightarrow 0^+)$. Hence $k = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best value of (3.1).

We confirm that the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in (3.2) ((3.3)) is the best possible. Otherwise we can come to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. $\square$

Remark 3.2. For $\lambda = 1$ in (3.1), (3.2) and (3.3), we have (1.7) and the following equivalent inequalities:

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\int_1^{\infty} \frac{f(x)dx}{\ln ex^n} \right]^p > \pi^p \int_1^{\infty} (1 - \theta_1(x)) x^{p-1} (\ln x)^{\frac{p}{2}-1} f^p(x) dx, \quad (3.9)$$

$$\int_1^{\infty} \frac{(\ln x)^{\frac{q}{2}-1}}{x(1 - \theta_1(x))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{\ln ex^n} \right]^q dx < \pi^q \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} a_n^q. \quad (3.10)$$

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