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ESTIMATING PARAMETRIC SURFACE AREAS USING SPLINE QUASI-INTERPOLANTS

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Abstract. This article introduces a novel method for estimating the area of parametric surfaces, utilizing numerical integration and bivariate spline quasi-interpolants (QI) over a uniform subdivision of a rectangular domain. The surfaces are approximated using QIs of bi-degree (k, k), and the area is computed via an expression involving a double integral. Our method demonstrates a convergence order of $k + 2$ for even k and $k + 1$ for odd k. These theoretical results were validated through numerical examples, focusing on both quadratic and cubic cases. Furthermore, we illustrate the practical applicability of our method by estimating the area of a human head using real 3D point data.

1. Introduction

The approximation of length, area, and volume plays a crucial role in real-world applications. Such measurements help optimize resource usage, create efficient structures, and contribute to solutions in various fields, including engineering, architecture, environmental sciences, ecology, medical imaging, radiology and geometric modeling. Our research specifically focuses on 3D parametric surface estimation, introducing a method based on the reconstruction of smooth surfaces from 3D point data.

Traditional numerical methods for surface area estimation, such as direct integration, often face challenges regarding computational efficiency and accuracy, especially when handling complex surfaces or higher-dimensional problems. Finite element and boundary element methods used to reconstruct surfaces provide high accuracy but can be computationally expensive and complex to implement. Other approaches, such as those based on B-splines and Non-Uniform Rational B-Splines (NURBS), offer flexibility and smoothness in surface representation but may still struggle to achieve optimal accuracy for highly complex geometries. The Monte Carlo method is appreciated for its simplicity and ease of implementation, especially for large-scale problems. However, it often requires a substantial number of samples to achieve the desired precision, making it less effective for certain applications. Subdivision surfaces, widely used in computer graphics for generating smooth surfaces from coarse meshes, represent another notable approach.

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Despite their effectiveness, these methods can face convergence issues and may require additional refinement steps.

Over the years, numerous methods have been proposed to estimate surface areas. In [12] and [9], the authors develop approaches to calculate the surface area of 3D binary objects. In [21], a method for surface area estimation is introduced, utilizing smooth reconstruction techniques in three-dimensional human body data analysis. The authors in [19] present a method for measuring the volume and area of digitized objects in 3D image analysis, while [13] proposes an approach based on polynomial interpolation to estimate the area of a parametric surface when a rectangular grid of point samples is available. Quasi-Monte Carlo methods for calculating the area of of a surface represented by sampled points apply the Cauchy-Crofton formula, as detailed in [10] and [11].

Quasi-interpolants (QIs) represent an attractive approach for constructing efficient local approximations of a given data set. They are particularly appealing due to their simplicity and efficiency. Unlike traditional interpolation methods, the construction of QIs does not require solving systems of equations. Instead, they are built using local information from the function and its derivatives, making them computationally efficient and easy to implement. Additionally, QIs can achieve high precision and smoothness, making them well-suited for approximating complex surfaces. Many authors have shown interest in the construction and application of spline-based QIs, as evidenced by various works, including [14], [15], [16], [1], [4], [6], [7], [8], [18], and [17].

The structure of this manuscript is as follows: In Section 2, titled "Preliminaries," we introduce a formula for estimating surface area and the univariate QI of degree k on a uniform partition. In Section 3, we present the bivariate QI approximant on rectangular domains, along with its error and the error associated with the corresponding cubature formula. In Section 4, we apply bivariate QIs to approximate the area of the objective function and analyze the resulting error. To confirm the theoretical results, Section 5 provides numerical examples using both quadratic and cubic cases. Finally, an application is presented in Section 6, and Section 7 concludes the manuscript.

2. Preliminary

2.1. **Formula of parametric surface area.** We denote $\mathcal{D} := [a, b] \times [c, d]$ to be a rectangular domain.
Let

$$f : (u, v) \in \mathcal{D} \to \mathbb{R}^3$$

$$f(u, v) := \big(x(u, v), y(u, v), z(u, v)\big) \tag{2.1}$$

be a parametric surface of $\mathbb{R}^3$.
We suppose that f is regular i.e
$D_u f(u, v) \neq 0$ and $D_v f(u, v) \neq 0$ for all $(u, v) \in \mathcal{D}$, where

$$D_u f = \Big(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\Big)^T, \; D_v f = \Big(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\Big)^T.$$

We consider the formula of computing area of a sufficiently smooth surface f (see [13],[20])

$$\mathcal{A}(f) := \int_{\mathcal{D}} \Big| D_u f \times D_v f \Big| du dv, \tag{2.2}$$

where $|\ .\ |$ denote the euclidean norm and $\times$ denote the vector (cross) product.

As the integrand function in (2.2) is expressed by a square root and contain derivatives, then it's difficult to calculate (2.2) exactly, so we give an approximation of (2.2) by applying a numerical cubature formula. As cubature rules, we first approximate each component of the function f by a bivariate QI of degree k obtained by the tensor product of two univariate QIs. Then, we will approximate the integral giving the area of the surface f by that which gives the area of its approximant.

2.2. C^{k-1} **QI of degree k on a uniform partition.** Given an interval $I := [\alpha, \beta]$ and an integer k, let $S_k = S_k^{k-1}(I)$ denote the space of splines in C^{k-1} and degree less than or equal to k on the uniform partition $X_n := \{x_i = \alpha + ih, 0 \leq i \leq n\}$ with mesh length $h = \dfrac{\beta - \alpha}{n}$.
We consider the basis functions $B_{i,k},\ i \in \tau_{n,k} := \{1, 2, \ldots, n+k\}$ of space S_k when multiple knots $x_{-k} = \ldots = x_{-1} = x_0 = \alpha$, and $\beta = x_n = x_{n+1} = \ldots = x_{n+k}$ are added at endpoints as presented in [7].
Let $T_n := \{t_i,\ 0 \leq i \leq n+1\}$ where $t_0 = \alpha$, $t_i = \frac{x_{i-1}+x_i}{2}$ for $i = 1, \cdots, n$ and $t_{n+1} = \beta$.

We consider a univariate spline QI defined in [15] as follows

$$Q_k f := \sum_{i \in \tau_{n,k}} \lambda_{i,k}(f)\ B_{i,k}, \tag{2.3}$$

where $\lambda_{i,k}(f)$ are linear functionals of the function f on either X_n for odd k or on T_n for even k.
Moreover, Q_k is constructed to be exact on $\mathbb{P}_k$ the polynomial space of degree at most k.

For a smooth function f, we consider the product integration formula based on Q_k with the weight function g

$$\int_\alpha^\beta f(u)g(u)du \approx \int_\alpha^\beta \mathcal{Q}_k f(u)g(u)du. \tag{2.4}$$

We define $\mathcal{E}_k(f, g)$ as the error associated to rule (2.4)

$$\mathcal{E}_k(f, g) := \int_\alpha^\beta \Big(f(u) - \mathcal{Q}_k f(u) \Big) g(u) du.$$

The following propositions provides the derivatives and product integral of errors associated with $\mathcal{Q}_k$, (for the proofs see [8, Cha.5] and [2]).

Proposition 2.1. *For $f \in C^{k+1}(I)$, we obtain the following estimation error*

$$||f^{(d)} - Q_k^{(d)} f||_\infty \leq C_{k,d} h^{k+1-d} ||f^{(k+1-d)}||_\infty,\ d = 0, 1, ..., k,$$

where $C_{k,d}$ denotes constants that can vary in different instances, but are independent of n.

Proposition 2.2. *For $f \in C^{r_k}(I)$, we have the following estimation error*

$$\mathcal{E}_k(f, g) \leq C_k\ h^{r_k}\ \|g\|_\infty\ \|f^{(r_k)}\|_\infty,\ \textit{if } k \textit{ is odd and } g \in L_1(I),$$

$$\mathcal{E}_k(f, g) \leq C_k\ h^{r_k}\ \|g^{(1)}\|_\infty\ \|f^{(r_k)}\|_\infty,\ \textit{if } k \textit{ is even and } g \in \mathcal{W}_1^1(I),$$

where $\mathcal{W}_1^1(I)$ denotes the Sobolev space of functions on I whose first derivatives are integrable (see [5]), C_k are constants independent of n, and

$$r_k := \begin{cases} k+1\ if\ k\ is\ odd, \\ k+2\ if\ k\ is\ even \end{cases} . \tag{2.5}$$

2.2.1. ***Quadratic case*** $(k = 2)$. We consider a C^1 quadratic QI

$$Q_2 f := \sum_{i \in \tau_{n,2}} \lambda_{i,2}(f)\ B_{i,2}, \tag{2.6}$$

with $\lambda_{i,3}(f)$ coefficients determined by the ensuing formulation

$$\begin{aligned}
&\lambda_{1,2}(f) = f_1,\ \lambda_{n+2,2}(f) = f_{n+1}, \\
&\lambda_{2,2}(f) = \frac{1}{6}(-2f_1 + 9f_2 - f_3), \\
&\lambda_{n+1,2}(f) = \frac{1}{6}(-f_n + 9f_{n+1} - 2f_{n+2}), \\
&\lambda_{i,2}(f) = \frac{1}{8}(-f_{i-1} + 10f_i - f_{i+1}),\ \text{for } 3 \leq i \leq n,
\end{aligned}$$

with $f_i = f(t_i)$, for $t_i \in T_n,\ i = 0, \cdots, n+1$.

$\mathcal{Q}_2$ is exact on $\mathbb{P}_2$, and for smooth functions f and g we have the following estimation errors

$$||f^{(d)} - Q_2^{(d)} f||_\infty \leq C_{2,d} h^{3-d} ||f^{(3-d)}||_\infty,\ d = 0, 1, 2,$$

$$\mathcal{E}_g^{(2)}(f) := \int_\alpha^\beta \Big(f(u) - \mathcal{Q}_2 f(u)\Big) g(u) du \leq C_2 h^4 \|g^{(1)}\|_\infty\ \|f^{(4)}\|_\infty,$$

where $C_{2,d}$ and C_2 are constants independents of n.

2.2.2. ***Cubic case*** $(k = 3)$. We consider a C^2 cubic QI

$$\mathcal{Q}_3 f := \sum_{i \in \tau_{n,3}} \lambda_{i,3}(f)\ B_{i,3}, \tag{2.7}$$

with $\lambda_{i,3}(f)$ coefficients determined by the ensuing formulation

$$\begin{aligned}
&\lambda_{1,3}(f) = f_0,\ \lambda_{n+3,3}(f) = f_n,\\
&\lambda_{2,3}(f) = \frac{1}{18}(7f_0 + 18f_1 - 9f_2 + 2f_3),\\
&\lambda_{n+2,3}(f) = \frac{1}{18}(2f_{n-3} - 9f_{n-2} + 18f_{n-1} + 7f_n),\\
&\lambda_{i,3}(f) = \frac{1}{6}(-f_{i-3} + 8f_{i-2} - f_{i-1}), \quad \text{for } 3 \le i \le n+1,
\end{aligned}$$

with $f_i = f(x_i)$, for $x_i \in X_n,\ i = 0, \cdots, n$.

$\mathcal{Q}_3$ is exact on $\mathbb{P}_3$, and for smooth functions f and g we have the estimation errors

$$||f^{(d)} - Q_3^{(d)} f||_\infty \le C_{3,d} h^{4-d} ||f^{(4-d)}||_\infty,\ d = 0, ..., 3,$$

$$\mathcal{E}_g^{(3)}(f) := \int_\alpha^\beta \Big(f(u) - \mathcal{Q}_3 f(u)\Big) g(u) du \le C_3 h^4 \|g\|_\infty\ \|f^{(3)}\|_\infty,$$

where $C_{3,\alpha}$ and C_3 are constants independent of n.

3. Bivariate QI on rectangle

Let f be a bivariate function defined on $\mathcal{D}$. We consider the two uniform partitions $X_m := \{x_i = a + ih_1, 0 \le i \le m\}$ and $Y_n := \{y_j = c + jh_2, 0 \le j \le n\}$ of the intervals $[a, b]$ and $[c, d]$ respectively, where $h_1 = \dfrac{b-a}{m}$ and $h_2 = \dfrac{d-c}{n}$. Let

$$T_m := \Big\{t_0 = a;\ t_{m+1} = b;\ t_i = \frac{x_{i-1} + x_i}{2},\ i = 1, \ldots, m\Big\}$$

and

$$S_n := \Big\{s_0 = c;\ s_{n+1} = d;\ s_j = \frac{y_{j-1} + y_j}{2},\ j = 1, \ldots, n\Big\}.$$

Let $\mathcal{T}_k$ the bivariate QI obtained by the tensor product of two QI of type (2.3) (see [3], [14]),

$$\begin{aligned}
\mathcal{T}_k f(s,t) :&= (Q_k \otimes Q_k) f(s,t) := Q_k(Q_k f(s,.)(t))\\
&= \sum_{i \in \tau_{m,k}} \sum_{j \in \tau_{n,k}} \lambda_{i,k}(\lambda_{j,k}(f)) B_{i,k}(s) B_{j,k}(t)\\
&= \sum_{i \in \tau_{m,k}} \sum_{j \in \tau_{n,k}} \lambda_{i,j,k}(f) B_{i,j,k}(s,t),
\end{aligned}$$

where $\lambda_{i,j,k}(f)$ are linear functionals of the function f on either $T_m \times S_n$ if k is even or on $X_m \times Y_n$ if k is odd.

For $\gamma = (\gamma_1, \gamma_2)$, $0 \leq \gamma_1, \gamma_2 \leq k$, let $D^{(\gamma)}$ denote the derivatives of order γ_1 with respect to first variable and order γ_2 with respect to second variable.

Theorem 3.1. *For $f \in C^{(k+1,k+1)}(\mathcal{D})$, the estimation errors associated with $\mathcal{T}_k$ and its derivatives are given by*

$$\|D^{\gamma}(\mathcal{T}_k f - f)\| \leq O(h^{k+1-\tilde{\gamma}}), \text{ as } h \to 0, \tag{3.1}$$

where $h = \max\{h_1, h_2\}$ and $\tilde{\gamma} = \max\{\gamma_1, \gamma_2\}$.

Proof. Let

$$\mathcal{Q}_{k,t} f(s,t) := \mathcal{Q}_k f(.,t)(s) = \sum_{i=1}^{m+k} \lambda_{i,k}(f(.,t)) \mathcal{B}_{i,k}(s),$$

and

$$\mathcal{Q}_{k,s} f(s,t) := \mathcal{Q}_k f(s,.)(t) = \sum_{j=1}^{n+k} \lambda_{j,k}(f(s,.)) \mathcal{B}_{j,k}(t).$$

For the sake of simplicity, we denote $l_k(s,t) := \mathcal{Q}_{k,s} f(s,t) - f(s,t)$.
Using the linearity of $\mathcal{Q}_k$, we obtain

$$\begin{aligned}
\mathcal{T}_k f(s,t) - f(s,t) &= \mathcal{Q}_k\Big(\mathcal{Q}_k f(s,.)(t)\Big) - f(s,t) = \mathcal{Q}_{k,t}\Big(\mathcal{Q}_{k,s} f(s,t)\Big) - f(s,t) \\
&= \mathcal{Q}_{k,t}\Big(\mathcal{Q}_{k,s} f(s,t)\Big) - \mathcal{Q}_{k,t} f(s,t) + \mathcal{Q}_{k,t} f(s,t) - f(s,t) \\
&= \mathcal{Q}_{k,t}\Big(\mathcal{Q}_{k,s} f(s,t) - f(s,t)\Big) + \mathcal{Q}_{k,t} f(s,t) - f(s,t) \\
&= \mathcal{Q}_{k,t} l_k(s,t) - l_k(s,t) + l_k(s,t) + \mathcal{Q}_{k,t} f(s,t) - f(s,t). \qquad (3.2)
\end{aligned}$$

Therefore

$$\Big\|D^{\gamma}\Big(\mathcal{T}_k f - f\Big)\Big\|_{\infty} \leq \Big\|D^{\gamma}\Big(\mathcal{Q}_{k,t} l_k - l_k\Big)\Big\|_{\infty} + \Big\|D^{\gamma} l_k\Big\|_{\infty} + \Big\|D^{\gamma}\Big(\mathcal{Q}_{k,t} f - f\Big)\Big\|_{\infty}.$$

According to Proposition 2.1, and for a sufficiently smooth function f there exist two constants C_{k,γ_1} and C_{k,γ_2} independents of n and m such that we have

$$\begin{aligned}
\Big\|D^{\gamma} l_k\Big\|_{\infty} &\leq C_{k,\gamma_2} h_2^{k+1-\gamma_2} \Big\|D^{(0,\gamma_2)} f\Big\|_{\infty}, \\
\Big\|D^{\gamma}\Big(\mathcal{Q}_{k,v} l_k - l_k\Big)\Big\|_{\infty} &\leq C_{k,\gamma_2} h_2^{k+1-\gamma_2} \Big\|D^{(0,\gamma_2} l_k\Big\|_{\infty} \\
&\leq C_{k,\gamma_1} C_{k,\gamma_2} h_2^{k+1-\gamma_2} h_1^{k+1-\gamma_1} \Big\|D^{\gamma} f\Big\|_{\infty}, \\
\Big\|D^{\gamma}\Big(\mathcal{Q}_{k,v} f - f\Big)\Big\|_{\infty} &\leq C_{k,\gamma_1} h_1^{k+1-\gamma_1} \Big\|D^{(\gamma_1,0)} f\Big\|_{\infty}.
\end{aligned}$$

Since

$$k+1-\tilde{\gamma} \leq k+1-\gamma_1 \text{ and } k+1-\tilde{\gamma} \leq k+1-\gamma_2,$$

then, for $h \to 0$ (especialy $h < 1$), we have

$$h_1^{k+1-\gamma_1} \leq h^{k+1-\tilde{\gamma}},\ h_2^{k+1-\alpha_2} \leq h^{k+1-\tilde{\gamma}},\ \text{then } h_1^{k+1-\gamma_1}\ h_2^{k+1-\gamma_2} \leq h^{k+1-\tilde{\gamma}}.$$

$$\left\|D^{\gamma}\Big(\mathcal{T}_k f - f\Big)\right\|_{\infty} \leq h^{k+1-\tilde{\gamma}}\Big(C_{k,\gamma_2}\left\|D^{(0,\gamma_2)}f\right\|_{\infty} + \leq C_{k,\gamma_1}C_{k,\gamma_2}\left\|D^{\gamma}f\right\|_{\infty} + C_{k,\gamma_1}\left\|D^{(\gamma_1,0)}f\right\|_{\infty}\Big).$$

Finally, we obtain

$$\left\|D^{\gamma}\Big(\mathcal{T}_k f - f\Big)\right\|_{\infty} \leq O(h^{k+1-\tilde{\gamma}}).$$

□

For the smooth function f and g defined on $\mathcal{D}$, we consider the product integration rule based on $\mathcal{T}_k$ with the weight bivariate function g

$$\int_{\mathcal{D}} f(s,t)g(s,t)dsdt \approx \int_{\mathcal{D}} \mathcal{T}_k f(s,t)g(s,t)dsdt. \tag{3.3}$$

Theorem 3.2. *For $f \in C^{k+2,k+2}(\mathcal{D})$, and for a smooth weight function g, we have the following estimation error*

$$\mathcal{E}_g^{(k)}(f) := \int_{\mathcal{D}} \Big(\mathcal{T}_k f(s,t) - f(s,t)\Big)g(s,t)\,dsdt \leq O(h^{r_k}), \quad as \quad h \to 0, \tag{3.4}$$

where r_k is an integer defined by (2.5).

Proof. According to (3.2), we have

$$\mathcal{T}_k f(s,t) - f(s,t) = \mathcal{Q}_{k,t}l_k(s,t) - l_k(s,t) + l_k(s,t) + \mathcal{Q}_{k,t}f(s,t) - f(s,t).$$

Then,

$$\begin{aligned}
\int_{\mathcal{D}} \Big(\mathcal{T}_k f(s,t) - f(s,t)\Big)g(s,t)\,dsdt &\leq \Big|\int_{\mathcal{D}} \Big(\mathcal{T}_k f(s,t) - f(s,t)\Big)g(s,t)\,dsdt\Big| \\
&= \Big|\int_a^b\int_c^d \Big(\mathcal{Q}_{k,t}l_k - l_k + l_k + \mathcal{Q}_{k,t}f - f\Big)(s,t)g(s,t)\,dsdt\Big| \\
&\leq \int_c^d dt \sup_{t\in[c,d]}\Big|\int_a^b \Big(\mathcal{Q}_{k,t}l_k(s,t) - l_k(s,t)\Big)g(s,t)\,ds\Big| + \\
&\quad \int_a^b ds \sup_{s\in[a,b]}\Big|\int_c^d l_k(s,t)g(s,t)\,dt\Big| + \\
&\quad \int_c^d dt \sup_{t\in[c,d]}\Big|\int_a^b \Big(\mathcal{Q}_{k,t}f(s,t) - f(s,t)\Big)g(s,t)\,ds\Big| \\
&\leq (d-c)\sup_{t\in[c,d]}\Big|\int_a^b \Big(\mathcal{Q}_{k,t}l_k(s,t) - l_k(s,t)\Big)g(s,t)\,ds\Big| + \\
&\quad (b-a)\sup_{s\in[a,b]}\Big|\int_c^d l_k(s,t)g(s,t)\,dt\Big| + \\
&\quad (d-c)\sup_{t\in[c,d]}\Big|\int_a^b \Big(\mathcal{Q}_{k,t}f(s,t) - f(s,t)\Big)g(s,t)\,ds\Big|.
\end{aligned}$$

Now, we use Proposition 2.1 and Proposition 2.2 to obtain the following bounds:

$$\sup_{s\in[a,b]}\left|\int_c^d l_k(s,t)g(s,t)dt\right| \le C_k h_2^{r_k}\|g^{(0,1)}\|_\infty\|f^{(0,r_k)}\|_\infty,$$

$$\sup_{t\in[c,d]}\left|\int_a^b \Big(\mathcal{Q}_{k,t}f(s,t)-f(s,t)\Big)g(s,t)ds\right| \le C_k h_1^{r_k}\|g^{(1,0)}\|_\infty\|f^{(r_k,0)}\|_\infty,$$

$$\sup_{t\in[c,d]}\left|\int_a^b \Big(\mathcal{Q}_{k,t}l_k(s,t)-l_k(s,t)\Big)g(s,t)ds\right| \le C_k h_1^{r_k}\|g^{(1,0)}\|_\infty\|l_k^{(r_k,0)}\|_\infty,$$

$$\le C_k C_{k,0} h_1^{r_k} h_2^{k+1}\|g^{(1,1)}\|_\infty\|f^{(r_k,k+1)}\|_\infty.$$

Then, the error estimation (3.4) is obtained by combining the three inequalities for each case.

□

4. Area surface approximation based on $\mathcal{T}_k$

Our aim in this section is to approximate area of a parametric surface $f(u,v) = (x(u,v), y(u,v), z(u,v))$ by using the operator $\mathcal{T}_k$. Only the values of f are available at points in sets $X_m \times Y_n$ or $T_m \times S_n$. To achieve this, we approximate the functions x, y and z by operators $\mathcal{T}_k x$, $\mathcal{T}_k y$ and $\mathcal{T}_k z$. Subsequently, we calculate $\mathcal{A}(\mathcal{T}_k f)$ the exact area of $\mathcal{T}_k f$ using the formula (2.2). This allows us to estimate the area of the original surface f, denoted as $\mathcal{A}(f)$, by approximating it with $\mathcal{A}(\mathcal{T}_k f)$, i.e

$$\mathcal{A}(f) \approx \mathcal{A}(\mathcal{T}_k f), \tag{4.1}$$

which means that,

$$\mathcal{A}(f) \approx I\Big(\big|D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f\big|\Big),$$

where

$$\mathcal{T}_k f = \big(\mathcal{T}_k x, \mathcal{T}_k y, \mathcal{T}_k z\big)$$

and

$$I(f) := \int_{\mathcal{D}} f(u,v)dudv.$$

For simplicity, we put

$$G = |D_u f \times D_v f| \text{ and } H_k = |D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f|.$$

Now, we are interested to estimate the error related to method given by (4.1). For that, we present two lemmas which allow us to prove theorem giving the convergence order.

Lemma 4.1. *If $f \in C^{(k+1,k+1)}(\mathcal{D})$, then the functions $D^\gamma G$ are bounded in $\mathcal{D}$ for all $\gamma_1 \le k$ and $\gamma_2 \le k$, where $\gamma = (\gamma_1, \gamma_2)$*

Proof. Let $\Gamma = \{(i,j) \in \mathbb{N}^2,\ (i,j) \le \gamma\}$ and $\Gamma^* = \Gamma \setminus \{(0,0), \gamma\}$.
We prove the lemma by induction. Since $D_u f \times D_v f$ is continuous on a compact

domain, G is bounded.
Suppose $D^{\gamma^*}G$ is bounded for $\gamma^* \in \Gamma^*$. By applying the Leibniz rule, we have

$$\begin{aligned} D^{\gamma}G^2 &= \sum_{(i,j)\in\Gamma} \binom{\gamma_1}{i}\binom{\gamma_2}{j} D^{(\gamma_1-i,\gamma_2-j)}G.D^{(i,j)}G \\ &= 2\ G\ D^{\gamma}G + \sum_{(i,j)\in\Gamma^*} \binom{\gamma_1}{i}\binom{\gamma_2}{j} D^{\gamma_1-i,\gamma_2-j}G.D^{(i,j)}G. \end{aligned}$$

Then

$$D^{\gamma}G = \frac{1}{2\ G}\left(D^{\gamma}G^2 - \sum_{(i,j)\in\Gamma^*} \binom{\gamma_1}{i}\binom{\gamma_2}{j} D^{(\gamma_1-i,\gamma_2-j)}G.D^{(i,j)}G \right).$$

We have $f \in C^{(k+1,k+1)}(\mathcal{D})$, so $D^{\gamma}G$ is bounded. Since f is regular, then G is bounded and does not vanish. The second term is bounded by induction.

Finally, we obtain the desired result.

□

Lemma 4.2. *If $f \in C^{(k+1,k+1)}(\mathcal{D})$, then the functions $D^{\gamma}H_k$ are bounded in $\mathcal{D}$ independently of h for sufficiently small h.*

Proof. According to Theorem 3.1, H_k is bounded for sufficiently small h. Since $D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f$ has derivatives of all orders and using the same reasoning as in Lemma 1, $D^{\gamma}H_k$ is also bounded.

□

In the following theorem, we provide the error estimation corresponding to the area estimation associated with the operator $\mathcal{T}_k$.

Theorem 4.3. *For a regular parametric surface $f \in C^{(k+1,k+1)}(\mathcal{D})$, we have the following error estimation associated with the formula (4.1)*

$$\mathcal{A}(f) - \mathcal{A}(\mathcal{T}_k f) \leq O(h^{r_k}), \ \text{as } h \to 0,$$

where the integer r_k is defined in (2.5).

Proof. We have

$$\begin{aligned} \mathcal{A}(f) - \mathcal{A}(\mathcal{T}_k f) &= \int_{\mathcal{D}} G(u,v)dudv - \int_{\mathcal{D}} H_k(u,v)dudv \\ &= \int_{\mathcal{D}} (G(u,v) - H_k(u,v))dudv \\ &= \int_{\mathcal{D}} \frac{G^2(u,v) - H_k^2(u,v)}{G(u,v) + H_k(u,v)}dudv. \end{aligned}$$

Let $E_k = f - \mathcal{T}_k f$.
Since $D_u f = D_u\mathcal{T}_k f + D_u f - D_u\mathcal{T}_k f = D_u\mathcal{T}_k f + D_u E_k$, then

$$\begin{aligned}
G^2 &= |D_u f \times D_v f|^2 = |(D_u\mathcal{T}_k f + D_u E_k) \times (D_v\mathcal{T}_k f + D_v E_k)|^2 \\
&= |D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f + D_u\mathcal{T}_k f \times D_v E_k + D_u E_k \times D_v\mathcal{T}_k f + D_u E_k \times D_v E_k|^2 \\
&= H_k^2 + |D_u\mathcal{T}_k f \times D_v E_k|^2 + |D_u E_k \times D_v\mathcal{T}_k f|^2 + |D_u E_k \times D_v E_k|^2 \\
&+ 2\Big[(D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f + D_u\mathcal{T}_k f \times D_v E_k).(D_u E_k \times D_v\mathcal{T}_k f + D_u E_k \times D_v E_k) \\
&+ (D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f).(D_u\mathcal{T}_k f \times D_v E_k) + (D_u E_k \times D_v\mathcal{T}_k f).(D_u E_k \times D_v E_k)\Big].
\end{aligned}$$

Denote

$$\begin{aligned}
\mathcal{R}_k =& |D_u\mathcal{T}_k f \times D_v E_k|^2 + |D_u E_k \times D_v\mathcal{T}_k f|^2 + |D_u E_k \times D_v E_k|^2 + \\
& 2(D_u\mathcal{T}_k f \times D_v E_k).(D_u E_k \times D_v\mathcal{T}_k f) + \\
& 2\Big[(D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f) + (D_u\mathcal{T}_k f \times D_v E_k) + (D_u E_k \times D_v\mathcal{T}_k f)\Big].(D_u E_k \times D_v E_k)
\end{aligned}$$

and

$$\mathcal{P}_k = 2(D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f).(D_u\mathcal{T}_k f \times D_v E_k + D_u E_k \times D_v\mathcal{T}_k f).$$

Then

$$G^2 - H_k^2 = \mathcal{P}_k + \mathcal{R}_k,$$

Therefore

$$\mathcal{A}(f) - \mathcal{A}(\mathcal{T}_k f) = \underbrace{\int_{\mathcal{D}} \frac{\mathcal{P}_k(u,v)}{G(u,v) + H_k(u,v)} dudv}_{I_1^{(k)}} + \underbrace{\int_{\mathcal{D}} \frac{\mathcal{R}_k(u,v)}{G(u,v) + H_k(u,v)} dudv}_{I_2^{(k)}}.$$

First, we will determine the convergence order of $I_2^{(k)}$.

We have $D_u\mathcal{T}_k f$ and $D_v\mathcal{T}_k f$ are bounded, and E_k is $O(h^{k+1})$. According to Theorem 3.1 $D_u E_k$ and $D_v E_k$ are $O(h^k)$. Therefore,

$$\begin{cases}
|D_u\mathcal{T}_k f \times D_v E_k|^2 \leq O(h^{2k}), \\
|D_u E_k \times D_v\mathcal{T}_k f|^2 \leq O(h^{2k}) \\
|D_u E_k \times D_v E_k|^2 \leq O(h^{4k}), \\
(D_u\mathcal{T}_k f \times D_v E_k).(D_u E_k \times D_v\mathcal{T}_k f) \leq O(h^{2k}), \\
\Big[(D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f) + (D_u\mathcal{T}_k f \times D_v E_k) + (D_u E_k \times D_v\mathcal{T}_k f)\Big].(D_u E_k \times D_v E_k) \leq O(h^{2k}).
\end{cases}.$$

Then $\|\mathcal{R}_k\| \leq O(h^{2k})$. According to Lemma 4.1 and Lemma 4.2, G and H_k are bounded on $\mathcal{D}$, then

$$\|I_2^{(k)}\| \leq O(h^{2k}).$$

It remains to evaluate the order of $I_1^{(k)}$.
By using the following propriety (called Lagrange formula)

$$(A \times B).(C \times D) = (A.C)(B.D) - (A.D)(B.C),$$

we obtain

$$\begin{aligned}\mathcal{P}_k &= 2(D_u\mathcal{T}_k f \times D_v\mathcal{T}_k f).(D_u\mathcal{T}_k f \times D_v E_k + D_u E_k \times D_v\mathcal{T}_k f)\\ &= 2\Bigg[\Big|D_u\mathcal{T}_k f\Big|^2(D_v\mathcal{T}_k f.D_v E_k) - (D_u\mathcal{T}_k f.D_v E_k)(D_v\mathcal{T}_k f.D_u\mathcal{T}_k f)\\ &+ (D_u\mathcal{T}_k f.D_u E_k)\Big|D_v\mathcal{T}_k f\Big|^2 - (D_u\mathcal{T}_k f.D_v\mathcal{T}_k f)(D_v\mathcal{T}_k f.D_u E_k)\Bigg].\end{aligned}$$

Then

$$I_1^{(k)} = \int_c^d\int_a^b \mathcal{F}_k(u,v).D_v E_k(u,v))dudv + \int_c^d\int_a^b \mathcal{L}_k(u,v).D_u E_k(u,v)dudv,$$

where

$$\mathcal{F}_k = 2\frac{\Big|D_u\mathcal{T}_k f\Big|^2 D_v\mathcal{T}_k f - \Big(D_u\mathcal{T}_k f.D_v\mathcal{T}_k f\Big)D_u\mathcal{T}_k f}{G+H_k},$$
$$\mathcal{L}_k = 2\frac{\Big|D_v\mathcal{T}_k f\Big|^2 D_u\mathcal{T}_k f - \Big(D_v\mathcal{T}_k f.D_u\mathcal{T}_k f\Big)D_v\mathcal{T}_k f}{G+H_k}.$$

Lemma 4.4. *If $f \in C^{(k+1,k+1)}(\mathcal{D})$, then $D^\gamma\mathcal{F}_k$ and $D^\gamma\mathcal{L}_k$ are bounded on $\mathcal{D}$, independently of h for sufficiently small h.*

Proof. The proof is simple, it suffices to use induction and apply the Leibniz rule on $D^\gamma(\mathcal{F}_k(G+H_k))$ and $D^\gamma(\mathcal{L}_k(G+H_k))$ and use Lemmas 4.1 and 4.2.
Since f is regular, then $\mathcal{F}_k$ is bounded.

Denote

$$\mathcal{B}_k = \Big|D_v\mathcal{T}_k f\Big|^2 D_u\mathcal{T}_k f - \Big(D_v\mathcal{T}_k f.D_u\mathcal{T}_k f\Big)D_v\mathcal{T}_k f$$

and

$$\mathcal{A}_k = \Big|D_u\mathcal{T}_k f\Big|^2 D_v\mathcal{T}_k f - \Big(D_u\mathcal{T}_k f.D_v\mathcal{T}_k f\Big)D_u\mathcal{T}_k f.$$

We have

$$\begin{aligned}D^\gamma(\mathcal{F}_k(G+H_k)) &= \sum_{(i,j)\in\Gamma}\binom{\gamma_1}{i}\binom{\gamma_2}{j}D^{(\gamma_1-i,\gamma_2-j)}\mathcal{F}_k\ D^{(i,j)}(G+H_k)\\ &= D^\gamma\mathcal{F}_k\ (G+H_k) + \sum_{\substack{(i,j)\in\Gamma\\(i,j)\neq(0,0)}}\binom{\gamma_1}{i}\binom{\gamma_2}{j}D^{\gamma_1-i,\gamma_2-j}\ \mathcal{F}_k\ D^{(i,j)}(G+H_k)\\ &= 2D^\gamma\mathcal{A}_k.\end{aligned}$$

Then

$$D^{\gamma}\mathcal{F}_k = \frac{1}{G+H_k}\left(2D^{\gamma}\mathcal{A}_k - \sum_{\substack{(i,j)\in\Gamma\\(i,j)\neq(0,0)}} \binom{\gamma_1}{i}\binom{\gamma_2}{j} D^{\gamma_1-i,\gamma_2-j}\ \mathcal{F}_k\ D^{(i,j)}(G+H_k)\right).$$

We have $D^{\gamma}\mathcal{A}_k$ is bounded, the second term is bounded by induction and Lemmas 4.1 and 4.2. Then $D^{\gamma}\mathcal{F}_k$ is bounded.
We follow the same reasoning to prove that $D^{\gamma}\mathcal{L}_k$ is bounded. □

The remainder of the proof of Theorem 4.3:

By applying integration by parts to the variable v for the first term and to the variable u for the second term of $I_1^{(k)}$, we obtain

$$\begin{aligned} I_1^{(k)} &= \int_a^b \left[\mathcal{F}_k(u,d).E_k(u,d) - \mathcal{F}_k(u,c).E_k(u,c) - \int_c^d D_v\mathcal{F}_k(u,v).E_k(u,v)dv\right]du \\ &- \int_c^d \left[\mathcal{L}_k(b,v).E_k(b,v) - \mathcal{L}_k(a,v).E_k(a,v) - \int_a^b D_u\mathcal{L}_k(u,v).E_k(u,v)du\right]dv \\ &= I_{1,1}^{(k)} + I_{1,2}^{(k)} + I_{1,3}^{(k)} + I_{1,4}^{(k)} + I_{1,5}^{(k)} + I_{1,6}^{(k)}, \end{aligned}$$

where

$$\begin{aligned} I_{1,1}^{(k)} &= \int_a^b \mathcal{F}_k(u,d).E_k(u,d)du,\ \ I_{1,2}^{(k)} = -\int_a^b \mathcal{F}_k(u,c).E_k(u,c)du, \\ I_{1,3}^{(k)} &= -\int_c^d \mathcal{L}_k(b,v).E_k(b,v)dv,\ \ I_{1,4}^{(k)} = \int_c^d \mathcal{L}_k(a,v).E_k(a,v)dv, \\ I_{1,5}^{(k)} &= -\int_a^b\int_c^d D_v\mathcal{F}_k(u,v).E_k(u,v)dvdu,\ \ I_{1,6}^{(k)} = \int_c^d\int_a^b D_u\mathcal{L}_k(u,v).E_k(u,v)dudv. \end{aligned}$$

For fixed numbers x and y, denote $f_y(u) = f(u,y)$ and $f_x(v) = f(x,v)$, then

$$\mathcal{T}_k f(x,v) = Q_k(Q_k f(x,.)(v)) = Q_k f(x,v) = Q_k f_x(v),$$

and

$$\mathcal{T}_k f(u,y) = Q_k(Q_k f(u,.)(y)) = Q_k f(u,y) = Q_k f_y(u).$$

As a result,

$$\begin{aligned} E_k(a,v) &= f(a,v) - \mathcal{T}_k f(a,v) = f_a(v) - Q_k f_a(v), \\ E_k(b,v) &= f(b,v) - \mathcal{T}_k f(b,v) = f_b(v) - Q_k f_b(v), \\ E_k(u,c) &= f(u,c) - \mathcal{T}_k f(u,c) = f_c(u) - Q_k f_c(u), \\ E_k(u,d) &= f(u,d) - \mathcal{T}_k f(u,d) = f_d(u) - Q_k f_d(u). \end{aligned}$$

By applying Proposition 2.1, we have

$$\|E_k(a,.)\| \leq O(h^{k+1}),\ \ \|E_k(b,.)\| \leq O(h^{k+1}),$$

$$\|E_k(.,c)\| \leq O(h^{k+1}),\ \|E_k(.,d)\| \leq O(h^{k+1}).$$

Also, according to Theorem 3.1, we have

$$\|E_k\| \leq O(h^{k+1}).$$

Moreover, according to Lemma 4.4, $\mathcal{F}_k$, $\mathcal{L}_k$, $D_v\mathcal{F}_k$ and $D_u\mathcal{L}_k$ are bounded on $\mathcal{D}$.

Then, by applying Proposition 2.2, we obtain

$$\|I_{1,i}^{(k)}\| \leq O(h^{r_k}),\ i = 1, ..., 6.$$

Finally we get the desired result. □

5. Numerical examples

In order to illustrate the method described in the previous sections, we use both the quadratic ($k = 2$) and cubic ($k = 3$) cases to approximate the areas of some examples of surfaces.
We consider the surface S_1 of quadrant unit sphere defined by :

$$S_1(u,v) := \Big(\cos u \sin v, \sin u \sin v, \cos v\Big),\ u \in [0,\pi],\ v \in [0,\frac{\pi}{2}],$$

a Möbius strip surface :

$$S_2(u,v) := \left(\left(1 + u\cos\frac{v}{2}\right)\cos v, \left(1 + u\cos\frac{v}{2}\right)\sin v, u\sin\frac{v}{2}\right),$$
$$u \in [0,1],\ v \in [0,2\pi],$$

a vase-shaped surface :

$$S_3(u,v) := \Big(u, \sin v(u^3 + 2u^2 - 2u + 2)/5, \cos v(u^3 + 2u^2 - 2u + 2)/5\Big),$$
$$u \in [-2.3, 1.3],\ v \in [0,2\pi],$$

and the parametric surface S_4 described by Franke's function f on $[0,1]^2$:

$$f(u,v) := \frac{3}{4}\exp\left(-\frac{(9u-2)^2 + (9v-2)^2}{4}\right) + \frac{3}{4}\exp\left(-\frac{(9u+1)^2}{49} - \frac{9v+1}{10}\right)$$
$$+ \frac{1}{2}\exp\left(-\frac{(9u-7)^2 + (9v-3)^2}{4}\right) - \frac{1}{5}\exp\left(-(9u-4)^2 - (9v-7)^2\right)\Bigg),$$

where $S_4(u,v) = \big(u, v, f(u,v)\big)$.

First, we approximate the components of each parametric surface using the operators $\mathcal{T}_2$ and $\mathcal{T}_3$. Then, we use the formula (2.2) to calculate the area of these approximants. Finally, we calculate the errors between the original surfaces and their approximants.

Tables 1, 2, 3, and 4 present the absolute numerical errors associated with area estimation of previous shapes using our method for both quadratic and cubic cases (columns 2 and 4) with several values of n. Columns 3 and 5 provide the

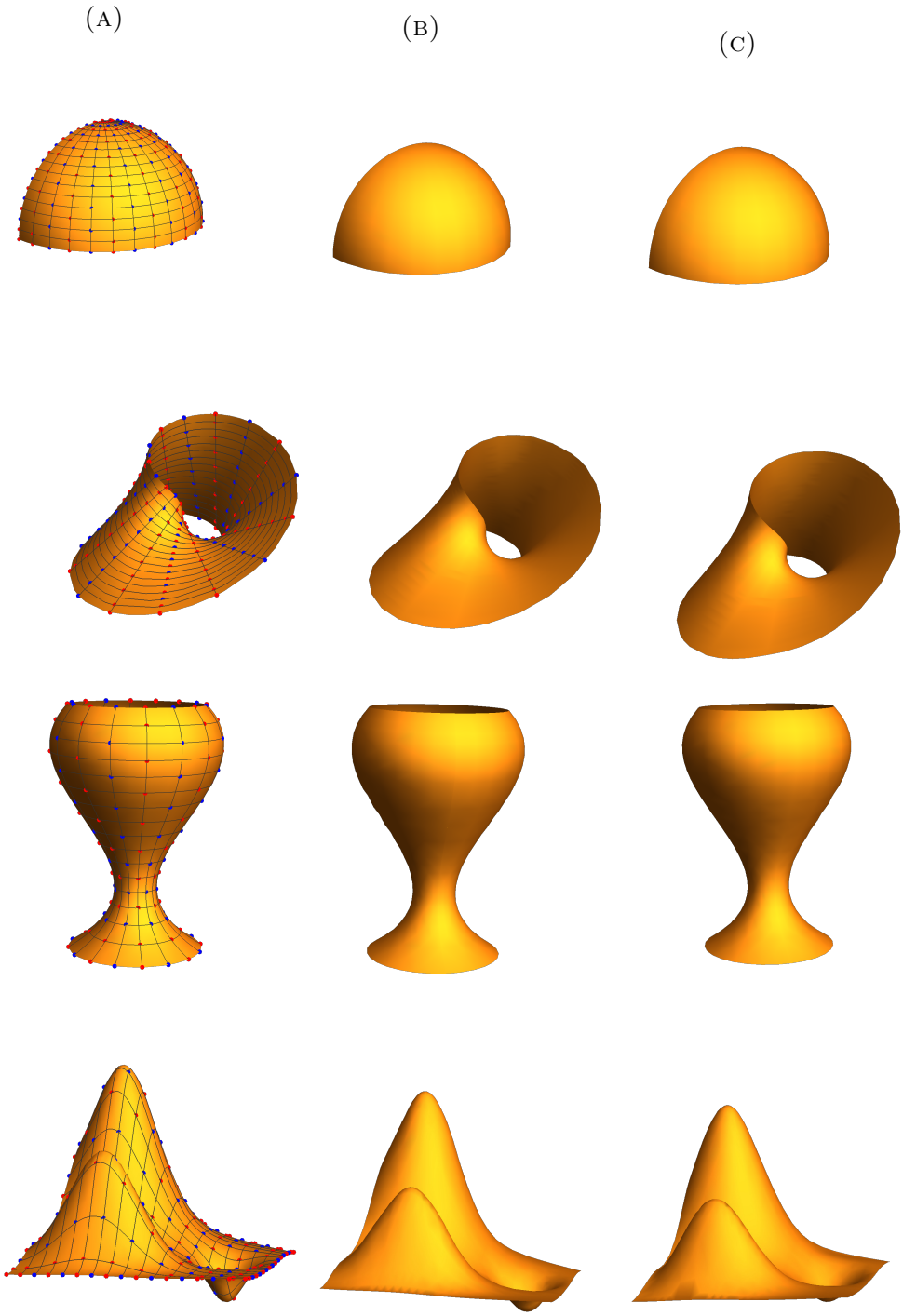


FIGURE 1. (A) The original surfaces S_1, S_2, S_3, S_4 with data points for $n = m = 8$ displayed in blue on subdivision domain $X_m \times Y_n$ and in red on $T_m \times S_n$, (B) the approximated surfaces using the quadratic QI $\mathcal{T}_2$, (C) the approximated surfaces using cubic QI $\mathcal{T}_3$.

numerical convergence order (NCO) for each case.
The numbers NCO are calculated by the formula

$$NCO = \frac{\log\left(\frac{e_p}{e_q}\right)}{\log\left(\frac{p}{q}\right)}.$$

where e_p and e_q are the absolute errors associated with the surface area approximation for $n = p$ and $n = q$ respectively.

TABLE 1. Absolute errors and numerical convergence orders associated with area approximation of S_1.

n	$\lvert A(\mathcal{T}_2 S_1) - A(S_1)\rvert$	NCO	$\lvert A(\mathcal{T}_3 S_1) - A(S_1)\rvert$	NCO
4	2.08139×10^{-2}	–	2.16841×10^{-2}	–
8	1.719108×10^{-3}	3.6	3.58299×10^{-3}	4.1
16	1.19793×10^{-4}	3.84	2.66801×10^{-4}	3.74
32	7.74563×10^{-6}	3.85	1.49264×10^{-5}	4.15
64	5.51457×10^{-7}	3.91	8.75894×10^{-5}	4.09

TABLE 2. Absolute errors and numerical convergence orders associated with area approximation of S_2.

n	$\lvert A(\mathcal{T}_2 S_2) - A(S_2)\rvert$	NCO	$\lvert A(\mathcal{T}_3 S_2) - A(S_2)\rvert$	NCO
4	1.96767×10^{-1}	–	1.54235×10^{-1}	–
8	1.78750×10^{-2}	3.46	1.31244×10^{-2}	3.55
16	1.39528×10^{-3}	3.67	1.01697×10^{-3}	3.69
32	1.07432×10^{-4}	3.7	6.59264×10^{-5}	3.95
64	7.32634×10^{-6}	3.87	4.07483×10^{-6}	4.02

TABLE 3. Absolute errors and numerical convergence orders associated with area approximation of S_3.

n	$\lvert A(\mathcal{T}_2 S_3) - A(S_3)\rvert$	NCO	$\lvert A(\mathcal{T}_3 S_3) - A(S_3)\rvert$	NCO
4	5.55638×10^{-1}	–	6.82351×10^{-1}	–
8	5.45787×10^{-2}	3.35	1.29201×10^{-1}	2.4
16	3.97699×10^{-3}	3.77	1.27778×10^{-2}	3.34
32	2.54563×10^{-4}	3.96	9.18546×10^{-4}	3.8
64	1.56534×10^{-5}	4.02	6.16843×10^{-6}	3.89

From previous examples and other numerical experiments, we deduce that the numerical orders of surface area approximation agree with those theoretically predicted. Furthermore, we observe that the original surfaces are well approximated by the spline surfaces reconstructed by $\mathcal{T}_2$ and $\mathcal{T}_3$ (see Figure 1).

TABLE 4. Absolute errors and numerical convergence orders associated with area approximation of S_4.

n	$\lvert A(\mathcal{T}_2 S_4) - A(S_4)\rvert$	NCO	$\lvert A(\mathcal{T}_3 S_4) - A(S_4)\rvert$	NCO
4	3.11159×10^{-1}	–	1.03252×10^{-1}	–
8	3.45286×10^{-2}	3.17	1.44973×10^{-3}	2.83
16	2.69224×10^{-3}	3.68	1.30041×10^{-4}	3.47
32	1.84361×10^{-4}	3.87	9.52187×10^{-6}	3.77
64	1.19343×10^{-5}	3.95	6.71533×10^{-6}	3.83

6. APPLICATION TO ESTIMATE AREA OF A HUMAN HEAD

To demonstrate the practical utility of our method, we estimated the surface area of a human head using quadratic and cubic. The computational domain was defined as $\mathcal{D} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$ with $m = 24$ and $n = 44$.

Our dataset comprised 20400 three-dimensional points $(f(u_i, v_j))$, where $i = 0, \ldots, 101$ and $j = 0, \ldots, 199$, originating from perfusion scintigraphy images. A coordinate transformation converted these spherical coordinates to a closed Cartesian surface representation.

By selectively sampling the original dataset, we evaluated 1196 points for the Quadratic case and 1125 points for the cubic case.

Utilizing our QI operators $\mathcal{T}_2$ and $\mathcal{T}_3$, we approximated the surface and computed its area using the formula (2.2), yielding the following estimates:

$$\mathcal{A}_2(f) \approx 757.52\,\mathrm{cm}^2 \quad \text{and} \quad \mathcal{A}_3(f) \approx 731.08\,\mathrm{cm}^2.$$

The original data points, the surfaces obtained using $\mathcal{T}_2$ and $\mathcal{T}_3$, as well as the 1196 and 1125 extracted points, are shown in Figure 2.

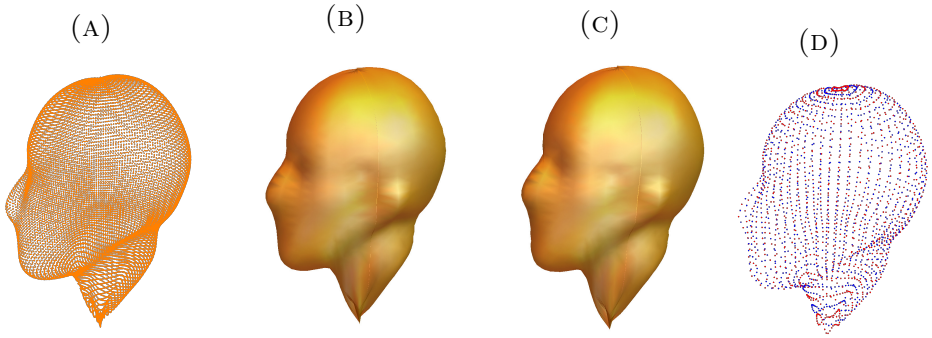


FIGURE 2. (A) The 17640 points of the human head, (B) the approximated surface using $\mathcal{T}_2$, (C) the approximated surface using $\mathcal{T}_3$, (D) the 1196 extracted points in red and the 1125 ones in Blue.

7. Conclusion

The surface area calculation method presented in this paper leverages the reconstruction of surfaces through the tensor product of QIs of degree k. This approach offers several distinct advantages over traditional methods such as polynomial interpolation. Foremost among these is the ability to locally reconstruct a perturbed subset of a given surface without affecting the entire surface data, enhancing both flexibility and efficiency. Furthermore, the reconstruction process circumvents the need to solve large linear systems, significantly reducing computational complexity and time.

One of the key strengths of our method lies in its high convergence order, attributed to the integration properties of quasi-interpolants, particularly for even degrees. Specifically, as demonstrated in the theoretical section and confirmed by numerical tests, our method achieves a convergence order of $k+2$ for even k and $k+1$ for odd k. This makes the approximation using even-order quasi-interpolants advantageous in terms of computational cost.

While our current analysis has focused on rectangular domains, the potential applications of this method are far-reaching. Future research directions include extending our approach to triangular domains and irregular shapes, thereby expanding its applicability to more complex and diverse surface geometries. Moreover, we anticipate that the principles underlying this method could be adapted to address other challenging problems in computational geometry and numerical analysis.

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