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ON CAPUTO CONFORMABLE FRACTIONAL NEUTRAL EVOLUTION EQUATIONS WITH FINITE DELAY

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Abstract. In this paper, we study a class of Caputo conformable fractional neutral evolution equations with finite delay. The existence and uniqueness for mild solutions are proved by fixed point theorems of Krasnoselskii and Banach contraction mapping, with some conditions. In the end, an example application is given for illustration purposes.

1. Introduction and preliminaries

Several authors have tried to replace the classical derivative with a fractional derivative in differential equations [3, 9, 17, 20, 21], because the model of many phenomena with memory with fractional differential become more precise and advanced than the classical model in various fields of science and engineering [10, 15] . The conformable derivative was introduced by Khalil et al. [14]. After Abdeljawad [1] defined the right and left conformable fractional derivatives, conformable fractional integrals and the fractional Laplace transform. The literature features various definitions of fractional differential operators, including Riemann-Liouville, Caputo, Hilfer-Riesz, Erdelyi-Kober, and Hadamard operators [15]. Notably, the Caputo and Riemann-Liouville fractional operators have attracted significant interest from researchers and are extensively used in complex fractional models, as highlighted by numerous studies [8].

Recently, Jarad et al. [13] introduced two integration and differentiation operators of arbitrary order based on Khalil's conformable operators [14]. These are Knoun as the Riemann-Liouville conformable integral and the Caputo conformable integral and derivative. They have been successfully applied to solve challenging problems in engineering sciences.

These conformal fractional operators combined with two parameters have become a subject of many contribution in several reserchers [2, 23]. Moreover, the nonlocal Cauchy problem with delay, associated with the Fractional derivative, have been investigated by many authors [3, 5, 6, 7, 10, 11, 15].

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In this paper, we are studying the following nonlocal-delay Cauchy problem with conformable fractional derivative,

$$\begin{cases} {}^{CC}\mathcal{D}_t^{\alpha,\beta}\left[u(t)-g((t,u_t)\right]=Au(t)+f\left(t,u_t\right) & t\in(0,a] \\ u_0(\tau)+\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(\tau)=\varphi(\tau), & \tau\in[-r,0] \end{cases} \tag{1.1}$$

where:
• ${}^{CC}\mathcal{D}_t^{\alpha,\beta}$ is the fractional derivative Conformable in the sense of Caputo of order $0<\beta<1$, with $0<\alpha<1$,
• A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of operators on E which E is a Banach space with the norm $|.|$,
• $f,g:[0,\infty)\times\mathcal{C}\to E$ and $\phi:\mathcal{C}^n\to\mathcal{C}$ are given functions, where $\mathcal{C}:=C(I,E)$ the continuous functions from $I\subset\mathbb{R}$ into$E$ with the norm $||u||=Sup_{t\in I}|u(t)|$
• $\varphi\in C([-r,0],E)$ and define u_t by $u_t(\tau)=u(t+\tau)$, for $\tau\in[-r,0]$.
This paper is divided into 4 sections. The second section is devoted to some concepts and basic properties of the conformable fractional calculus in the Caputo and Reimann-Liouville setting, in the third Section we define the mild solution of the problem and prove the existence and uniqueness under some conditions, and in the last section we give an example of application of the main results.

1.1. **Preliminaries.** We start, in this section, by recaling some basic concepts and theorems of the conformable fractional calculus.

Definition 1.1. [14] The left conformable derivative of a function $x:[t_0,\infty)\to\mathbb{R}$ with $\alpha\in(0,1]$ at the initial point t_0 is defined by the following relation

$$T_{t_0}^{\alpha}x(t)=\lim_{\epsilon\to 0}\frac{x\left(t+\epsilon\left(t-t_0\right)^{1-\alpha}\right)-x(t)}{\epsilon}$$

for all $t>0,\alpha\in(0,1)$. If x is α-differentiable in some $(0,\tau),\tau>0$, and $\lim_{t\to 0^+}T_0^{\alpha}x(t)$ exists, then define:

$$T_0^{\alpha}x(0)=\lim_{t\to 0^+}T_0^{\alpha}x(t)$$

The fractional integral $I_a^{\alpha}(.)$ associated with the conformable fractional derivative is defined by

$$I_a^{\alpha}x(t)=\int_a^t(s-a)^{\alpha-1}x(s)\mathrm{d}s \tag{1.2}$$

We now recall the definition of the fractional conformable operators of order $\beta, Re(\beta)>0$ extended by Jarad and al. in both the Caputo and the Riemann-Liouville settings as follows:

Definition 1.2. [13]. Let $\beta\in\mathbb{C}$;$Re(\beta)>0$.
The Riemann-Liouville conformable integral about a given function x of order β with $\alpha\in(0,1)$ is formulated by:

$${}^{\mathcal{RC}}\mathcal{I}_a^{\alpha,\beta}x(t)=\frac{1}{\Gamma\left(\beta\right)}\int_a^t\left(\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{\beta-1}x(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \tag{1.3}$$

The Riemann-Liouville conformable derivative for a given function x of order β with $\alpha \in (0,1]$ is introduced by

$$\begin{aligned} {}^{\mathcal{RC}}\mathfrak{D}_a^{\alpha,\beta}x(t) &= T_a^{\alpha,n}\left({}^{\mathcal{RC}}\mathcal{I}_a^{\alpha,n-\beta}x\right)(t) \\ &= \frac{T_a^{\alpha,n}}{\Gamma(n-\beta)}\int_a^t \left(\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{\beta-1} x(t)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \end{aligned} \tag{1.4}$$

where $n = [\mathrm{Re}(\beta)] + 1$ and $T_a^{\alpha,n} = \overbrace{T_a^{\alpha}T_a^{\alpha}\dots T_a^{\alpha}}^{n \text{ times}}$ and T_a^{α} is the left conformable derivative with $\alpha \in (0,1]$.

When $a = 0$ and $\beta = 1$, the fractional operators in (1.3), (1.4) coincides with the Riemann-Liouville fractional integral and derivative. To define the generalized notion of the conformable derivative in the Caputo sense, we needed to introduce some functions spaces:

Definition 1.3. [1]. For $\alpha \in (0,1]$ and $n = 1,2,3,\dots$, define

$$C^n_{\alpha,a}([a,b]) = \left\{f : [a,b] \to \mathbb{R} \text{ such that } T_a^{\alpha,n-1}f \in \mathbb{I}_{\alpha}([a,b])\right\},$$

with, $\mathbb{I}_{\alpha}([a,b])$ is the space defined by:

$$\mathbb{I}_{\alpha}([a,b]) = \{f : [a,b] \to \mathbb{R} : f(x) = (I_a^{\alpha}\psi)(t) + f(a), \text{ for some } \psi \in \mathbb{L}_{\alpha}(a)\},$$

where,

$$\mathbb{L}_{\alpha}(a) = \{\psi : [a,b] \to \mathbb{R} : (I_a^{\alpha}\psi)(t) \text{ exists for all } t \in [a,b]\}$$

Definition 1.4. [13]. The Caputo conformable derivative for a given function $x \in \mathcal{C}^n_{\alpha,a}([a,T])$ of order β with $\alpha \in (0,1]$ is defined by

$$\begin{aligned} {}^{CC}\mathcal{D}_a^{\alpha,n-\beta}x(t) &= {}^{\mathcal{RC}}\mathcal{I}_a^{\alpha,\beta}(T_a^{\alpha,n})x)(t) \\ &= \frac{1}{\Gamma(n-\beta)}\int_a^t \left(\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{n-\beta-1} T_a^{\alpha,n}x(s)\frac{ds}{(s-a)^{1-\alpha}} \end{aligned} \tag{1.5}$$

such that $n = [\mathrm{Re}(\beta)] + 1$.

If $a = 0$ and $\alpha = 1$, then ${}^{CC}\mathcal{D}_a^{\alpha,\beta}x(t) = {}^{c}\mathfrak{D}_a^{\beta}x(t)$.
In the following lemma, we give the applied properties of Caputo and Riemann-Liouville fractional conformable operators.

Lemma 1.5. [13].*Let* $\mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\nu) > 0$. *Then for* $\alpha \in (0,1]$ *and for any* $t > a$, *we have the following statements:*
(S1) ${}^{RC}\mathcal{I}_a^{\alpha,\beta}\left({}^{RC}\mathcal{I}_a^{\alpha,\gamma}x\right)(t) = \left({}^{RC}\mathcal{I}_a^{\alpha,\beta+\gamma}x\right)(t)$,
(S2) ${}^{RC}\mathcal{I}_a^{\alpha,\beta}(t-a)^{\alpha(\nu-1)}(s) = \frac{1}{\alpha^{\beta}}\frac{\Gamma(\nu)}{\Gamma(\nu+\beta)}(s-a)^{\alpha(\nu+\beta-1)}$,
(S3) ${}^{\mathcal{RC}}\mathfrak{D}_a^{\alpha,\beta}(t-a)^{\alpha(\nu-1)}(s) = \alpha^{\beta}\frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}(s-a)^{\alpha(\nu-\beta-1)}$,
(S4) ${}^{RC}\mathfrak{D}_a^{\alpha,\beta}\left({}^{RC}\mathcal{I}_a^{\alpha,\gamma}x\right)(t) = \left({}^{RC}\mathcal{I}_a^{\alpha,\gamma-\beta}x\right)(t), \quad (\mathrm{Re}(\beta) < \mathrm{Re}(\gamma))$.

Lemma 1.6. [13]. *Let* $n-1<\operatorname{Re}(\beta)<n$ *and* $x\in\mathcal{C}^n_{\alpha,a}([a,\tau])$. *Then for* $\alpha\in(0,1]$, *the following identity holds*

$$
\begin{aligned}
{}^{RC}\mathcal{I}_a^{\alpha,\beta}\left({}^{C}\mathfrak{D}_a^{\alpha,\beta}x\right)(t) &= x(t)-\sum_{k=0}^{n-1}\frac{T_a^{\alpha,k}x\,(a)}{\alpha^k k!}(t-a)^{k\alpha}.\\
&= x(t)-\sum_{k=0}^{n-1}c_k\,(t-a)^{k\alpha},
\end{aligned}
$$

such that: $c_0,c_1,\ldots,c_{n-1}\in\mathbb{R}$.

The adapted transform is given by the following definition.

Definition 1.7. [1]. The conformable fractional Laplace transform of order $\alpha\in\,]0,1]$ of a function $f(.)$ is defined by

$$
\mathcal{L}_\alpha(f(t))(\lambda):=\int_0^{+\infty}t^{\alpha-1}e^{-\lambda(t^\alpha/\alpha)}f(t)\mathrm{d}t,\quad \lambda>0 \tag{1.6}
$$

Proposition 1.8. [12] *If* $f(.)$ *is* α *differentiable function, then we have the following results:*

$$
\mathcal{L}_\alpha\left(D^\alpha f(t)\right)(\lambda)=\lambda L_\alpha(f(t))(\lambda)-f(0) \tag{1.7}
$$

$$
\mathcal{L}_\alpha\left(f\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda)=\mathcal{L}(f(t))(\lambda) \tag{1.8}
$$

The conformable Laplace transforms of some using functions is given by the following Lemma:

Lemma 1.9. [1]

(1) $\mathcal{L}_\alpha\{1\}(\lambda)=\frac{1}{\lambda}$
(2) $\mathcal{L}_\alpha\left\{(\frac{t^\alpha}{\alpha})^\beta\right\}(\lambda)=\frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}$
(3) $\mathcal{L}_\alpha\left\{e^{\mu(\frac{t^\alpha}{\alpha})}\right\}(\lambda)=\frac{1}{\lambda-\mu}$

Proof. We show these results by using (1.8) □

Definition 1.10. Let x and y be two continuous functions at each interval $[0,\ T]$. We define the α-conformable convolution of x and y by

$$
(x*y)_\alpha(t)=\int_a^t x(s)y((t^\alpha-s^\alpha)^{\frac{1}{\alpha}})d_\alpha s
$$

Now, we present the α-conformable Laplace transform of the α- conformable convolution

Remark 1.11. . Let x and y be two functions continuous at interval $[a,T]$. Then,

$$
\mathcal{L}_\alpha\left\{(x*y)_\alpha\right\}=\mathcal{L}_\alpha\left\{x\right\}\mathcal{L}_\alpha\left\{y\right\}
$$

Proof.

$$\begin{aligned}\mathcal{L}_\alpha\{x\}\mathcal{L}_\alpha\{y\} &= \int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}}t^{\alpha-1}x(t)dt\int_0^\infty e^{-\lambda\frac{s^\alpha}{\alpha}}s^{\alpha-1}x(s)ds\\ &= \int_0^\infty\int_0^\infty e^{-\lambda(\frac{t^\alpha}{\alpha}+\frac{s^\alpha}{\alpha})}x(t)y(s)t^{\alpha-1}s^{\alpha-1}dtds\end{aligned}$$

we take u such that $u^\alpha = t^\alpha + s^\alpha$, then

$$\begin{aligned}\mathcal{L}_\alpha\{x\}\mathcal{L}_\alpha\{y\} &= \int_0^\infty e^{-\lambda\frac{u^\alpha}{\alpha}}\left[\int_0^u x((u^\alpha-s^\alpha)^{\frac{1}{\alpha}})y(s)d_\alpha s\right]u^{\alpha-1}du\\ &= \mathcal{L}_\alpha\{(x*y)_\alpha\}\end{aligned}$$

□

Lemma 1.12. *(Krasnoselskii FPT, [21]).Let E be a Banach space, let B be a bounded closed and convex subset of E and let $P1; P2$ be maps of B into E such that $P_1x + P_2y \in B$ for every pair $x; y \in B$. If P_1 is a contraction and P_2 is completely continuous, then the equation $P_1x + P_2x = x$ has a solution on B.*

In this paper, let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.
For $0 < \beta \leq 1$, we define the fractional power A^β as a closed linear operator on its domain $D\left(A^\beta\right)$.
For analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used [18].
(a) There is a $M \geq 1$ such that

$$M := \sup_{t\in[0,+\infty)} |T(t)| < \infty$$

(b) for any $\beta \in (0,1]$, there exists a positive constant C_β such that

$$\left|A^\beta T(t)\right| \leq \frac{C_\beta}{t^\beta}, \quad 0 < t \leq a$$

2. Main Results

First, we give the definition of mild solutions for the Cauchy problem (1.1). So, using the definition of the Riemann-Liouville Conformablel Integral and Lemma 1.6, we obtain the equivalent integral solution of the fractional equation (1.1)

$$\begin{cases} u(t) = \varphi(0) - \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(0) - g(0,u_0) + g(t,u_t) \\ \qquad + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1}(Au(s) + f(s,u_s))s^{\alpha-1}ds. \quad s \leq t \in [0,a] \\ u_0(\tau) + \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(\tau) = \varphi(\tau), \quad \tau \in [-r, 0] \end{cases} \tag{2.1}$$

Lemma 2.1. *If (2.1) is verified, then we have*

$$\begin{cases} u(t) = \int_0^\infty \mathcal{M}_\beta(\theta) T((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta \theta) \left(\varphi(0) - \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(0) - g(0, u_0)\right) d\theta \\ \qquad + g(t, u_t) + \beta \int_0^t \int_0^\infty \mathcal{M}_\beta(\theta) (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} A T((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta \theta) g(s,\ u_s) s^{\alpha-1} d\theta ds. \\ \qquad + \beta \int_0^t \int_0^\infty \mathcal{M}_\beta(\theta) (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} T((\frac{t^\alpha}{\alpha})^\beta \theta) f(s,\ u_s) s^{\alpha-1} d\theta ds, \quad t \in [0, a]. \\ u_0(\tau) + \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(\tau) = \varphi(\tau), \quad \tau \in [-r, 0] \end{cases} \tag{2.2}$$

Where $\mathcal{M}_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\psi_\beta(\theta^{-\frac{1}{\beta}})$ *is the probability density function of Mainardi defined on* $(0,\ \infty)$. □

Proof. Let $\lambda > 0$. Applying the conformable Laplace transforms to (1.1), we have

$$\hat{u}_\alpha(\lambda) = \frac{u_0}{\lambda} + \frac{1}{\lambda^\beta}(A\hat{u}_\alpha(\lambda) + \hat{f}_\alpha(\lambda)) \ ,$$

where

$$\hat{u}_\alpha(\lambda) = \mathcal{L}_\alpha\left\{u(t)\right\}(\lambda) = \int_0^\infty e^{-\lambda(\frac{t^\alpha}{\alpha})} u(t) t^{\alpha-1} dt$$

and

$$\hat{f}_\alpha(\lambda) = \int_0^\infty e^{-\lambda(\frac{t^\alpha}{\alpha})} f(t, u_t) t^{\alpha-1} dt.$$

therefore

$$\begin{aligned} \hat{u}_\alpha(\lambda) &= \lambda^{\beta-1}(\lambda^\beta I - A)^{-1} u_0 + (\lambda^\beta I - A)^{-1}\hat{f}_\alpha(\lambda) \\ &= \lambda^{\beta-1}\int_0^\infty e^{-\lambda^\beta t} T(t) u_0 dt + \int_0^\infty e^{-\lambda^\beta t} T(t)\hat{f}_\alpha(\lambda) dt \end{aligned}$$

We take $t = (\frac{s^\alpha}{\alpha})^\beta$, then

$$\begin{aligned} \hat{u}_\alpha(\lambda) = \beta & \int_0^\infty \lambda^{\beta-1}(\frac{s^\alpha}{\alpha})^{\beta-1} e^{-(\lambda(\frac{s^\alpha}{\alpha})^\beta} T((\frac{s^\alpha}{\alpha})^\beta) u_0 s^{\alpha-1} ds \\ & + \int_0^\infty \beta(\frac{s^\alpha}{\alpha})^{\beta-1} e^{-(\lambda(\frac{s^\alpha}{\alpha})^\beta} T((\frac{s^\alpha}{\alpha})^\beta)\hat{f}_\alpha(\lambda) s^{\alpha-1} ds \\ & = \int_0^\infty -\frac{1}{\lambda}\frac{d}{ds}(e^{-(\lambda(\frac{s^\alpha}{\alpha})^\beta}) T((\frac{s^\alpha}{\alpha})^\beta) u_0 ds \\ & + \int_0^\infty \int_0^\infty \beta(\frac{s^\alpha}{\alpha})^{\beta-1} e^{-(\lambda(\frac{s^\alpha}{\alpha})^\beta} T((\frac{s^\alpha}{\alpha})^\beta) e^{-(\lambda(\frac{t^\alpha}{\alpha}))} f(t,\ u_t) s^{\alpha-1} t^{\alpha-1} ds dt. \\ & =: I_1 + I_2 \end{aligned}$$

We consider the following one-sided stable probability density in [16, 22]:

$$\psi_\beta(\theta) = \frac{1}{\pi}\sum_{k=1}^\infty \frac{(-1)^{k-1}\theta^{-\alpha k-1}}{k!}\Gamma(\beta k + 1)\sin(k\pi\beta)$$

whose integration is given by

$$\int_0^\infty e^{-\lambda\theta}\psi_\beta(\theta)d\theta = e^{-\lambda^\beta}, \quad \beta\in(0,1)$$

.

Using this relation, we get

$$\begin{aligned}
I_1 &= \int_0^\infty -\frac{1}{\lambda}\frac{d}{ds}(e^{-(\lambda(\frac{s^\alpha}{\alpha})^\beta})T((\frac{s^\alpha}{\alpha})^\beta)u_0 ds \\
&= \int_0^\infty\int_0^\infty \theta\psi_\beta(\theta)e^{-\lambda(\frac{s^\alpha}{\alpha})\theta}T((\frac{s^\alpha}{\alpha}))^\beta)u_0 s^{\alpha-1}d\theta ds \\
&= \int_0^\infty e^{-\lambda(\frac{s^\alpha}{\alpha})}(\int_0^\infty \psi_\beta(\theta)T(\frac{(\frac{s^\alpha}{\alpha})^\beta}{\theta^\beta})u_0 d\theta)s^{\alpha-1}ds
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^\infty\int_0^\infty \beta(\frac{s^\alpha}{\alpha})^{\beta-1}e^{-\lambda(\frac{s^\alpha}{\alpha})^\beta}T((\frac{s^\alpha}{\alpha})^\beta)e^{-(\lambda(\frac{t^\alpha}{\alpha}))}f(t,\ u_t)s^{\alpha-1}t^{\alpha-1}dsdt. \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta(\frac{s^\alpha}{\alpha})^{\beta-1}\psi_\beta(\theta)e^{-\lambda(\frac{s^\alpha}{\alpha})\theta}T((\frac{s^\alpha}{\alpha})^\beta)e^{-(\lambda(\frac{t^\alpha}{\alpha}))}f(t,\ u_t)s^{\alpha-1}t^{\alpha-1}d\theta dsdt. \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta e^{-\lambda(\frac{t^\alpha}{\alpha}+\frac{s^\alpha}{\alpha})}\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)T(\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta})f(t,\ u_t)t^{\alpha-1}s^{\alpha-1}d\theta dsdt. \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \beta e^{-\lambda(\frac{t^\alpha}{\alpha}+\frac{s^\alpha}{\alpha})}\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)T(\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta})f((\nu^\alpha-s^\alpha)^{\frac{1}{\alpha}},u((\nu^\alpha-s^\alpha)^{\frac{1}{\alpha}})\nu^{\alpha-1}s^{\alpha-1}d\theta dsd\nu. \\
&= \int_0^\infty\int_0^\nu\int_0^\infty \beta e^{-\lambda(\frac{t^\alpha}{\alpha}+\frac{s^\alpha}{\alpha})}\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)T(\frac{(\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta})f((\nu^\alpha-s^\alpha)^{\frac{1}{\alpha}},u((\nu^\alpha-s^\alpha)^{\frac{1}{\alpha}})\nu^{\alpha-1}s^{\alpha-1}d\theta d\nu ds. \\
&= \int_0^\infty e^{-\lambda(\frac{t^\alpha}{\alpha})}\times\left[\int_0^t\int_0^\infty \beta\psi_\beta(\theta)\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}T\left(\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}\right)f(s,u_s)s^{\alpha-1}d\theta ds)\right]t^{\alpha-1}dt
\end{aligned}$$

Therfore we obtain

$$\begin{aligned}
\hat{u}_\alpha(\lambda) &= \int_0^\infty e^{-\lambda(\frac{s^\alpha}{\alpha})}\left(\int_0^\infty \psi_\beta(\theta)T(\frac{(\frac{t^\alpha}{\alpha})^\beta}{\theta^\beta})u_0 d\theta\right)s^{\alpha-1}ds \\
&\quad + \int_0^\infty e^{-\lambda(\frac{t^\alpha}{\alpha})}\times\left[\int_0^t\int_0^\infty \beta\psi_\beta(\theta)\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}T\left(\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}\right)f(s,u_s)s^{\alpha-1}d\theta ds)\right]t^{\alpha-1}dt
\end{aligned}$$

Using the inverse conformable Laplace transform, we get

$$\begin{aligned}
u(t) &= \int_0^\infty \psi_\beta(\theta)T(\frac{(\frac{t^\alpha}{\alpha})^\beta}{\theta^\beta})u_0 d\theta \\
&\quad + \int_0^t\int_0^\infty \beta\psi_\beta(\theta)\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}}{\theta^\beta}T\left(\frac{(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^\beta}{\theta^\beta}\right)f(s,u_s)s^{\alpha-1}d\theta ds
\end{aligned}$$

Therefore

$$\begin{aligned}
u(t) &= \int_0^\infty \mathcal{M}_\beta(\theta)T((\frac{t^\alpha}{\alpha})^\beta\theta)u_0 d\theta \\
&\quad + \int_0^t\int_0^\infty \beta\theta\mathcal{M}_\beta(\theta)(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}T\left((\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^\beta\theta\right)f(s,u_s)s^{\alpha-1}d\theta ds
\end{aligned}$$

where $\mathcal{M}_\beta(\theta) = \dfrac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\psi_\beta(\theta^{-\frac{1}{\beta}})$ is the probability density function of Mainardi defined on $(0,\ \infty)$ [16, 22]. □

For any $x \in E$, we define two operators $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ for $0 \le s \le t \le T$ by

$$S_{\alpha,\beta}(t,s)x = \int_0^\infty \mathcal{M}_\beta(\theta)T((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta\theta)xd\theta$$

and

$$P_{\alpha,\beta}(t,s)x = \beta\int_0^\infty \theta\mathcal{M}_\beta T\left((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta\theta\right)xd\theta$$

□

Now, we can introduce the following definition.

Definition 2.2. We say that $u \in E$ is an integral solution of the delay Cauchy problem (1.1) if:

$$\begin{cases} u(t) = S_{\alpha,\beta}(t,0)\left(\varphi(0) - \phi\left(u_{t_1}, u_{t_2}, \dots, u_{t_n}\right)(0) - g(0,u_0)\right) + g(t,u_t) \\ \qquad + \displaystyle\int_0^t (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} AP_{\alpha,\beta}(t,s)g(s,\ u_s)s^{\alpha-1}ds. \\ \qquad + \displaystyle\int_0^t (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} P_{\alpha,\beta}(t,s)f(s,\ u_s)s^{\alpha-1}ds., \quad t \in [0,a]. \\ u_0(\tau) + \phi\left(u_{t_1}, u_{t_2}, \dots, u_{t_n}\right)(\tau) = \varphi(\tau), \quad \tau \in [-r, 0] \end{cases} \tag{2.3}$$

Where, $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ are two operators defined by, for all $x \in E$ and $0 \le s \le t \le T$,

$$S_{\alpha,\beta}(t,s)x = \int_0^\infty \mathcal{M}_\beta(\theta)T((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta\theta)xd\theta$$

and

$$P_{\alpha,\beta}(t,s)x = \int_0^\infty \beta\theta\mathcal{M}_\beta(\theta)T\left((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta\theta\right)xd\theta$$

Lemma 2.3. *The operators $S_{\alpha,\beta}$ and $P_{\alpha,\beta}$ have the following properties*

(a). *For any $t \ge s \ge 0$, fixed $S_{\alpha,\beta}$ and $P_{\alpha,\beta}$ are bounded linear operators, i.e. for any $x \in E$, $\|S_{\alpha,\beta}(t,s)(x)\| \le M\|x\|$ and $\|P_{\alpha,\beta}(t,s)(x)\| \le \dfrac{M}{\Gamma(\beta)}\|x\|$.*

(b). *The operators $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ are strongly continuous for all $t \ge s \ge 0$, such that, for all $x \in E$ and $0 \le s \le t_1 < t_2 \le T$ we have, $\|S_\alpha(t_2,s)x - S_{\alpha,\beta}(t_1,s)x\| \to 0$ and $\|P_{\alpha,\beta}(t_2,s)x - P_{\alpha,\beta}(t_1,s)x\| \to 0$ when $t_1 \to t_2$.*

(c). *If $T(t)$ is a compact operator for every $t > 0$, then $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ are compact for all $t, s > 0$.*

(d). *If $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ are compact and strongly continuous semigroups of bounded linear operators for $t, s > 0$, then $S_{\alpha,\beta}(t,s)$ and $P_{\alpha,\beta}(t,s)$ are continuous in the uniform operator topology.*

(e) *For $t > 0$, $\frac{d}{dt}\left(S_{\alpha,\beta}(t,0)x\right) = t^{\alpha-1}AP_{\alpha,\beta}(t,0)x$*

2.1. **Existence.** To obtain the existence of mild solutions, we will need the foollowing assumptions:

(H_1) : The function $f(t,\cdot):\mathcal{C}\to E$ is continuous for almost all $t\in[0,a]$, and for every each $u\in\mathcal{C}$, the function $f(\cdot,u):[0,a]\to E$ is strongly measurable,

(H_2) : for every $m>0$ there exists a function $\varepsilon_m\in L^{\infty}([0,a],\mathbb{R}^+)$ such that: $\sup_{\|u\|\leq m}\|f(t,u)\|\leq\varepsilon_m(t)$, for almost all $t\in[0,a]$

(H_3) : there exists a constant $R_0>0$ such that: $\|\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})-\phi(v_{t_1},v_{t_2},\ldots,v_{t_n})\|\leq R_0\|u-v\|$, for $u,v\in C([-r,a],E)$,

(H_4) : $g:[0,a]\times\mathcal{C}\to E$ is a continuous function and there is $\eta\in(0,1)$; $K_0,K_1>0$ such that $g\in D(A^{\eta})$ and for any $u,v\in\mathcal{C},t\in[0,a]$, the function $A^{\eta}g(\cdot,u)$ is strongly measurable and $A^{\eta}g(t,\cdot)$ satisfies the Lipschitz condition

$$|A^{\eta}g(t,u)-A^{\eta}g(t,v)|\leq K_0\|u-v\|$$

and inequality:

$$|A^{\eta}g(t,u)|\leq K_1(\|u\|+1)$$

Theorem 2.4. *If $T(t)_{t>0}$ is compact and $(H_1)-(H_4)$ are satisfied, then the nonlocal Cauchy problem with delay (1.1) has a mild solution provided that:*

$$R_i:=MR+(M+1)\left|A^{-\eta}\right|K_i+\frac{C_{1-\eta}K_i}{\beta\eta\alpha^{\eta}}a^{\alpha\beta\eta}<1,\quad i\in\{0;1\}\tag{2.4}$$

Proof. We prove the existence of mild solution for the nonlocal Cauchy problem (1.1) by applying Krasnoselskii's fixed point theorem.

First, we define the function $w\in C([-r,a],E)$ such that $|w(t)|\equiv 0, t\in[-r,a]$.for any positive constant m and $u\in B_m=\{u\in C([-r,a],E):\|u\|\ |\leq m\}$, and according to the hypothesis (H_3), for $t\in[0,a]$, we have:

$$\begin{aligned}|S_{\alpha,\beta}(t,0)(\varphi(0)-\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})(0)-g(0,u_0))|&\leq M(|\varphi(0)|+R\|u-w\|)\\&+M|\phi(w_{t_1},w_{t_2},\ldots,w_{t_n})|\\&+M\left|A^{-\eta}A^{\eta}g(0,u_0)\right|\\&\leq M[\|\varphi\|+Rm+\|\phi(w_{t_1},w_{t_2},\ldots,w_{t_n})\|]\\&+M\left|A^{-\eta}\right|K_1(m+1)\end{aligned}\tag{2.5}$$

$P_{\alpha,\beta}(t,s)$ is a strongly continuous semigroup of bounded linear operator for $t,s>0$, then, $s^{\alpha-1}AP_{\alpha,\beta}\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)g(s,u_s)$ is continuous on $[0,a]$.

For every $u\in B_m;t\in[0,t]$, using the properties (a), (b) and the hypothesis (H_4) we have:

$$\begin{aligned}&\int_0^t\left|(\frac{t^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha})^{\beta-1}AP_{\alpha,\beta}(t,s)g(s,\ u_s)s^{\alpha-1}\right|\mathrm{d}s=\int_0^t\left|(\frac{t^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha})^{\beta-1}A^{1-\eta}P_{\alpha,\beta}(t,s)A^{\eta}g(s,\ u_s)s^{\alpha-1}\right|ds\\&\leq\int_0^t(\frac{t^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha})^{\beta-1}s^{\alpha-1}\frac{\beta\alpha^{\beta(1-\eta)}C_{(1-\eta)}}{(t^{\alpha}-s^{\alpha})^{\beta(1-\eta)}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}K_1(m+1)ds\\&=K_1(m+1)C_{1-\eta}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}\int_0^t\frac{\beta\alpha^{1-\beta\eta}s^{\alpha-1}}{(t^{\alpha}-s^{\alpha})^{(1-\beta\eta)}}ds,\end{aligned}$$

then

$$\int_0^t \left|(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}AP_{\alpha,\beta}(t,s)g(s,\ u_s)s^{\alpha-1}\right|\mathrm{d}s \leq \frac{C_{1-\eta}K_1(m+1)}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta} \tag{2.6}$$

therefore : $\left|(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}s^{\alpha-1}AP_{\alpha,\beta}(t,s)g(s,\ u_s)\right|$ is integrable in the Lebesgue sense for all $s\in[0,t]$ and $t\in[0,a]$

For every $m>0$, we define two operators (Q_1) et (Q_2) on B_m as follows:

$$\begin{cases}(Q_1u)(t)=S_{\alpha,\beta}(t,0)\left(\varphi(0)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(0)-g(0,u_0)\right)+g(t,u_t)\\ \qquad\qquad +\displaystyle\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}AP_{\alpha,\beta}(t,s)g(s,\ u_s)s^{\alpha-1}ds; \qquad t\in[0,a]\\ (Q_1u)(\tau)=\varphi(\tau)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(\tau); \qquad \tau\in[-r,0]\end{cases} \tag{2.7}$$

and

$$\begin{cases}(Q_2u)(t)=\displaystyle\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}P_{\alpha,\beta}(t,s)f(s,\ u_s)s^{\alpha-1}ds; \qquad t\in[0,a]\\ (Q_2u)(\tau)=0; \qquad \tau\in[-r,0]\end{cases} \tag{2.8}$$

Then, u is a mild solution of (1.1) if and only if $u=Q_1u+Q_2u$ has a solution $u\in B_m$. So we determine m_0 such that Q_1+Q_2 has a fixed point in B_{m_0}.

We choose m_0 such that

$$m_0\geq\frac{\Lambda}{1-R_1} \tag{2.9}$$

where,

$$\Lambda=M\left[\|\varphi\|+\left\|\phi\left(w_{t_1},w_{t_2},\ldots,w_{t_n}\right)\right\|\right]+(M+1)\left|A^{-\eta}\right|K_1+\frac{M\left\|\epsilon_{m_0}\right\|_{L^\infty([0,a];\mathbb{R}^+)}}{\alpha^\beta\Gamma(\beta+1)}a^{\alpha\beta}$$

We show that Q_1+Q_2 has a fixed point, by using Krasnoselskii's Fixed Point Theorem, in three steps

Step 1: $Q_1u+Q_2v\in B_{m_0}$, for any $u,v\in B_{m_0}$.

Clearly, for every pair $u;v\in B_{m_0}$, $(Q_1u)(t)$ and $(Q_2v)(t)$ are continuous in $t\in[-r,a]$.

Let $u,v\in B_{m_0}$ and $t\in[0,a]$,according to (H_1), (H_2) and (3.4),(3.5) we get:

$$\begin{aligned}\left|(Q_1u)(t)+(Q_2v)(t)\right| &\leq\left|S_{\alpha,\beta}(t,0)\right|\left|\left(\varphi(0)-\phi\left(u_{t_1},u_{t_2},\ldots,u_{t_n}\right)(0)-g(0,u_0)\right)\right|\\ &+\left|\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}AP_{\alpha,\beta}(t,s)g(s,\ u_s)s^{\alpha-1}ds\right|\\ &+\left|g\left(t,u_t\right)\right|+\left|\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}P_{\alpha,\beta}(t,s)f(s,\ u_s)s^{\alpha-1}ds\right|\\ &\leq M\left[\|\varphi\|+Rm_0+\left\|\phi\left(w_{t_1},w_{t_2},\ldots,w_{t_n}\right)\right\|\right]+(M+1)\left|A^{-\eta}\right|K_1\left(m_0+1\right)\\ &+\frac{M}{\alpha^\beta\Gamma(\beta+1)}\left|\epsilon_{m_0}\right|_{L^\infty([0,a];\mathbb{R}^+)}a^{\alpha\beta}+\frac{C_{1-\eta}K_1(m_0+1)}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta}\\ &\leq m_0.\end{aligned}$$

For $\tau \in [-r,0]$

$$\begin{aligned}|(Q_1u)(\tau)+(Q_2v)(\tau)| &\le M\left[\|\varphi\|+Rm_0+\|\phi(w_{t_1},w_{t_2},\ldots,w_{t_n})\|\right]\\ &\le m_0\end{aligned}$$

Therefore, $Q_1u+Q_2v\in B_{m_0}$, for every $u,v\in B_{m_0}$
Step 2: Q_1 is contraction on B_{m_0} .
Let $u,v\in B_{m_0}$ and $t\in[0,a]$,

$$\begin{aligned}|(Q_1u)(t)-(Q_1v)(t)| &\le \left|S_{\alpha,\beta}(t,0)\left[\phi(u_{t_1},u_{t_2},\ldots,u_{t_n})(0)-\phi(v_{t_1},v_{t_2},\ldots,v_{t_n})(0)\right]\right|\\ &\quad+\left|S_{\alpha,\beta}(t,0)\left[g(0,u_0)-g(0,v_0)\right]\right|+|g(t,u_t)-g(t,v_t)|\\ &\quad+\int_0^t\left|A^{1-\eta}P_{\alpha,\beta}(t,s)\left[A^\eta g(s,u_s)-A^\eta g(s,v_s)\right]\right|(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}s^{\alpha-1}ds\\ &\le MR\|u-v\|+(M+1)\left|A^{-\eta}\right|K_0\|u-v\|+\frac{C_{1-\eta}K_0}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta}\|u-v\|\\ &=\left(MR+(M+1)\left|A^{-\beta}\right|K_0+\frac{C_{1-\eta}K_0}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta}\right)\|u-v\|\end{aligned}$$

For $\tau\in[-r,0]$ we have,

$$|(Q_1u)(\tau)-(Q_1v)(\tau)|\le MR\|u-v\|,\quad \tau\in[-r,0]$$

then, $\|Q_1u-Q_1v\| \le \left(MR+(M+1)|A^{-\eta}|K_0+\frac{C_{1-\eta}K_0}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta}\right)\|u-v\|$.
Therefore according to (2.4), Q_1 is a contraction.
Step 3 : We show that, (Q_2) is completely continuous .
First, we show that Q_2 is continuous on B_{m_0}.
Let $\{u^n\}\subseteq B_{m_0}$ where $u^n\to u$ on B_{m_0}. then: $u_t^n\to u_t$ for $t\in[0,a]$ according to (H_1),we have:

$$\xi(s,u_s^n)\to\xi(s,u_s),\, s\in[0,a],\qquad \text{as } n\to\infty.$$

Since $s^{\alpha-1}|\xi(s,u_s^n)-\xi(s,u_s)|\le 2s^{\alpha-1}\epsilon_{m_0}(s)$, according to the dominated convergence theorem, we have:

$$\begin{aligned}|(Q_2u^n)(t)-(Q_2u)(t)| &= \left|\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}s^{\alpha-1}P_{\alpha,\beta}(t,s)\left[f(s,u_s^n)-f(s,u_s)\right]\mathrm{d}s\right|\\ &\le\frac{M}{\Gamma(\beta)}\int_0^t(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha})^{\beta-1}s^{\alpha-1}\left|f(s,u_s^n)-f(s,u_s)\right|\mathrm{d}s\to0,\ \text{as}n\to\infty\end{aligned}$$

which implies Q_2 is continuous.

Next, we will show that $\{Q_2 u, u \in B_{m_0}\}$ is relatively compact,
For $u \in B_{m_0}$ and $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$. we have :

$$\begin{aligned}
|(Q_2 u)(t_2) - (Q_2 u)(t_1)| &= \int_{t_1}^{t_2} (\frac{t_2^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} s^{\alpha-1} P_{\alpha,\beta}(t_2, s) f(s, u_s) \mathrm{d}s. \\
&+ \int_0^{t_1} (\frac{t_2^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} s^{\alpha-1} \left| P_{\alpha,\beta}(t_2, s) f(s, u_s) - P_{\alpha,\beta}(t_1, s) f(s, u_s) \right| \mathrm{d}s \\
&+ \int_0^{t_1} \left[(\frac{t_2^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} - (\frac{t_1^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} \right] s^{\alpha-1} \left| P_{\alpha,\beta}(t_1, s) f(s, u_s) \right| \mathrm{d}s. \\
&= |J_1 + J_2 + J_3|
\end{aligned}$$

Using the assumption (H_3) and property (a) and (b) of operator $P_{\alpha,\beta}$ of Lemma (2.3), we obtain that $|J_1|, |J_2|\, et\, |J_3| \longrightarrow 0$ as $t_1 \longrightarrow t_2$.
Then $\{Q_2 u, u \in B_{m_0}\}$ is equicontinuous , and uniformly bounded.
Next we show that for every $t \in [-r, a], V(t) = \{(P_2 u)(t), u \in B_{m_0}\}$ is relatively compact on E.
For $t \in [-r, 0], V(t)$ is relatively compact on E.
For $0 < t \leq a$, and for every $\varepsilon \in (0, t)$, we define $Q_{2,\varepsilon}$ on B_{m_0}:

$$\begin{aligned}
(Q_{2,\varepsilon,\delta} u)(t) &= \int_0^{t-\varepsilon} (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} s^{\alpha-1} P_{\alpha,\beta}(t, s) f(s, u_s) \mathrm{d}s \\
&= \beta \int_0^{t-\epsilon} \int_\delta^\infty \theta \mathcal{M}_\beta(\theta) (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} T\left((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta \theta \right) f(s, u_s) s^{\alpha-1} d\theta ds \\
&= \beta T(\epsilon^\beta \delta) \int_0^{t-\epsilon} \int_\delta^\infty \theta \mathcal{M}_\beta(\theta) (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} T\left((\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^\beta \theta - \epsilon^\beta \delta \right) f(s, u_s) s^{\alpha-1} d\theta ds.
\end{aligned}$$

Since $T\left(\frac{\varepsilon^\alpha}{\alpha}\right)$ is compact, then $V_\varepsilon(t) = \{(Q_{2,\varepsilon,\delta} u)(t), u \in B_{m_0}\}$ is relatively compact.
And, for every $\varepsilon > 0$ et $\delta > 0$, by using (H_2) and proprety (a) of Lemma(2.3) of $P_{\alpha,\beta}$ we obtain,

$$|(Q_{2,\varepsilon,\delta} u)(t) - (Q_{2,\varepsilon,\delta} u)(t)| \leq \frac{M \|\epsilon_{m_0}\|_{L^\infty}}{\Gamma(\beta+1)} (\frac{t^\alpha}{\alpha} - \frac{(t-\epsilon)^\alpha}{\alpha})^\beta$$

So there are relatively compact sets close to the set $V(t), t > 0$. Hence the set $V(t), t > 0$ is also relatively compact.
According to the theorem of Ascoli-Arzela, $\{Q_2 u, u \in B_{m_0}\}$ is relatively compact.
Therefore, Q_2 is completely continuous .
Finally, the conditions of the Krasnoselskii theorem are verified, then $Q_1 u + Q_2 u$ has a fixed point, and the nonlocal neutral evolution problem(1.1) has a mild solution. □

2.2. **Existence and Uniqueness of mild solution.** To show the existence and uniqueness result for the non-local Cauchy problem (1.1), we apply the Banach contraction principle and we add the following assumptions:

(H_5) : $f(t, u_t)$ is strongly measurable for every $u \in C([-r, a], B_m)$ and almost all $t \in [0, a]$.

(H_6) : there exists a function $\rho \in L^\infty([0,a],\mathbb{R}^+)$ such that; for every $u, v \in C([-r,a],B_m)$, where m is a positive constant.we have:

$$|f(t,u_t) - f(t,v_t)| \leq \rho(t)\|u-v\|, t \in [0,a]$$

,

Theorem 2.5. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_6)$ *are satisfied, then the nonlocal Cauchy problem with delay (1.1) has a unique mild solution provided that:*

$$R_2 := MR + (M+1)\left|A^{-\eta}\right| K_0 + \frac{C_{1-\eta}K_0}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta} + \frac{MM_1 a^{\alpha\beta}}{\alpha^\beta\Gamma(\beta+1)} < 1 \quad (2.10)$$

where, $M_1 = \|\rho\|_{L^\infty([0,a];\mathbb{R}^+)}$.

Proof. We define the operator Q on B_m by:

$$\begin{cases} (Qu)(t) = S_{\alpha,\beta}(t,0)\left(\varphi(0) - \phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)(0) - g(0,u_0)\right) + g(t,u_t) \\ \qquad + \int_0^t (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} AP_{\alpha,\beta}(t,s) g(s,\ u_s) s^{\alpha-1} ds. \\ \qquad + \int_0^t (\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} P_{\alpha,\beta}(t,s) f(s,\ u_s) s^{\alpha-1} ds, \quad t \in [0,a] \\ \\ (Qu)(\tau) = \varphi(\tau) - \left(\phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)\right)(\tau), \qquad \tau \in [-r,0]. \end{cases} \quad (2.11)$$

it suffices to prove that Q has a unique fixed point on B_{m_0}, where m_0 is defined by (2.9).
For every $u, v \in B_{m_0}$ and $t \in [0;a]$, according to $(H_3), (H_4)$ and (H_6), and applying the Hölder's inequality, we have:

$$\begin{aligned} |(Qu)(t) - (Qv)(t)| &\leq \left|S_{\alpha,\beta}(t,0)\left[\left(\phi\left(u_{t_1}, u_{t_2}, \ldots, u_{t_n}\right)\right)(0) - \left(\phi\left(v_{t_1}, v_{t_2}, \ldots, v_{t_n}\right)\right)(0)\right]\right| \\ &+ \left|S_{\alpha,\beta}(t,0)\left[g(0,u_0) - g(0,v_0)\right]\right| + |g(t,u_t) - g(t,v_t)| \\ &+ \int_0^t \left| A^{1-\eta} P_{\alpha,\beta}(t,s)\left[A^\eta g(s,u_s) - A^\eta g(s,v_s)\right](\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} s^{\alpha-1}\right| ds \\ &+ \int_0^t \left|(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha})^{\beta-1} s^{\alpha-1} P_{\alpha,\beta}(t,s)\left[f(s,u_s) - f(s,v_s)\right]\right| \mathrm{d}s \\ &\leq MR\|u-v\| + (M+1)\left|A^{-\eta}\right| K_0 \|u-v\| + \frac{C_{1-\eta}K}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta}\|u-v\| \\ &+ \frac{M}{\alpha^\beta\Gamma(\beta+1)}\|\rho\|_{L^\infty([0,a];\mathbb{R}^+)}\|u-v\| \end{aligned}$$

Then,

$$|(Qu)(t) - (Qv)(t)| \leq \left(MR + (M+1)\left|A^{-\eta}\right| K_0 + \frac{C_{1-\eta}K_0}{\beta\eta\alpha^{\beta\eta}}\frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}a^{\alpha\beta\eta} + \frac{MM_1 a^{\alpha\beta}}{\alpha^\beta\Gamma(\beta+1)}\right)\|u-v\|$$

which means that Q is a contraction according to (2.10). By applying the Banach contraction mapping principle, Q has a unique fixed point on B_{m_0}, the proof is complete.

2.3. **Application.** Let $E = L^2([0,\pi], R)$. Consider the following fractional partial differential equations.

$$\begin{cases} \partial_t^{\alpha,\beta}\left(u(t,x) - \int_0^\pi N(x,y)u_t(\tau,y)\mathrm{d}y\right) = \partial_x^2 u(t,x) + \partial_x G\left(t, u_t(\tau,x)\right), \quad t \in (0,a] \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,a] \\ u(\tau,x) + \sum_{i=0}^n \int_0^\pi b(x,y)u_{t_i}(\tau,y)\mathrm{d}y = (\varphi(\tau))(x), \quad \tau \in [-r,0], \end{cases} \tag{2.12}$$

Where: $\partial_t^{\alpha,\beta}$ is a conformable Caputo fractional partial derivative of order β and of type α such that $0<\alpha<1, 0<\beta<1, a>0$, and $t \in [0,a]$.
We define : $E = L^2([0,\pi], R)$,

- G is a given function,
- $0 < t_0 < t_1 < \cdots < t_n < a, \varphi \in C([-r,0], E)$, that is $\varphi(\tau) \in E$.
- $b(x,y) \in L^2([0,\pi]\times[0,\pi], R)$ and $u_t(\tau,z) = u(t+\tau,x), t \in [0,a], \tau \in [-r,0]$.

We take: $A = \partial_x^2(.)$ with the domain:

$$D(A) = \{v(\cdot) \in E : v, v'\text{absolutly continuous}, v'' \in E, v(0) = v(\pi) = 0\}$$

Then A generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint
The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}v = \sum_{n=1}^{\infty} n\langle v, u_n\rangle u_n$$

on the space $D\left(A^{\frac{1}{2}}\right) = \{v(\cdot) \in E, \sum_{n=1}^\infty n\langle v,u_n\rangle u_n \in E\}$. The problem (2.12) can take the form in E

$$\begin{cases} D_t^{\alpha,\beta}\left(U(t) - g\left(t, U_t\right)\right) = AU(t) + f\left(t, U_t\right) \quad t \in (0,a] \\ U_0(\tau) + \left(\phi\left(U_{t_1}, \ldots, U_{t_n}\right)\right)(\tau) = \varphi(\tau) \quad \tau \in [-r,0], \end{cases} \tag{2.13}$$

where $U_t = u_t(\tau,\cdot)$, that is $(U(t+\tau))(x) = u(t+\tau,x), t \in [0,a], x \in [0,\pi]$.
The function $g : [0,a]\times\mathcal{C} \to E$ is given by

$$\left(g\left(t, U_t\right)\right)(x) = \int_0^\pi N(x,y)u_t(\tau,y)\mathrm{d}y.$$

Let $\left(N_h v\right)(x) = \int_0^\pi N(x,y)v(y)\mathrm{d}y$, for $v \in E = L^2([0,\pi],R), x \in [0,\pi]$. The function $f : [0,a]\times\mathcal{C} \to E$ is given by

$$\left(f\left(t, U_t\right)\right)(x) = \partial_x G\left(t, u_t(\tau,x)\right),$$

and the function $\phi : \mathcal{C}^n \to \mathcal{C}$ is given by

$$\left(\phi\left(U_{t_1}, \ldots, U_{t_n}\right)\right)(\tau) = \sum_{i=0}^n b_\phi U_{t_i}(\tau)$$

where $\left(b_\phi v\right)(x) = \int_0^\pi b(x,y)v(y)\mathrm{d}y$, for $v \in E = L^2([0,\pi],R), x \in [0,\pi]$. For $\alpha = 1/2, \beta = 1/2$ and $f\left(t, U_t\right) = \frac{1}{t^{1/3}}\sin U_t$, then $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_5)$ and (H_6) are satisfied. Furthermore, for $v_1, v_2 \in E$, we have

$$\begin{aligned}\|b_\phi v_1 - b_\phi v_2\|_{L^2[0,\pi]} &= \left(\int_0^\pi \left(\int_0^\pi b(x,y)\left(v_1(y)-v_2(y)\right)\mathrm{d}y\right)^2 \mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_0^\pi \left(\int_0^\pi b^2(x,y)\mathrm{d}y \int_0^\pi \left[v_1(y)-v_2(y)\right]^2 \mathrm{d}y\right)\mathrm{d}x\right)^{1/2} \\ &= \left(\int_0^\pi \int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}z\right)^{1/2} \|v_1 - v_2\|_{L^2[0,\pi]}\end{aligned}$$

So, $R_0 = (n+1)\left[\int_0^\pi \int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}x\right]^{1/2}$, then (H$_3$) is satisfied.

Moreover, we assume that the following conditions hold. (c) The function $N(x,y), x, y \in [0,\pi]$ is measurable and

$$\int_0^\pi \int_0^\pi N^2(x,y)\mathrm{d}y\ \mathrm{d}x < \infty$$

(d) the function $\partial_x N(z,y)$ is measurable, $N(0,y) = N(\pi,y) = 0$, and let

$$\bar{N} = \left(\int_0^\pi \int_0^\pi \left(\partial_x N(x,y)\right)^2 \ \mathrm{d}y\mathrm{dx}\right)^{\frac{1}{2}} < \infty$$

we can take $\eta = \frac{1}{2}$ Then (H_4) is satisfied. We conclude that Eq (4.2) has a mild solution provided that:

$$R_0 + \left(2 + 2\sqrt{2}C_{1/2}\right)\sup_{t\in[0,1]}\left(\int_0^\pi \left(\int_0^\pi \partial_x N(x,y)u_t(\tau,y)\right)^2 \mathrm{d}y\right) < 1 \qquad (2.14)$$

where $R_0 = (n+1)\left[\int_0^\pi \int_0^\pi b^2(x,y)\mathrm{d}y\ \mathrm{d}x\right]^{1/2}$. Hence, by applying Theorem 3.2, the system (4.1) admits a unique mild solution.

3. Conclusion

In this article, we have considered a class of conformable Caputo fractional neutral evolution equations with nonlocal-delay conditions. First, we have introduced a mild solution method by using the fractional operators $P_{\alpha,\beta}$ and $S_{\alpha,\beta}$, along with the Conformable-Laplace transform, the existence and uniqueness results are established by using Krasnoselskii fixed point theorem, and Banach fixed point theorem. Finally, the investigation of the result has been illustrated by providing suitable example.

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