

FIXED POINT RESULTS TO TVS-CONE B-METRIC SPACES

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ABSTRACT. In this paper we introduce the concept of tvs-cone b-metric space over solid cone. Our results improve, generalize, extend, unify, enrich and complement recent fixed point results established by H. Huang and S. Xu, M. P. Stanić et al., A. S. Cvetković et al. and M. H. Shah et al. The examples are given to illustrate the usability of our results.

1. INTRODUCTION

In this paper, by using Minkowski functional and convex envelope, we improve and generalize several results of [18], [27], [29], [46] and [50]. We must mention here that non-convex analysis, especially ordered normed spaces, normal cones and topical functions ([6], [22], [24] [41], [42]) have some applications in optimization theory. For more details see [6], [41], [51], [53]. In these cases an order is introduced by using vector space cones. Huang and Zang [22] used this approach and they have replaced the real numbers $\mathbb{R}$ by an ordered Banach space E and defined a cone metric space. For fixed point results in cone metric and b-metric space see [1]-[5], [7]-[40], [42]-[54].

The following definitions and results will be needed in the sequel.

Let E be a real locally convex Hausdorff space with zero vector θ . A proper nonempty and closed subset P of E is called a cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{\theta\}$. We shall always assume that the cone P has a nonempty interior $\text{int}P$ (such cones are called solid).

Given a cone $P \subset E$, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P). The pair (E, P) is an ordered topological vector space.

For a pair of elements x, y in E such that $x \preceq y$, put $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset A of E is said to be order-convex if $[x, y] \subset A$, whenever $x, y \in A$ and $x \preceq y$.

Ordered topological vector space (E, P) is order-convex if it has a base neighborhoods of θ consisting of order-convex subsets. In this case, the cone P is said to be normal. If E is a normed space, this condition means that the unit ball is

Date: Received: Jul 5, 2013; Accepted: Jul 24, 2013.

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2010 *Mathematics Subject Classification.* Primary 54H25, Secondary 47H10.

Key words and phrases. tvs-cone b-metric space; common fixed point; coincidence point; weakly compatible, Minkowski functional.

order-convex, which is equivalent to the condition that there is a number k such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leq k \|y\|$ (for details see [51]).

Let E be a real linear space, A be a subset of E , with coA we denote the convex envelope (convex hull) of A . For some details see [14] and [54].

If V is an absolutely convex and absorbing subset of a tvs E , its Minkowski functional q_V is defined by

$$E \ni x \mapsto q_V(x) = \inf \{ \lambda > 0 : x \in \lambda V \}. \quad (1.1)$$

It is a semi-norm on E , that is, $q_V(x+y) \leq q_V(x) + q_V(y)$ for all $x, y \in E$ and $q_V(\lambda x) = |\lambda| q_V(x)$ for $x \in E, \lambda$ a scalar. If V is an absolutely convex neighborhood of θ in E , then q_V is continuous and

$$\{x \in E : q_V(x) < 1\} = \text{int}V \subset V \subset \overline{V} = \{x \in E : q_V(x) \leq 1\}. \quad (1.2)$$

Note also that $q_V(coA) \subset co(q_V(A))$ for arbitrary $A \subset E$.

Let now (E, P) be an ordered tvs and let $e \in \text{int}P$. Then $[-e, e] = (P - e) \cap (e - P) = \{z \in E : -e \preceq z \preceq e\}$ is an absolutely convex neighborhood of θ ; its Minkowski functional $q_{[-e, e]}$ will be denoted by q_e . It is an increasing function on P . If the cone P is solid and normal, q_e is a norm in E .

Definition 1.1. Let X be a non-empty set and E a real locally convex Hausdorff space with solid cone P . A vector-valued function $d : X \times X \rightarrow E$ is said to be a *tv-cone b-metric function* on X with the constant $s \geq 1$ (in [18] and [50] $K \geq 1$) if the following conditions are satisfied:

- (d₁) $\theta \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$;
- (d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d₃) $d(x, z) \preceq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

The pair (X, d) is called *the tv-cone b-metric space or tv-cone metric type space*. Observe that if $s = 1$ then the ordinary triangle inequality in a tv-cone metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of tv-cone b-metric spaces is effectively larger than that of the ordinary tv-cone metric spaces. That is, every cone metric space is a tv-cone b-metric space, but the converse need not be true. The following examples shows the above remarks.

Example 1.2. Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \rightarrow P$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = \theta$, $x \in X$ and $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, d) is a complete tv-cone b-metric space but the triangle inequality is not satisfied. Indeed, we have that $d(-1, 1) + d(1, 0) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = d(-1, 0)$. It is not hard to verify that $s = \frac{3}{2}$.

Example 1.3. Let $X = \mathbb{N} \cup \{\infty\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$.

Define $d : X \times X \rightarrow E$ by

$$d(x, y) = \begin{cases} (0, 0), & \text{if } x = y, \\ \left(\left| \frac{1}{x} - \frac{1}{y} \right|, \left| \frac{1}{x} - \frac{1}{y} \right| \right), & \text{if } x \text{ and } y \text{ are even or } xy = \infty, \\ (5, 5), & \text{if } x \text{ and } y \text{ are odd and } x \neq y, \\ (2, 2), & \text{otherwise.} \end{cases}$$

Then (X, d) is a tvs-cone b-metric space with the coefficient $s = 3$. But it is not a cone metric spaces since the triangle inequality is not satisfied, Indeed, $(5, 5) = d(1, 3) \succ d(1, 2) + d(2, 3) = (2, 2) + (2, 2) = (4, 4)$.

For the given cone b-metric space (X, d) one can construct a b-metric space (X, D, sk) in sense of [38], [39] and [43] where b-metric $D : X \times X \rightarrow [0, +\infty)$ is given by $D(x, y) = \|d(x, y)\|$.

Example 1.4 Let X be the set of Lebesgue measurable functions $f(x)$ on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$, that is, as in example 3.2. from [38]. Define $D : X \times X \rightarrow [0, +\infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx.$$

Then (X, D) is a b-metric space with the constant $s = 2$. Further, let us take $E = C_{\mathbb{R}}^1[0, 1]$, $\|u\| = \|u\|_{\infty} + \|u'\|_{\infty}$, $u \in E$, $P = \{u \in E : u(t) \geq 0 \text{ on } [0, 1]\}$. Let us define $d : X \times X \rightarrow P$ by $d(f, g)(t) := D(f, g) \cdot \varphi(t)$, for each $f, g \in X$, where φ is a fixed function from P , for instance $\varphi(t) = e^t$. Since $(X, D, 2)$ is a b-metric space so (X, d) is a cone b-metric space with $s = 2$, hence, tvs-cone b-metric space.

Definition 1.5 Let (X, d) be a tvs-cone b-metric space and let be $\{x_n\}$ sequence in E . Then it is said that:

(i) $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$;

(ii) $\{x_n\}$ is Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$;

(iii) (X, d) is complete if every Cauchy sequence is convergent in X .

If the cone P is normal we have the following:

Proposition 1.6 The sequence $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \rightarrow \theta$ as $n \rightarrow \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow \theta$ as $n, m \rightarrow \infty$.

Remark 1.7 Let (X, d) be a tvs-cone b-metric space over ordered real Hausdorff locally convex space E with a cone P . Then the following properties are often used ($a, b, c \in E$):

- (1) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
- (2) If $\theta \preceq u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.

(3) If $c \in \text{int}P$, $a_n \in E$ and $a_n \rightarrow \theta$ in locally convex spaces E , then there exists a natural number n_0 such that $a_n \ll c$ for all $n > n_0$.

(4) If E is a real topological vector space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

The following result was proved in [32].

Theorem 1.8. Let (X, d) be a tvs-cone metric space over a solid cone P and let $e \in \text{int}P$. Let q_e be the corresponding Minkowski functional of $[-e, e]$. Then $d_q = q_e \circ d$ is a (real valued) metric on X . Moreover:

$$1^0 x_n \xrightarrow{d} x \Leftrightarrow x_n \xrightarrow{d_q} x.$$

$2^0 \{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is Cauchy sequence in (X, d_q) .

$3^0 (X, d)$ is complete if and only if (X, d_q) is complete.

Remark 1.9. The previous Theorem is also true if (X, d) is a tvs-cone b-metric space. In this case (X, d_q) is a b-metric space. The proof is similar to the previous one, and so is omitted.

For the sake of convenience we mention the following definition.

Definition 1.10 [1] Let f and g be self mappings of a nonempty set X and $C(f, g) = \{x \in X : fx = gx\}$. The pair (f, g) is called weakly compatible if $f gx = g fx$ for all $x \in C(f, g)$.

Definition 1.11 [1] Let f and g be self mappings of a nonempty set X . If $w = fx = gx$ for some x in X , then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g .

Lemma 1.12 [1] Let f and g be weakly compatible self maps of a nonempty set X . If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .

In [50] M. P. Stanić et al. proved the following results and formulated as Theorem 3.1., 3.2., and Corollary 3.1.

Theorem 1.12. Let (X, d) be a complete CMTS (Cone metric type space), with constant $s \in [1, 2]$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, \frac{1}{2s})$ for all $x, y \in X$ there exists

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (1.3)$$

such that the following inequality $d(Fx, Ty) \preceq \lambda u(x, y)$ holds. Then F and T have a unique common fixed point.

Theorem 1.13. Let (X, d) be a complete CMTS (Cone metric type space), with constant $s > 2$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, \frac{1}{s^2})$ for all $x, y \in X$ there

exists

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (1.4)$$

such that the inequality $d(Fx, Ty) \preceq \lambda u(x, y)$ holds. Then F and T have a unique common fixed point.

Corollary 1.14. Let (X, d) be a complete cone metric space relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, \frac{1}{2})$ for all $x, y \in X$ there exists

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (1.5)$$

such that the inequality $d(Fx, Ty) \preceq \lambda u(x, y)$ holds. Then S and T have a unique common fixed point.

In [29] H. Huang and S. Xu proved the following two theorems and formulated as Theorem 2.1. and 2.3.

Theorem 1.15. Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty) \preceq \lambda d(x, y), \text{ for } x, y \in X, \quad (1.6)$$

where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X . Furthermore, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.16. Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty) \preceq \lambda_1 d(x, Tx) + \lambda_2 d(y, Ty) + \lambda_3 d(x, Ty) + \lambda_4 d(y, Tx), \text{ for } x, y \in X, \quad (1.7)$$

where the constant $\lambda_i \in [0, 1)$ and

$$\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min \left\{ 1, \frac{2}{s} \right\}, i = 1, 2, 3, 4. \quad (1.8)$$

Then T has a unique fixed point in X . Furthermore, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 1.17. We note that the proof of Theorem 1.15 in [29] is not correct (see page 6, line 13+). Indeed, for $\lambda = \frac{1}{s}$, $s > 1$ and $p = m$ we have that

$$\frac{s^p \lambda^{m+1}}{s - \lambda} d(x_1, x_0) + s^{p-1} \lambda^m d(x_1, x_0) \nrightarrow \theta, \quad (1.9)$$

as $m \rightarrow \infty$. Also, note that if $s > 1$ and $\lambda > \frac{1}{s}$, $m = p$ and $d(x_1, x_0) \neq \theta$ then

$$\frac{s^p \lambda^{m+1}}{s - \lambda} d(x_1, x_0) + s^{p-1} \lambda^m d(x_1, x_0) > \frac{1}{s(s - \lambda)} d(x_1, x_0) + \frac{1}{s} d(x_1, x_0) \succ \theta, \quad (1.10)$$

so

$$\frac{s^p \lambda^{m+1}}{s - \lambda} d(x_1, x_0) + s^{p-1} \lambda^m d(x_1, x_0) \nrightarrow \theta. \quad (1.11)$$

Therefore and the proof of Theorem 1.16 is doubtful when $\lambda = \frac{1}{s}$ and $\lambda > \frac{1}{s}$ and $s > 1$ (see page 8, line 12+). Instead (1.8) must $\frac{\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4)}{2 - \lambda_1 - \lambda_2 - s(\lambda_3 + \lambda_4)} < \frac{1}{s}$, that is, $(s+1)(\lambda_1 + \lambda_2) + (s+s^2)(\lambda_3 + \lambda_4) < 2$. It is worth to notice that Theorem 1.16. follows by ([46], Theorem 2.10) taking $a_1 = 0$ and $g = i_X$ identity mapping on X . $\square$

2. MAIN RESULTS

In this section we shall prove some fixed point theorems of contractive mappings in the setting of a tvs-cone b-metric space. First we give the following results as the continuation researches from [14], [18], [23], [27], [32], [43], [46], [50]. All these results are new in the context of b -metric spaces (that is, metric type spaces by [30] and [39]).

Theorem 2.1. *Let (X, d) be a complete b -metric space with the constant $s \in [1, 2]$. Let $\{F, T\}$ be a pair of self-mappings on X such that for all $x, y \in X$*

$$d(Fx, Ty) \leq \lambda \max \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (2.1)$$

where $\lambda \in (0, \frac{1}{2s})$. Then F and T have a unique common fixed point.

Theorem 2.2. *Let (X, d) be a complete b -metric space with the constant $s > 2$. Let $\{F, T\}$ be a pair of self-mappings on X such that for all $x, y \in X$*

$$d(Fx, Ty) \leq \lambda \max \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (2.2)$$

where $\lambda \in (0, \frac{1}{s^2})$. Then F and T have a unique common fixed point.

Theorem 2.3. *Let (X, d) be a complete metric space. Let $\{F, T\}$ be a pair of self-mappings on X such that for all $x, y \in X$*

$$d(Fx, Ty) \leq \lambda \max \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (2.3)$$

where $\lambda \in (0, \frac{1}{2})$. Then F and T have a unique common fixed point.

Proofs: Taking $E = \mathbb{R}, P = [0, +\infty), \|\cdot\| = |\cdot|$ in Theorems 1.12, 1.13 and Corollary 1.14 we obtain the claims of Theorems 2.1, 2.2. and 2.3. $\square$

Remark 2.4. Similarly, we can obtain several new results in the context of b -metric spaces by using results from [18], [27] and [46].

The following statement extends and generalizes Theorem 1.12.

Theorem 2.5. *Let (X, d) be a complete tvs-cone b -metric space with constant $s \in [1, 2]$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, \frac{1}{2s})$ and for all $x, y \in X$ there exists*

$$u(x, y) \in co \{\theta, d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \quad (2.6)$$

such that the inequality

$$d(Fx, Ty) \preceq \lambda u(x, y) \quad (2.7)$$

holds. Then F and T have a unique common fixed point.

Proof. Let $e \in \text{int}P$ and let q_e and d_q have the meaning as in Theorem 1.8. Applying Minkowski functional q_e , we obtain from (2.6) and (2.7) that for fixed $x, y \in X$,

$$d_q(Fx, Ty) = q_e(d(Fx, Ty)) \leq \lambda q_e(u_{x,y}).$$

Here,

$$\begin{aligned} q_e(u_{x,y}) &\in q_e(\text{co}\{\theta, d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}) \\ &\subset \text{co}(q_e\{\theta, d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}) \\ &= \text{co}\{q_e(\theta), q_e(d(x, y)), q_e(d(x, Fx)), q_e(d(y, Ty)), q_e(d(x, Ty)), \\ &\quad q_e(d(y, Fx))\} \\ &= \text{co}\{0, d_q(x, y), d_q(x, Fx), d_q(y, Ty), d_q(x, Ty), d_q(y, Fx)\} \\ &= [0, M], \end{aligned}$$

where $M = \max\{d_q(x, y), d_q(x, Fx), d_q(y, Ty), d_q(x, Ty), d_q(y, Fx)\}$. We have proved that

$$d_q(Fx, Ty) \leq \lambda \max\{d_q(x, y), d_q(x, Fx), d_q(y, Ty), d_q(x, Ty), d_q(y, Fx)\}.$$

The proof further follows by Remark 1.9. and Theorem 2.1. $\square$

The following result extends and generalizes Theorem 1.13 and its proof is omitted.

Theorem 2.6. *Let (X, d) be a complete tvs-cone b-metric space with constant $s > 2$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, \frac{1}{s^2})$ for all $x, y \in X$ there exists*

$$u(x, y) \in \text{co}\{\theta, d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\},$$

such that the inequality $d(Fx, Ty) \preceq \lambda u(x, y)$ holds. Then F and T have a unique common fixed point.

The result which follows extends and generalizes Theorem 2.1. from [46].

Theorem 2.7 *Let (X, d) be a tvs-cone b-metric space with constant $s \geq 1$ relative to a solid cone P . Suppose that the mappings $f, g : X \rightarrow X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X , and that for some constant $\lambda \in [0, \frac{1}{s})$ and for every $x, y \in X$, there exists*

$$u \equiv u_{x,y} \in \text{co}\left\{\theta, sd(gx, gy), sd(gx, fx), sd(gy, fy), \frac{d(gy, fx) + d(gx, fy)}{2}\right\}, \quad (2.8)$$

such that

$$d(fx, fy) \preceq \frac{\lambda}{s} u_{x,y}. \quad (2.9)$$

Then f and g have a unique point of coincidence in X . Moreover if f, g are weakly compatible, then f, g have a unique common fixed point.

Proof. Let $e \in \text{int}P$ and let q_e and d_q have the meaning as in previous theorem. Applying Minkowski functional q_e , we obtain from (2.8) and (2.9) that for fixed

$x, y \in X$,

$$d_q(fx, fy) = q_e(d(fx, fy)) \leq \frac{\lambda}{s} q_e(u_{x,y}),$$

where

$$\begin{aligned} q_e(u_{x,y}) &\in q_e\left(\text{co}\left\{\theta, sd(gx, gy), sd(gx, fx), sd(gy, fy), \frac{d(gy, fx) + d(gx, fy)}{2}\right\}\right) \\ &\subset \text{co}\left(q_e\left\{\theta, sd(gx, gy), sd(gx, fx), sd(gy, fy), \frac{d(gy, fx) + d(gx, fy)}{2}\right\}\right) \\ &\subset \text{co}\left\{q_e(\theta), sq_e(d(gx, gy)), sq_e(d(gx, fx)), sq_e(d(gy, fy)), \frac{q_e(d(gy, fx)) + q_e(d(gx, fy))}{2}\right\} \\ &= \text{co}\left\{0, sd_q(gx, gy), sd_q(gx, fx), sd_q(gy, fy), \frac{d_q(gy, fx) + d_q(gx, fy)}{2}\right\} \\ &= [0, m], \end{aligned}$$

where $m = \max\left\{sd_q(gx, gy), sd_q(gx, fx), sd_q(gy, fy), \frac{d_q(gy, fx) + d_q(gx, fy)}{2}\right\}$. Hence, we have proved that

$$d_q(fx, fy) \leq \lambda \max\left\{sd_q(gx, gy), sd_q(gx, fx), sd_q(gy, fy), \frac{d_q(gy, fx) + d_q(gx, fy)}{2}\right\}.$$

According Remark 1.9. and ([46], Theorem 2.1.) follows that f and g have a unique common fixed point. $\square$

In the next theorem we generalize Theorem 2.4. from [46]. The proof is similar to the previous one, and so is omitted.

Theorem 2.8. *Let (X, d) be a tvs-cone b-metric space with constant $s \geq 1$ relative to a solid cone P . Suppose that the mappings $f, g : X \rightarrow X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X , and that for some constant $\lambda \in [0, \frac{1}{s})$ and for every $x, y \in X$, there exists*

$$u \equiv u_{x,y} \in \text{co}\left\{\theta, sd(gx, gy), \frac{d(gx, fx) + d(gy, fy)}{2}, \frac{d(gy, fx) + d(gx, fy)}{2}\right\}, \quad (2.10)$$

such that

$$d(fx, fy) \preceq \frac{\lambda}{s} u_{x,y}. \quad (2.11)$$

Then f and g have a unique point of coincidence in X . Moreover if f, g are weakly compatible, then f, g have a unique common fixed point.

The following result generalizes Corollary 2.5. from [46].

Corollary 2.9. *Let (X, d) be a tvs-cone b-metric space with constant $s \geq 1$ relative to a solid cone P . Suppose that the mappings $f, g : X \rightarrow X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X , and that for some*

constant $\lambda \in [0, \frac{1}{s})$ and for every $x, y \in X$, we have that

$$d(fx, fy) \preceq \lambda d(gx, gy). \quad (2.12)$$

Then f and g have a unique point of coincidence in X . Moreover if f, g are weakly compatible, then f, g have a unique common fixed point.

Finally, we prove theorem of Hardy-Rogers type in the context of tvs-cone b-metric spaces. This result extends and generalizes Theorem 2.10. from [46].

Theorem 2.10. *Let (X, d) be a tvs-cone b-metric space with constant $s \geq 1$ relative to a solid cone P . Suppose that the mappings $f, g : X \rightarrow X$ are such that $fX \subseteq gX$ and fX or gX is a complete subspace of X and that there exists nonnegative constants $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ with*

$$2sa_1 + (s+1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2 \quad (2.13)$$

such that, for each $x, y \in X$,

$$d(fx, fy) \preceq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx). \quad (2.14)$$

Then f and g have a unique point of coincidence in X . Moreover if f, g are weakly compatible, then f, g have a unique common fixed point.

Proof. Taking $E = \mathbb{R}, P = [0, +\infty), \|\cdot\| = |\cdot|$ in Theorems 2.10. from [46]. we obtain its truthless in the context of b-metric spaces. Since (2.14) implies

$$d_q(fx, fy) \leq a_1 d_q(gx, gy) + a_2 d_q(gx, fx) + a_3 d_q(gy, fy) + a_4 d_q(gx, fy) + a_5 d_q(gy, fx), \quad (2.15)$$

then the proof follows according to Remark 1.9. $\square$

According to Remark 1.9 and ([30], Theorem 3.3) it follows the next result.

Theorem 2.11. *Let (X, d) be a complete tvs-cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition*

$$d(Tx, Ty) \preceq \lambda d(x, y), x, y \in X \quad (2.16)$$

where $\lambda \in [0, \frac{1}{s})$ is a constant. Then T has a unique fixed point in X . Furthermore, the iterative sequence $\{T^n x\}$ converges to the fixed point.

The following example satisfies the conditions of Theorem 2.11, but does not satisfy the conditions of Theorem 1.15.

Example 2.12. Let $E = C_{\mathbb{R}}^1[0, 1]$ with $\|u\| = \|u\|_{\infty} + \|u'\|_{\infty}, u \in E$ and let $P = \{u \in E : u(t) \geq 0 \text{ on } [0, 1]\}$. It is well known that this cone is solid but it is not normal. Now consider the space $E = C_{\mathbb{R}}^1[0, 1]$ endowed with the strongest locally convex topology τ^* . Then P is also τ^* -solid, but not τ^* -normal. (For details see [[31], Example 2.2.).

Let $X = [0, 1]$ and define a tvs-cone metric $d : X \times X \rightarrow P$ by $d(x, y)(t) = |x - y|^2 e^t$. Then (X, d) is a complete tvs-cone b-metric space with the coefficient $s = 2$. Let us define $T : X \rightarrow X$ as $Tx = \frac{1}{2}x - \frac{1}{4}x^2$ for all $x \in X$.

Now we obtain that

$$\begin{aligned}
 d(Tx, Ty)(t) &= \left| \frac{1}{2}x - \frac{1}{4}x^2 - \left(\frac{1}{2}y - \frac{1}{4}y^2 \right) \right|^2 e^t \\
 &= \frac{1}{4} |x - y|^2 \left| 1 - \frac{x + y}{2} \right|^2 e^t \\
 &\leq \frac{1}{4} |x - y|^2 e^t \\
 &= \frac{1}{4} d(x, y)(t).
 \end{aligned}$$

Hence, $d(Tx, Ty) \preceq \frac{1}{4}d(x, y)$ for all $x, y \in X, \lambda = \frac{1}{4} \in [0, \frac{1}{2})$. So T satisfies the conditions of Theorem 2.11. Here $0 \in X$ is the unique fixed point of T . However, T does not satisfy the conditions of Theorem 1.15. Indeed, (X, d) is not cone b-metric space because locally convex space (E, τ^*) is not normed (see [[31], Example 2.2).

This example shows that our result is a genuine generalization of the main theorem in [29].

The following example shows the usability of our previous results in the context of tvs-cone b-metric spaces.

Example 2.13. Let us take E and P as in the previous example. Let be $X = \{a, b, c\}$ and define tvs-cone b-metric $d : X \times X \rightarrow P$ by $d(a, b)(t) = 2 + 3t$, $d(b, c)(t) = 5 - 3t$, $d(a, c)(t) = 3$, $d(x, y)(t) = d(y, x)(t)$ and $d(x, x)(t) = \theta(t) = 0$ for all $x, y \in X$. Then (X, d) is a complete cone metric space. Consider now the mappings $T = F : X \rightarrow X$ given by $Fa = a, Fb = c, Fc = a$. It is not hard to check that all conditions of Theorem 2.5. are satisfied (for details see [14], Example 4.1.). Here, the unique common fixed point is a . However, this example does not satisfy the conditions of ([50], Theorem 3.1.), that is, Theorem 1.12 above. Indeed, take $x = a, y = b$. We obtain that $d(Fa, Tb) = d(Fa, Fb) = d(a, c)$ and there is not $u(a, b) \in \{d(a, b), d(a, a), d(b, c), d(a, c), d(b, a)\}$ satisfying $d(Fa, Tb) \preceq \lambda u(a, b)$ with some $\lambda \in [0, \frac{1}{2s})$.

Following theorem extends the result of Theorem 2.11. for $\lambda \in [0, \frac{1}{s}]$, $s > 1$.

Theorem 2.14. *Let (X, d) be a complete tvs-cone b-metric space with the coefficient $s > 1$. Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition*

$$d(Tx, Ty) \preceq \lambda d(x, y), x, y \in X \quad (2.17)$$

where $\lambda \in [0, \frac{1}{s}]$ is a constant. Then T has a unique fixed point in X .

Proof: If $\lambda \in [0, \frac{1}{s})$ the proof follows from a similar process as in ([30], Lemma 3.1), ([46], Theorem 2.1) and ([50], Theorem 3.1). If $\lambda = \frac{1}{s}$, we define a sequence $\{x_n\}$ by $x_{n+1} = T^2x_n$ for all $n \geq 0$ with arbitrary $x_0 \in X$. Denote $d(x_n, x_{n+1})$ by

d_n for all $n \geq 0$. Then it follows from (2.16) that

$$\begin{aligned} d_n &= d(x_n, x_{n+1}) = d(T^2 x_{n-1}, T^2 x_n) \\ &\preceq \lambda d(T x_{n-1}, T x_n) \\ &\preceq \lambda^2 d(x_{n-1}, x_n) = \lambda^2 d_{n-1}. \end{aligned}$$

By induction we obtain that

$$d_n \preceq \lambda^{2n} d_0.$$

Therefore, for any $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= s d_n + s d(x_{n+1}, x_m) \\ &\preceq s d_n + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= s d_n + s^2 d_{n+1} + s^2 d(x_{n+2}, x_m) \\ &\vdots \\ &\preceq s d_n + s^2 d_{n+1} + s^3 d_{n+2} + \cdots + s^{m-n} d_{m-1} \\ &\preceq s \lambda^{2n} d_0 + s^2 \lambda^{2n+2} d_0 + s^3 \lambda^{2n+4} d_0 + \cdots \\ &= \lambda^{2n-1} d_0 + \lambda^{2n} d_0 + \lambda^{2n+1} d_0 + \cdots \\ &= \frac{\lambda^{2n-1} d_0}{1 - \lambda}. \end{aligned}$$

Since $\frac{\lambda^{2n-1} d_0}{1 - \lambda} \rightarrow \theta$ as $n \rightarrow \infty$ in locally convex space E , therefore, by (1) and (3) of Remark 1.7, we obtain, for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d) and by completeness of (X, d) , there exists $u \in X$ such that $\lim_{n \rightarrow \infty} x_n = u$.

We shall show that u is the fixed point of T . It follows from (2.17) that

$$\begin{aligned} d(u, T^2 u) &\preceq s[d(u, x_{n+1}) + d(x_{n+1}, T^2 u)] = s[d(u, x_{n+1}) + d(T^2 x_n, T^2 u)] \\ &\preceq s[d(u, x_{n+1}) + \lambda d(T x_n, T u)] \preceq s[d(u, x_{n+1}) + \lambda^2 d(x_n, u)] \end{aligned}$$

i.e.

$$d(u, T^2 u) \preceq s d(u, x_{n+1}) + \lambda d(x_n, u), x, y \in X \quad (2.18)$$

As, $\lim_{n \rightarrow \infty} x_n = u$, there exists $n_1 \in \mathbb{N}$ such that for every $c \in E$ with $\theta \ll c$, $d(u, x_{n+1}) \ll \frac{c}{s}$ and $d(x_n, u) \ll \frac{c}{\lambda}$, for all $n > n_1$. So, it follows from (2.18) that $d(u, T^2 u) \ll c$ for all $n > n_1$, therefore, $T^2 u = u$, i.e., u is a fixed point of T^2 .

Again, (2.17) shows that this fixed point must be unique.

Now $T^2 T u = T T^2 u = T u$, therefore $T u$ is another fixed point of T^2 and by uniqueness we have $T u = u$.

Thus, u is the unique fixed point of T . $\square$

Open problem: It is not yet known if the results of Theorem 2.14, can be proved for $\frac{1}{s} < \lambda < 1$, $s > 1$, with or without some additional condition on T or space (X, d) .

Acknowledgement. First and second authors are thankful to the Ministry

of Education, Science and Texhnological Development of Serbia. Third author is thankful to Professor Mahesh Dubey for his encouragement.

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