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IDENTIFICATION OF THE RIGHT HAND SIDE OF EQUATION OF TORSIONAL VIBRATIONS OF A ROD

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Abstract. We analyze the solvability of the inverse problem with an unknown coefficient for a fourth-order hyperbolic equation. To investigate the solvability of the inverse problem, the problem is considered in a rectangle area, with set conditions being typical of the boundary-value problem and an auxiliary condition being necessary of the unknown coefficient searching. Using the Fourier method, the problem is reduced to solving a system of integral equations, and using the compressed mapping method, the existence and uniqueness of a solution to the system of integral equations is proved. The existence and uniqueness of a classical solution to the original problem is shown.

1. Introduction

A distinctive feature of the presented paper is the study of the inverse problem for the 2D equation of transverse oscillation of the rod from knowledge of additional condition in time variable. It should be noted that one of the classes of qualitatively new problems is problems with nonlocal conditions. From physical considerations, the additional conditions are completely natural, and they arise in mathematical modelling in cases where it is impossible to obtain information about the process occurring at the boundary of the region of its flow using direct measurements. The term “nonlocal conditions” and their classification were introduced by Dezin [2].

The applied significance of problems with nonlocal conditions is associated with the study of processes occurring in the turbulent plasma, the processes of diffusion, heat conductivity problems of mathematical biology, as well as some inverse problems of mathematical physics [6], [7]. Ramazanova [12] proved the existence and uniqueness of the solution for the inverse problem of the fourth-order hyperbolic equation. In this work, the authors considered the identification problems for recovering the kernels in a hyperbolic integro-differential operator equations. Moreover, Huntul, Mehraliyev, Ramazanova [3], [4], [9] studied the solvability of inverse problem for the fourth-order pseudo-hyperbolic equation with an unknown time-dependent coefficient and numerically solved several inverse problems concerning the determination time-dependent coefficients.

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This article is outlined as follows. The inverse problem for the 2D equation of torsional vibrations of the rod from knowledge of additional condition for determining the coefficient of the right-hand side along with the function $u(x,t)$ has been investigated. The considered problem has been reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem was shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorems for auxiliary problems have been proved.

2. Formulation of the problem.

Inverse problem: find functions $u(x,t), a(t), b(t), c(t)$ that are relate in the Domain $D_T = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\}$ by the equation

$$u_{tt}(x,t) + u_{xxxx}(x,t) = a(t)u(x,t) + b(t)u_t(x,t) + c(t)g(x,t) + f(x,t) \quad (2.1)$$

with the initial condition (IC)

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad (0 \leq x \leq 1), \quad (2.2)$$

with boundary conditions

$$u(1,t) = u_x(0,t) = u_{xx}(0,t) = u_{xxx}(1,t) = 0 \quad (0 \leq t \leq T), \quad (2.3)$$

and with auxiliary condition

$$u(x_i,t) = h_i(t) \quad (i = 1, 2.3; 0 \leq t \leq T), \quad (2.4)$$

where $x_i \in (0,1)\,(i = 1,2,3; x_1 \neq x_2 \neq x_3)$ - fixed functions $f(x,t), \varphi(x), \psi(x)$, $h_i(t)$ $(i = 1,2,3)$ are given functions.

Definition 2.1. By the classical solution of the inverse boundary value problem (2.1)-(2.4) we mean the $\{u(x,t),\ a(t),\ b(t),\ c(t)\}$ functions of $u(x,t)$, $a(t)$, $b(t)$ and $c(t)$ satistying equation (2.1)-(2.4) in ordinary sense, where:

(1) *functions $u(x,t)$ and the derivatives included in the equation are continuous in D_T;*
(2) *$a(t), b(t)$ and $c(t)$ are continious on $[0,T]$;*
(3) (2.1) *and* (2.2)-(2.4) are satisfied in the usual sense.

Theorem 2.2. *Let be*

$$\varphi(x), \psi(x) \in C[0,1], \quad h_i(t) \in C^2[0,T]\ (i = 1,2,3),$$
$$f(x,t), g(x,t) \in C(D_T)$$

and the agreement condition be satisfied

$$\varphi(x_i) = h_i(0), \ \ \psi(x_i) = h_i'(0)\ (i = 1,2,3). \quad (2.5)$$

Then the problem (2.1)-(2.4) *are equivalent to the finding the functions $u(x,t), a(t), b(t)$ and $c(t)$ in* (2.1)-(2.3) *from* (2.5),

$$h_i''(t) + u_{xxxx}(x_i,t) = a(t)h_i(t) + b(t)h_i(t) + c(t)g(x_i,t) + \\ + f(x_i,t) \ \ (i = 1,2,3; 0 \leq t \leq T), \quad (2.6)$$

and

$$h(t) \equiv \begin{vmatrix} h_1(t) & h_1'(t) & g(x_1,t) \\ h_2(t) & h_2'(t) & g(x_2,t) \\ h_3(t) & h_3'(t) & g(x_3,t) \end{vmatrix} \neq 0 \ (0 \le t \le T).$$

Proof. Let be $h_i(t) \in C^2[0,T]$ $(i = 1,2,3)$, classical solution to problem (2.1)-(2.4). Integrating equation (2.4) twice, we get:

□

$$u_t(x_i,t) = h_i'(t), \quad u_{tt}(x_i,t) = h_i''(t) \ \ (i = 1,2,3; 0 \le t \le T). \tag{2.7}$$

Substituting $x = x_i$ in (2.1), we get:

$$u_{tt}(x_i,t) + u_{xxxx}(x_i,t) =$$

$$= a(t)u(x_i,t) + b(t)u_t(x_i,t) + c(t)g(x_i,t) + f(x_i,t) \ \ (i = 1,2,3; 0 \le t \le T). \tag{2.8}$$

From (2.8), due to (2.4) and (2.7), the fulfillment of (2.6) follows.
Now, suppose that $\{u(x,t)\,,\, a(t),\, b(t),\, c(t)\}$ is the solution to problem (2.1) - (2.3), (2.6). Then from (2.6) ,(2.8), we have:

$$\frac{d^2}{d^2}(u(x_i,t) - h_i(t)) - b(t)\frac{d}{d}(u(x_i,t) - h_i(t)) -$$

$$- a(t)(u(x_i,t) - h_i(t)) = 0 \quad (i = 1,2,3; 0 \le t \le T). \tag{2.9}$$

By virtue of (2.2) and agreement condition (2.5), we obtain:

$$u(x_i,0) - h_i(0) = \varphi(x_i) - h_i(0) = 0,$$

$$u_t(x_i,0) - h_i'(0) = \psi(x_i) - h_i'(0) = 0 \, (i = 1,2,3). \tag{2.10}$$

From (2.9), (2.10) we conclude that condition (2.4) is satisfied. The theorem is proven.

3. Solvability of the inverse boundary value problem.

First component $u(x,t)$ of solution $\{u(x,t)\,,\, a(t),\, b(t),\, c(t)\}$ of the problem (2.1)-(2.3), (2.6) will be sought in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi), \tag{3.1}$$

where

$$u_k(t) = 2\int_0^1 u(x,t) \cos \lambda_k x dx$$

Then, using the formal Fourier scheme, from (2.1) and (2.2) we have:

$$u_k''(t) + \lambda_k^4 u_k(t) = F_k(t; u, a, b, c) \ \ (0 \le t \le T;\ k = 1,2,...), \tag{3.2}$$

$$u_k(0) = \varphi_k, \quad u_k'(0) = \psi_k \ \ (k = 1,2,...), \tag{3.3}$$

and

$$F_k(t; u, a, b, c) = f_k(t) + a(t)u_k(t) \ + b(t)u_k'(t) \ + c(t)g_k(t)\,,$$

$$f_k(t) = 2\int_0^1 f(x,t) \cos \lambda_k x dx \,,\, g_k(t) = 2\int_0^1 g(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \quad \psi_k = 2\int_0^1 \psi(x)\cos\lambda_k x dx, \quad (k = 1, 2, ...). \tag{3.4}$$

Solving problem (3.2), (3.3), we find:

$$u_k(t) = \varphi_k \cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k \sin\lambda_k^2 t +$$

$$+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau; u, a, b, c)\sin\lambda_k^2(t-\tau)d\tau \ \ (k = 1, 2, ...). \tag{3.5}$$

After substituting the expression from $u_k(t)$ $(k = 1, 2, ...)$ in (3.1), to define a component $u(x,t)$ solution of problem (2.1)-(2.3), (2.6) we obtain:

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\varphi_k\cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k\sin\lambda_k^2 t +\right.$$

$$\left.+\frac{1}{\lambda_k^2}\int_0^t F_k(\tau; u, a, b, c)\sin\lambda_k^2(t-\tau)d\tau\right\}\cos\lambda_k x. \tag{3.6}$$

Now, from (3.1), we get:

$$h_i''(t) + \sum_{k=1}^{\infty}\lambda_k^4 u_k(t)\cos\lambda_k x_i =$$

$$= a(t)h_i(t) + b(t)h_i(t) + c(t)g(x_i,t) + f(x_i,t) \ \ (i = 1, 2, 3; 0 \le t \le T).$$

Let's assume that

$$h(t) \equiv \begin{vmatrix} h_1(t) & h_1'(t) & g(x_1,t) \\ h_2(t) & h_2'(t) & g(x_2,t) \\ h_3(t) & h_3'(t) & g(x_3,t) \end{vmatrix} \neq 0 \ (0 \le t \le T).$$

Then from the last relation we have:

$$a(t) = [h(t)]^{-1}\left\{q_1(t) + \sum_{k=1}^{\infty}\lambda_k^4 u_k(t) q_{1k}(t)\right\}, \tag{3.7}$$

$$b(t) = [h(t)]^{-1}\left\{q_2(t) + \sum_{k=1}^{\infty}\lambda_k^4 u_k(t) q_{2k}(t)\right\}, \tag{3.8}$$

$$c(t) = [h(t)]^{-1}\left\{q_3(t) + \sum_{k=1}^{\infty}\lambda_k^4 u_k(t) q_{3k}(t)\right\}, \tag{3.9}$$

where

$$q_1(t) \equiv \begin{vmatrix} h_1''(t) - f(x_1,t) & h_1'(t) & g(x_1,t) \\ h_2''(t) - f(x_2,t) & h_2'(t) & g(x_2,t) \\ h_3''(t) - f(x_3,t) & h_3'(t) & g(x_3,t) \end{vmatrix}, \ q_{1k}(t) \equiv \begin{vmatrix} \cos\lambda_k x_1 & h_1'(t) & g(x_1,t) \\ \cos\lambda_k x_2 & h_2'(t) & g(x_2,t) \\ \cos\lambda_k x_3 & h_3'(t) & g(x_3,t) \end{vmatrix},$$

$$q_2(t) \equiv \begin{vmatrix} h_1(t) & h_1''(t) - f(x_1,t) & g(x_1,t) \\ h_2(t) & h_2''(t) - f(x_2,t) & g(x_2,t) \\ h_3(t) & h_3''(t) - f(x_3,t) & g(x_3,t) \end{vmatrix}, q_{2k}(t) \equiv \begin{vmatrix} h_1(t) & \cos\lambda_k x_1 & g(x_1,t) \\ h_2(t) & \cos\lambda_k x_2 & g(x_2,t) \\ h_3(t) & \cos\lambda_k x_3 & g(x_3,t) \end{vmatrix},$$

$$q_3(t) \equiv \begin{vmatrix} h_1(t) & h'_1(t) & h''_1(t) - f(x_1,t) \\ h_2(t) & h'_2(t) & h''_2(t) - f(x_2,t) \\ h_3(t) & h'_3(t) & h''_3(t) - f(x_3,t) \end{vmatrix}, \; q_{3k}(t) \equiv \begin{vmatrix} h_1(t) & h'_1(t) & \cos\lambda_k x_1 \\ h_2(t) & h'_2(t) & \cos\lambda_k x_2 \\ h_3(t) & h'_3(t) & \cos\lambda_k x_3 \end{vmatrix}.$$

Substituting expression (3.5) into (3.7), (3.8) and (3.9), respectively, we obtain:

$$a(t) = [h(t)]^{-1} \left\{ q_1(t) + \sum_{k=1}^{\infty} \lambda_k^4 q_{1k}(t) \times \right.$$
$$\left. \times \left[\varphi_k \cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k \sin\lambda_k^2 t + \frac{1}{\lambda_k^2}\int_0^t F_k(\tau;u,a,b,c)\sin\lambda_k^2(t-\tau)d\tau \right] \right\}, \tag{3.10}$$

$$b(t) = [h(t)]^{-1} \left\{ q_2(t) + \sum_{k=1}^{\infty} \lambda_k^4 q_{2k}(t) \times \right.$$
$$\left. \times \left[\varphi_k \cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k \sin\lambda_k^2 t + \frac{1}{\lambda_k^2}\int_0^t F_k(\tau;u,a,b,c)\sin\lambda_k^2(t-\tau)d\tau \right] \right\}, \tag{3.11}$$

$$c(t) = [h(t)]^{-1} \left\{ q_3(t) + \sum_{k=1}^{\infty} \lambda_k^4 q_{3k}(t) \times \right.$$

,

$$\left. \times \left[\varphi_k \cos\lambda_k^2 t + \frac{1}{\lambda_k^2}\psi_k \sin\lambda_k^2 t + \frac{1}{\lambda_k^2}\int_0^t F_k(\tau;u,a,b,c)\sin\lambda_k^2(t-\tau)d\tau \right] \right\} \tag{3.12}$$

Thus, the solution to problem (2.1)-(2.3),(2.6) was reduced to solving the system (3.6), (3.10)-(3.12), with respect to unknown functions $u(x,t)$,$a(t)$, $b(t)$ *and* $c(t)$.

Lemma 3.1. *If* $\{u(x,t)\,,\, a(t),\, b(t),\, c(t)\}$ *– any classical solution to problem* (2.1)-(2.3), (2.6), *then the functions*

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \;\; (k = 0,1,2,...)$$

satisfy the system consisting of equations (3.5).

Corollary 3.2. *Let system* (3.6), (3.10)-(3.12) *have a unique solution. Then problem* (2.1)-(2.3), (2.6) *cannot have more than one solution, i.e. if problem* (2.1)-(2.3), (2.6) *has a solution, then it is unique.*

Let us denote $B_{2,T}^{5,3}$ [12] the set of all functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x \quad (\lambda_k = k\pi)$$

considered inD_T, where $u_k(t) \in C^1[0;T]$ $(k = 1,2...)$ and

$$I(u) = \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_k(t)\|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u'_k(t)\|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty$$

We define the norm in this set as follows:

$$\|(u(x,t)\|_{B^{5,3}_{2,T}} = I(u).$$

Let us denote $E^{5,3}_T$ [9] the space $B^{5,3}_{2,T} \times C\,[0,T] \times C\,[0,T]\ \times C\,[0,T]$ of a vector of functions $z(x,t) = \{u(x,t), a(t), b(t), c(t)\}$ with norm

$$\|z(x,t)\|_{E^{5,4}_{2,T}} = \|u(x,t)\|_{B^{5,3}_{2,T}} + \|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} + \|c(t)\|_{C[0,T]}\,.$$

It is known that $B^{5,3}_{2,T}$ and $E^{5,3}_T$ are Banach spaces.

Consider the operator $\Phi(u,a,b,c) = \{\Phi_1(u,a,b,c), \Phi_2(u,a,b,c), \Phi_3(u,a,b,c), \Phi_4(u,a,b,c)\}$ in the space $E^{5,3}_T$ and

$$\Phi_1(u,a,b,c) = \tilde{u}(x,t) = \sum_{k=1}^{\infty} \tilde{u}_k(t) \sin \lambda_k x, \Phi_2(u,a,b,c) = \tilde{a}(t),$$

$$\Phi_3(u,a,b,c) = \tilde{b}(t), \Phi_3(u,a,b,c) = \tilde{c}(t)\,,$$

$\tilde{u}_k(t)$ $(k = 1,2...)$, $\tilde{a}(t)$, $\tilde{b}(t)$, $\tilde{c}(t)$ are equal to the right-hand sides of (3.6), respectively, (3.10), (3.11),(3.12). From (3.5) we get

$$\tilde{u}'_k(t) = -\lambda_k^2 \varphi_k \sin \lambda_k^2 t + \psi_k \cos \lambda_k^2 t +$$

$$+ \int_0^t F_k(\tau, u, a, b, c) \cos \lambda_k^2 (t-\tau) dt\, (k = 1,2...) \tag{3.13}$$

It is easy to see that

$$\|q_{1k}(t)\|_{C[0,T]} \le \|h'_2(t)\, g(x_3,t) - h'_3(t)\, g(x_2,t)\|_{C[0,T]} +$$

$$+ \|h'_1(t)\ g(x_3,t) -\ h'_3(t)\ g(x_1,t)\|_{C[0,T]} + \|h'_1(t)\ g(x_2,t) -\ h'_2(t)\ g(x_1,t)\|_{C[0,T]} \equiv p_1(T),$$

$$\|q_{2k}(t)\|_{C[0,T]} \le \|h_2(t)\, g(x_3,t) - h_3(t)\, g(x_2,t)\|_{C[0,T]} +$$

$$+ \|h_1(t)\ g(x_3,t) -\ h_3(t)\ g(x_1,t)\|_{C[0,T]} + \|h_1(t)\ g(x_2,t) -\ h_2(t)\ g(x_1,t)\|_{C[0,T]} \equiv p_2(T)\,,$$

$$\|q_{3k}(t)\|_{C[0,T]} \le \|h_2(t)\, h'_3, t) - h_3(t) h'_2(t)\|_{C[0,T]} +$$

$$+ \|h_1(t)\ h'_3(t) -\ h_3(t)\ h'_1(t)\|_{C[0,T]} + \|h_1(t)\ h'_2(t) -\ h_2(t)\ h'_1(t)\,\|_{C[0,T]} \equiv p_3(T)\,,$$

In order to show the existence and uniqueness of the solution of the system of integral equations (3.6),(3.10),(3.12) using the last inequality, we make the following estimates:

$$\left\{\sum_{k=1}^{\infty} \left(\lambda_k^5 \|\tilde{u}_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} \le \sqrt{5} \left\{\sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_k|\right)^2\right\}^{\frac{1}{2}} + \sqrt{5} \left\{\sum_{k=1}^{\infty} \left(\lambda_k^3 |\psi_k|\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5T} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 |f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{5}T \|a(t)\|_{C[0,T]} \left\{\sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5}T \|b(t)\|_{C[0,T]} \left\{\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u'_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}, \tag{3.14}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|\tilde{u}_k'(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} \le \sqrt{5}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,|\varphi_k|\right)^2\right\}^{\frac{1}{2}} + \sqrt{5}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\psi_k|\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{5}T\,\|a(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5}T\,\|b(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k'(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+\sqrt{5T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}, \tag{3.15}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|q_1(t)\|_{C[0,T]} +$$

$$+p_1(T)\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,|\varphi_k|\right)^2\right\}^{\frac{1}{2}} + \left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\psi_k|\right)^2\right\}^{\frac{1}{2}} +\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + T\,\|a(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+T\,\|b(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k'(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$\left.+\sqrt{T}\,\|c(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{3.16}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|q_2(t)\|_{C[0,T]} +$$

$$+p_2(T)\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,|\varphi_k|\right)^2\right\}^{\frac{1}{2}} + \left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\psi_k|\right)^2\right\}^{\frac{1}{2}} +\right.$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + T\,\|a(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$+T\,\|b(t)\|_{C[0,T]}\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k'(t)\|_{C[0;T]}\right)^2\right\}^{\frac{1}{2}} +$$

$$\left.+\sqrt{T}\,\|c(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right]\Bigg\}, \tag{3.17}$$

$$\|\tilde{c}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{ \|q_3(t)\|_{C[0,T]} + \right.$$

$$+p_3(T) \left(\sum_{K=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 |\varphi_k|\right)^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 |\psi_k|\right)^2 \right\}^{\frac{1}{2}} + \right.$$

$$+\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 |f_k(\tau)|\right)^2 d\tau \right)^{\frac{1}{2}} + T\|a(t)\|_{C[0,T]} \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \|u_k(t)\|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} +$$

$$+T\|b(t)\|_{C[0,T]} \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k'(t)\|_{C[0;T]} \right)^2 \right\}^{\frac{1}{2}} +$$

$$\left.\left. + \sqrt{T}\|c(t)\|_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 |g_k(\tau)|\right)^2 d\tau \right)^{\frac{1}{2}} \right] \right\}, \tag{3.18}$$

Let the given problems (2.1)-(2.3), (2.6) satisfy the following conditions:

(1) $\varphi(x) \in C^4[0,1],\ \ \varphi^{(5)}(x) \in L_2(0,1), \varphi'(0) = \varphi(1) = \varphi'''(0) = \varphi''(1) = \varphi^{(4)}(1) = 0;$
(2) $\psi(x) \in C^2[0,1],\ \ \psi'''(x) \in L_2(0,1), \psi'(0) = \psi(1) = \psi''(1) = 0;$
(3) $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in L_2(D_T), f_x(0,t) = f(1,t) = f_{xx}(1,t) = 0\ \ (0 \leq t \leq T);$
(4) $g(x,t), g_x(x,t), g_{xx}(x,t) \in C(D_T),\ g_{xxx}(x,t) \in L_2(D_T), g_x(0,t) = g(1,t) = g_{xx}(1,t) = 0\ \ (0 \leq t \leq T);$
(5) $h(t) \neq 0\ (0 \leq t \leq T).$

Then from (3.14) and (3.15) we have:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}} \leq$$

$$\leq A_1(T) + B_1(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}} + D_1(T)\ \|c(t)\|_{C[0,T]}, \tag{3.19}$$

where

$$A_1(T) = 2\sqrt{5}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + 2\sqrt{5}\left\|\psi'''(x)\right\|_{L_2(0,1)} + 2\sqrt{5T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T) = 2\sqrt{5}T,\ D_1(T) = 2\sqrt{5T}\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}.$$

Further, from (3.19) and (3.20), respectively, we obtain:

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) +$$
$$+B_2(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}} + D_2(T)\ \|c(t)\|_{C[0,T]}, \tag{3.20}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \leq A_3(T) +$$
$$+B_2(T) \left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} \right) \|u(x,t)\|_{B_{2,T}^{5,3}} + D_3(T)\ \|c(t)\|_{C[0,T]}, \tag{3.21}$$

$$\|\tilde{c}(t)\|_{C[0,T]} \leq A_4(T)+$$
$$+B_4(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B^{5,3}_{2,T}}+D_4(T)\;\|c(t)\|_{C[0,T]}\;, \tag{3.22}$$

where

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|q_1(t)\|_{C[0,T]}+\right.$$
$$\left.+\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_1(T)\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\|\psi'''(x)\|_{L_2(0,1)}+\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$
$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_1(T)T\;,\;D_2(T)=$$
$$=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_1(T)\sqrt{T}\,\|g_{xxx}(x,t)\|_{L_2(D_T)}\,,$$
$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|q_2(t)\|_{C[0,T]}+\right.$$
$$\left.+\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_2(T)\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\|\psi'''(x)\|_{L_2(0,1)}+\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$
$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_2(T)T$$
$$D_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_2(T)\sqrt{T}\,\|g_{xxx}(x,t)\|_{L_2(D_T)}\,,$$
$$A_4(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|q_3(t)\|_{C[0,T]}+\right.$$
$$\left.+\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_3(T)\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\|\psi'''(x)\|_{L_2(0,1)}+\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$
$$B_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_3(T)T$$
$$D_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{K=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}p_3(T)\sqrt{T}\,\|g_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

From inequalities (3.19)-(3.22) we conclude:

$$\|\tilde{u}(x,t)\|_{B^{5,3}_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq \tag{3.23}$$
$$\leq A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B^{5,3}_{2,T}}+D(T)\;\|c(t)\|_{C[0,T]}\,,$$
$$A(T)=A_1(T)+A_2(T)+A_3(T),B(T)=B_1(T)+B_2(T)+B_3(T),$$
$$D(T)=D_1(T)+D_2(T)+D_3(T).$$

So, we can prove the following theorem.

Theorem 3.3. *Let the conditions (1)-(5) be satisfied and*

$$(B(T)\,(A(T)+2)+D(T))\,(A(T)+2)<1. \tag{3.24}$$

Then problem (2.1)-(2.3), (2.6) has a unique solution in the ball $K=K_R\left(\|z\|_{E_T^{5,3}}\leq A(T)+2\right)$ *of space* $E_T^{5,3}$.

Proof. In space $E_T^{5,3}$ consider the operator equation

$$z=\Phi z, \tag{3.25}$$

where $z=\{u,a,b,c\}$, the components $\Phi_i\,(u,a,b,c)$ $(i=\overline{1,4})$ of the operator $\Phi\,(u,a,b,c)$ are determined by the right-hand sides of equations (3.6), (3.10)-(3.12), respectively.

Consider the operator $\Phi\,(u,a,b,c)$in the ball $K=K_R\left(\|z\|_{E_T^{5,4}}\leq A(T)+2\right)$ from $E_T^{5,3}$.

Similarly, (3.23) we obtain that for any $z,z_1,z_2\in K_R$ the following estimates are valid:

$$\|z\|_{E_T^{5,3}}\leq A(T)+B(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\|u(x,t)\|_{B_{2,T}^{5,3}}+D(T)\,\|c(t)\|_{C[0,T]}\leq$$

$$\leq A(T)+(B(T)\,(A(T)+2)+D(T))\,(A(T)+2)\,, \tag{3.26}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,3}}\leq\; D(T)\,\|c_1(t)-c_2(t)\|_{C[0,T]}+ \tag{3.27}$$

$$+B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|b_1(t)-b_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{5,3}}\right).$$

From estimates (3.26) and (3.27), taking into account (3.24), it is clear that the operator Φ acts in the ball $K=K_R$ and is compressive. Therefore, in the ball $K=K_R$ the operator Φ has a single fixed point $\{u,a,b,c\}$, which is a solution to equation (3.25), that is $\{u,a,b,c\}$ is the only solution to system (3.6), (3.10)-(3.12) in the ball $K=K_R$.

A function $u(x,t)$, as an element of space $B_{2,T}^{5,3}$, is continuous and has continuous derivatives $u_x(x,t),u_{xx}(x,t),u_{xxx}(x,t),u_{xxxx}(x,t),u_t(x,t),u_{tx}(x,t),u_{txx}(x,t)$ in D_T.

From (3.2) it is easy to see that $u_k''(t)\in C\,[0,T]$and

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\,\|u_k''(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq\sqrt{2}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{2}\left\|\|a(t)u_x(x,t)+b(t)u_{tx}(x,t)+c(t)g_x(x,t)+f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}$$

Hence it follows $u_{tt}(x,t)\in C(D_T)$ and by Lemma 3.1 solution of problem is unique in $K=K_R$. □

The theorem is proven.

Theorem 3.4. *Let all the conditions of Theorem 3.3 be satisfied and*

$$\varphi(x_i) = h_i(0), \;\; \psi(x_i) = h_i'(0) \;\; (i = 1, 2, 3).$$

Then problem (2.1)-(2.4) has a unique classical solution in the ball $K = K_R\left(\|z\|_{E_T^{5,4}} \leq A(T) + 2\right)$ *of space* $E_T^{5,3}$.

4. Conclusion

In the work, the classical solvability of a nonlinear coefficient identification problem for a fourth-order hyperbolic equation with nonlocal conditions was investigated. Using the Fourier series method, the solutions of inverse boundary value problems are constructed in the form of a Fourier series. The inverse problem was formulated as an auxiliary problem which was reduced to the operator equation in a specified Banach space using the method of spectral analysis. Using these facts, we have proved the existence, uniqueness and solvability of a classical solution to one inverse boundary value problem for a fourth-order hyperbolic equation with an auxiliary condition. The proposed work is novel and has never been solved theoretically.

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