Gulf Journal of Mathematics
Vol 1 (2013) 65-71

A COMMON FIXED POINT RESULT FOR (μ, ψ)-WEAKLY CONTRACTIVE MAPPINGS

SUMIT CHANDOK

Abstract. A common fixed point result satisfying a (μ, ψ)-weak contractive condition for single-valued mappings in the framework of metric spaces is derived. The results presented in the paper generalize and extend several well-known results in the literature.

1. Introduction and Preliminaries

The Banach contraction mapping is one of the pivotal results of analysis. It is very popular tool for solving existence problems in different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature (see [1]-[14]).

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [14] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [2] and of Reich types [13]. Recently, Doric [10] proved a common fixed point theorem for generalized (ψ, ϕ)-weakly contractive mappings. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been studied by many authors (see [1], [3]-[10], [14], [15] and references cited therein). In this paper, we generalize the Chatterjea type contraction mappings to (μ, ψ)-generalized Chatterjea type contraction mappings and derived a common fixed point theorem for single-valued mappings in metric spaces for such contraction mappings.

First, we recall some basic definitions and notations.

Let (X, d) be a metric space. A map $T : X \to X$ is said to be:

(a) *Kannan type* (see [11]) if there exists a $k \in (0, \frac{1}{2}]$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$, for all $x, y \in X$;

(b) *Chatterjea type* ([8]) if there exists a $k \in (0, \frac{1}{2}]$ such that $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$, for all $x, y \in X$;

Date: Received: Jul 2, 2013; Accepted: Jul 24, 2013.

2010 *Mathematics Subject Classification.* 41A50, 47H10, 54H25.

Key words and phrases. Common fixed point, (μ, ψ)-generalized Chatterjea type contraction mappings, Ordered metric spaces.

Khan et al. [12] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

A function $\mu : [0,\infty) \to [0,\infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) μ is monotone increasing and continuous;

(ii) $\mu(t) = 0$ if and only if $t = 0$.

Using, the control function, we generalize the Chatterjea type contraction mappings as follows:

Suppose that T and f are self-mappings of a metric space (X, d). A mapping T is said to be (μ,ψ)-*generalized Chatterjea type contraction* if for all $x, y \in X$,

$$\mu(d(Tx, fy)) \leq \mu(\frac{1}{2}[d(x, fy) + d(y, Tx)]) - \psi(d(x, fy), d(y, Tx)),$$

holds, where $\mu : [0,\infty) \to [0,\infty)$ is an altering distance function and $\psi : [0,\infty)^2 \to [0,\infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Let M be a nonempty subset of a metric space (X, d), a point $x \in M$ is a *common fixed (coincidence) point* of f and T if $x = fx = Tx$ $(fx = Tx)$. The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively, $C(f, T)$). The mappings $T, f : M \to M$ are called *commuting* if $Tfx = fTx$ for all $x \in M$.

2. Main Results

Theorem 2.1. *Let M be a subset of a complete metric space (X, d) and f and T are self-mappings of M. Suppose that T and f satisfy*

$$\mu(d(Tx, fy)) \leq \mu(\frac{1}{2}[d(x, fy) + d(y, Tx)]) - \psi(d(x, fy), d(y, Tx)), \quad (2.1)$$

where $\mu : [0,\infty) \to [0,\infty)$ is an altering distance function and $\psi : [0,\infty)^2 \to [0,\infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$, for all $x, y \in M$. Then T and f have a unique common fixed point.

Proof. Let $x_0 \in M$. We can choose $x_1, x_2 \in M$ such that $x_1 = Tx_0$ and $x_2 = fx_1$. By induction, we construct a sequence $\{x_n\}$ in M such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = fx_{2n+1}$, for every $n \geq 0$. Consider

$$\begin{aligned}
\mu(d(x_{2n+1}, x_{2n+2})) &= \mu(d(Tx_{2n}, fx_{2n+1})) \\
&\leq \mu(\frac{1}{2}[d(x_{2n}, fx_{2n+1}) + d(x_{2n+1}, Tx_{2n})]) - \\
&\quad \psi(d(x_{2n}, fx_{2n+1}), d(x_{2n+1}, Tx_{2n})) \\
&= \mu(\frac{1}{2}d(x_{2n}, x_{2n+2})) - \psi(d(x_{2n}, x_{2n+2}), 0) \\
&\leq \mu(\frac{1}{2}d(x_{2n}, x_{2n+2})).
\end{aligned}$$

Since μ is a monotone increasing function, for all $n = 1, 2, \ldots$, we have $d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$. This implies that $d(x_{2n+1}, x_{2n+2}) \leq$

$d(x_{2n}, x_{2n+1})$. Following the similar arguments, we obtain $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$. Hence, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Thus $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$.

As $d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$, on letting $n \to \infty$, we have $r \leq \lim \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}r + \frac{1}{2}r$, $\lim d(x_{2n}, x_{2n+2}) = 2r$. Using the continuity of μ and lower semi-continuity of ψ, we have $\mu(r) \leq \mu(r) - \psi(2r, 0)$. This implies that $\psi(2r, 0) = 0$ and hence $r = 0$ i.e. $d(x_{n+1}, x_n) \to 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$, $d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon$. This means that $d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon$. So, we have

$$\begin{aligned}
\epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\
&\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\
&< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).
\end{aligned}$$

On letting $k \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we have

$$\lim d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{2.2}$$

Also,

$$\begin{aligned}
\epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\
&\leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}) \\
&\leq 2d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)}, x_{2n(k)}).
\end{aligned}$$

On letting $k \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we have

$$\lim d(x_{2m(k)-1}, x_{2n(k)}) = \epsilon = \lim d(x_{2m(k)}, x_{2n(k)}). \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
d(x_{2m(k)}, x_{2n(k)}) &\leq d(x_{2m(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \\
&\leq d(x_{2m(k)}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).
\end{aligned}$$

On letting $k \to \infty$, we have $\lim d(x_{2m(k)}, x_{2n(k)+1}) = \epsilon$. Also,

$$\begin{aligned}
d(x_{2m(k)-1}, x_{2n(k)}) &\leq d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \\
&\leq d(x_{2m(k)-1}, x_{2n(k)}) + 2d(x_{2n(k)+1}, x_{2n(k)}).
\end{aligned}$$

On letting $k \to \infty$, we have $\lim d(x_{2m(k)-1}, x_{2n(k)+1}) = \epsilon$.

Consider

$$\begin{aligned}
\mu(\epsilon) &\leq \mu(d(x_{2m(k)}, x_{2n(k)})) = \mu(d(Tx_{2m(k)-1}, fx_{2n(k)-1})) \\
&\leq \mu(\frac{1}{2}[d(x_{2m(k)-1}, fx_{2n(k)-1}) + d(x_{2n(k)-1}, Tx_{2m(k)-1})]) - \\
&\quad \psi(d(x_{2m(k)-1}, fx_{2n(k)-1}), d(x_{2n(k)-1}, Tx_{2m(k)-1})) \\
&= \mu(\frac{1}{2}[d(x_{2m(k)-1}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)})]) - \\
&\quad \psi(d(x_{2m(k)-1}, x_{2n(k)}), d(x_{2n(k)-1}, x_{2m(k)})).
\end{aligned}$$

Taking $k \to \infty$, and using the continuity of μ and lower semi-continuity of ψ, we have $\mu(\epsilon) \leq \mu(\frac{1}{2}[\epsilon + \epsilon]) - \psi(\epsilon, \epsilon)$ and consequently $\psi(\epsilon, \epsilon) \leq 0$, which is contradiction since $\epsilon > 0$. Thus $\{x_{2n}\}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence and therefore is convergent in the complete metric space X, so there exists $t \in M$ such that $\lim\{x_n\} = t$. Consider

$$\begin{aligned}
\mu(d(x_{2n+1}, ft)) &= \mu(d(Tx_{2n}, ft)) \\
&\leq \mu(\frac{1}{2}[d(x_{2n}, ft) + d(t, Tx_{2n})]) - \psi(d(x_{2n}, ft), d(t, Tx_{2n})) \\
&= \mu(\frac{1}{2}[d(x_{2n}, ft) + d(t, x_{2n+1})]) - \psi(d(x_{2n}, ft), d(t, x_{2n+1})).
\end{aligned}$$

On letting $n \to \infty$, we have $\mu(d(t, ft)) \leq \mu(\frac{1}{2}d(t, ft)) - \psi(d(t, ft), 0)) \leq \mu(\frac{1}{2}d(t, ft))$, which is a contradiction unless $d(t, ft) = 0$ and hence $t = ft$.

Again, consider

$$\begin{aligned}
\mu(d(Tt, t)) &= \mu(d(Tt, ft)) \\
&\leq \mu(\frac{1}{2}[d(t, ft) + d(t, Tt)]) - \psi(d(t, ft), d(t, Tt)) \\
&= \mu(\frac{1}{2}d(t, Tt)) - \psi(0, d(t, Tt)) \\
&\leq \mu(\frac{1}{2}d(t, Tt)).
\end{aligned}$$

This is a contradiction, hence $d(Tt, t) = 0$, $Tt = t$. Therefore, $t = Tt = ft$, i.e. t is a common fixed point of T and f. Now, we establish the uniqueness of common fixed point of T and f.

If z_1, z_2 are two common fixed points of T and f, then

$$\begin{aligned}
\mu(d(z_1, z_2)) &= \mu(d(T(z_1), f(z_2))) \\
&\leq \mu(\frac{1}{2}[d(z_1, fz_2) + d(z_2, Tz_1)]) - \psi(d(z_1, fz_2), d(z_2, Tz_1)) \\
&= \mu(\frac{1}{2}[d(z_1, z_2) + d(z_2, z_1)]) - \psi(d(z_1, z_2), d(z_2, z_1)) \\
&= \mu(d(z_1, z_2)) - \psi(d(z_1, z_2), d(z_2, z_1)).
\end{aligned}$$

This implies that $d(z_1, z_2) = 0$ i.e. $z_1 = z_2$, by the property of ψ.

□

If $T = f$, we have the following result.

Corollary 2.2. *(see [4]) Let M be a subset of a complete metric space (X, d) and T is a self-mapping of M. Suppose that T satisfies*

$$\mu(d(Tx, Ty)) \leq \mu(\frac{1}{2}[d(x, Ty) + d(y, Tx)]) - \psi(d(x, Ty), d(y, Tx)), \quad (2.4)$$

where $\mu : [0, \infty) \to [0, \infty)$ is an altering distance function and $\psi : [0, \infty)^2 \to [0, \infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$, for all $x, y \in M$. Then T has a unique fixed point.

If $\mu(t) = t$, we have the following result.

Corollary 2.3. *(see [3], [9]) Let M be a subset of a complete metric space (X,d) and T is a self-mapping of M. Suppose that T satisfies*

$$d(Tx,Ty) \;\leq\; \frac{1}{2}[d(x,Ty)+d(y,Tx)]-\psi(d(x,Ty),d(y,Tx)), \tag{2.5}$$

where $\psi:[0,\infty)^2\to[0,\infty)$ is a lower semi-continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$, for all $x,y\in M$. Then T has a unique fixed point.

If $\psi(s,t))=(\frac{1}{2}-\frac{k}{2})(s+t)$, where $k\in[0,1)$, in Corollary 2.3, we have the following result.

Corollary 2.4. *Let M be a subset of a complete metric space (X,d) and T is a self-mapping of M. Suppose that T satisfies*

$$d(Tx,Ty) \;\leq\; \frac{k}{2}[d(x,Ty)+d(y,Tx)], \tag{2.6}$$

for all $x,y\in M$, where $k\in[0,1)$. Then T has a unique fixed point.

Remarks 2.1. Corollary 2.3 is a special case of Theorems 4.1 and 4.3 in [15]. Also, the results of Rhoades [14], Boyd and Wang [2] and Reich [13] are all special cases of the results in [15] and as a consequence, we have some examples for the altering distance function.

Example 2.5. *Let $M=[0,1]$. Let d be defined by $d(x,y)=|x-y|$. We set $Tx=0$ and $fx=\frac{x^2}{8}$ for all $x\in M$. Define $\mu:[0,\infty)\to[0,\infty)$ and $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ by*

$$\mu(t)=\frac{t}{2},\ \textit{and}\ \psi(t,s)=\frac{t+s}{16}.$$

Then for $x,y\in M$, we have

$$\mu(d(Tx,fy))=\mu(d(0,\frac{y^2}{8}))=\mu(\frac{y^2}{8})=\frac{y^2}{16},$$

and

$$\begin{aligned}
&\mu(\frac{1}{2}[d(x,fy)+d(y,Tx)])-\psi(d(x,fy),d(y,Tx))\\
=\;&\mu(\frac{1}{2}[d(x,\frac{y^2}{8})+d(y,0)])-\psi(d(x,\frac{y^2}{8}),d(y,0))\\
=\;&\mu(\frac{1}{2}[|x-\frac{y^2}{8}|+y])-\psi(|x-\frac{y^2}{8}|,y)\\
=\;&\frac{1}{4}[|x-\frac{y^2}{8}|+y]-\frac{|x-\frac{y^2}{8}|+y}{16}\\
=\;&\frac{3}{16}[|x-\frac{y^2}{8}|+y]\\
\geq\;&\frac{3y}{16}\\
\geq\;&\frac{y^2}{16}.
\end{aligned}$$

Hence

$$\mu(d(Tx,fy)) \;\leq\; \mu(\frac{1}{2}[d(x,fy)+d(y,Tx)]) - \psi(d(x,fy),d(y,Tx)),$$

Thus by Theorem 2.1, T and f have a unique common fixed point 0.

Example 2.6. *Let $M=[0,1]$. Let d be defined by $d(x,y)=|x-y|$. We set $Tx=0$ and $fx=\frac{x^2}{8}$ for all $x\in M$. Define $\mu:[0,\infty)\to[0,\infty)$ and $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ by*

$$\mu(t)=t, \text{ and } \psi(t,s)=\frac{t}{2}+\frac{3s}{8}.$$

Then for $x,y\in M$, we have

$$\mu(d(Tx,fy))=\mu(d(0,\frac{y^2}{8}))=\mu(\frac{y^2}{8})=\frac{y^2}{8},$$

and

$$\begin{aligned}
& \mu(\frac{1}{2}[d(x,fy)+d(y,Tx)]) - \psi(d(x,fy),d(y,Tx)) \\
=\;& \mu(\frac{1}{2}[d(x,\frac{y^2}{8})+d(y,0)]) - \psi(d(x,\frac{y^2}{8}),d(y,0)) \\
=\;& \mu(\frac{1}{2}[|x-\frac{y^2}{8}|+y]) - \psi(|x-\frac{y^2}{8}|,y) \\
=\;& \frac{1}{2}[|x-\frac{y^2}{8}|+y] - [\frac{|x-\frac{y^2}{8}|}{2}+\frac{3y}{8}] \\
=\;& \frac{5}{8}y \\
\geq\;& \frac{y^2}{8}.
\end{aligned}$$

Hence

$$\mu(d(Tx,fy)) \;\leq\; \mu(\frac{1}{2}[d(x,fy)+d(y,Tx)]) - \psi(d(x,fy),d(y,Tx)),$$

Thus by Theorem 2.1, T and f have a unique common fixed point 0.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces, New results in Operator Theory*, Advances Appl., Vol. 8 (I. Gohberg and Yu. Lyubich, eds.), Birkhauser, Basel, 1997, 7-22.

[2] D.W. Boyd and T.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.

[3] S. Chandok, *Some common fixed point theorems for generalized f-weakly contractive mappings*, J. Appl. Math. Informatics **29** (2011), 257-265.

[4] S. Chandok, *Some common fixed point theorems for generalized nonlinear contractive mappings*, Computers Math. Appl. **62** (2011), 3692-3699 doi: 10.1016/j.camwa.2011.09.009.

[5] S. Chandok, *Common fixed points, invariant approximation and generalized weak contractions*, Internat. J. Math. Math. Sci. Vol. 2012 (2012) Article ID 102980.

[6] S. Chandok, *Common fixed points for generalized nonlinear contractive mappings in metric spaces*, Mat. Vesnik **65** (2013), 29-34.

[7] S. Chandok, *Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces*, J. Nonlinear Anal. Opt. **4** (2013), 45-52.

[8] S.K. Chatterjea, *Fixed point theorem*, C.R. Acad. Bulgare Sci. **25** (1972), 727-730.

[9] B.S. Choudhury, *Unique fixed point theorem for weakly C-contractive mappings*, Kathmandu University J. Sci. Engg. Tech. **5** (2009), 6-13.

[10] D. Doric, *Common fixed point for generalized (ψ, ϕ)-weak contractions*, Appl. Math. Letters **22** (2009), 1896-1900.

[11] R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly **76** (1969), 405-408.

[12] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1-9.

[13] S. Reich, *Some fixed point problems*, Atti Acad. Naz. Lincei Ren. Cl. Sci. Fis. Mat. Natur. **57** (1975), 194-198.

[14] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal **47** (2001), 2683-2693.

[15] B. Djafari Rouhani and S. Moradi, *Common fixed point of multivalued generalized ϕ-weak contractive mappings*, Fixed Point Theory Appl., Volume 2010, Article ID 708984, 13 pages.

Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar-143001,India.

E-mail address: chansok.s@gmail.com; chandhok.sumit@gmail.com