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ON SOME PROJECTIVELY FLAT (α, β)-METRICS

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Abstract. The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β. We consider the projective flatness of Finsler spaces with some (α, β)-metrics. The purpose of the present paper is to find the conditions for a Finsler space with $L = \frac{\alpha^2}{\alpha-\beta} + \beta$ and $L = \alpha + \frac{\beta^{n+1}}{\alpha^n}$ to be projectively flat.

1. Introduction and preliminaries

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of an (α, β)-metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto [4]. A Finsler metric $L(x, y)$ is called an (α, β)-metric $L(\alpha, \beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n. The most interesting examples of (α, β)-metric are the Randers metric, Kropina metric and Matsumoto metric.

A Finsler space $F^n = (M^n, L)$ is called a locally Minkowski space [6] if M^n is covered by coordinate neighborhood system (x^i) in each of which L is a function of y^i only. A Finsler space $F^n = (M^n, L)$ is called projectively flat if F^n is projective to a locally Minkowski space.

The condition for a Finsler space with (α, β)-metric to be projectively flat was studied by M. Matsumoto [5]. Aikou, Hashiguchi and Yamauchi [1] gave interesting results on the projective flatness of Matsumoto space. A locally Minkowski space with (α, β)-metric is called flat-parallel [2] if α is locally flat and β is parallel with respect to α.

The purpose of the present paper is to consider the projective flatness of Finsler space with $L = \frac{\alpha^2}{\alpha-\beta} + \beta$ and $L = \alpha + \frac{\beta^{n+1}}{\alpha^n}$.

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric and $R^n = (M^n, \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^iy^j$. Let γ^i_{jk} be the Christoffel symbols with respect to α and denoted by the semicolon $(;)$, the covariant differentiation with respect to γ^i_{jk}. From the differential

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1-form $\beta(x,y) = b_i(x)y^i$ we define

$$\begin{aligned} 2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i}, \quad s^i_j = a^{ir}s_{rj}, \\ s_i = b^r s_{ri}, \quad \gamma_{jhk} = a_{hr}\gamma^r_{jk}, \quad b^2 = a^{rs}b_r b_s. \end{aligned} \tag{1.1}$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity and the subscript 0 means contraction by y^i, for instance; $t_0 = t_i y^i$.

Now the following Matsumoto's theorem [5] is well-known:

Theorem 1.1. *A Finsler space $F^n = (M^n, L)$ with an (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that γ^i_{jk} satisfies:*

$$\begin{aligned} &\left(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2\right)/2 + (\alpha L_\beta/L_\alpha)\, s^i_0 \\ &\quad + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)\left(\alpha^2 b^i/\beta - y^i\right) = 0, \end{aligned} \tag{1.2}$$

where a subscript 0 means a contraction by $y^i, L_\alpha = \partial L/\partial\alpha, L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha, L_{\beta\beta} = \partial L_\beta/\partial\beta$ and C is given by

$$\begin{aligned} &C + \left(\alpha^2 L_\beta/\beta L_\alpha\right) s_0 \\ &\quad + \left(\alpha L_{\alpha\alpha}/\beta^2 L_\alpha\right)\left(\alpha^2 b^2 - \beta^2\right)(C + \alpha r_{00}/2\beta) = 0. \end{aligned} \tag{1.3}$$

By the homogeneity of L we know $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, so the formula (1.3) can be rewritten in the following form:

$$\begin{aligned} &\left\{1 + (L_{\beta\beta}/\alpha L_\alpha)\left(\alpha^2 b^2 - \beta^2\right)\right\}(C + \alpha r_{00}/2\beta) \\ &\quad = (\alpha/2\beta)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)\, s_0\right\} \end{aligned} \tag{1.4}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (1.2) and it is written as the form:

$$\begin{aligned} &\left\{1 + L_{\beta\beta}\left(\alpha^2 b^2 - \beta^2\right)/(\alpha L_\alpha)\right\}\left\{\left(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2\right)/2 + (\alpha L_\beta/L_\alpha)\, s^i_0\right\} \\ &\quad + (L_{\alpha\alpha}/L_\alpha)\left(\frac{\alpha}{2\beta}\right)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)\, s_0\right\}\left(\alpha^2 b^i/\beta - y^i\right) = 0. \end{aligned} \tag{1.5}$$

Thus we have

Theorem 1.2. *If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then a Finsler space F^n with an (α, β)-metric is projectively flat if and only if (1.5) is satisfied.*

It is known [3] that if α^2 contains β as a factor, then the dimension is equal to two and $b^2 = 0$.

Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0 \ (mod\ \beta)$.

2. Finsler Space with the Metric $L = \frac{\alpha^2}{\alpha - \beta} + \beta$

Let F^n be a Finsler space with an (α, β)-metric given by

$$L = \frac{\alpha^2}{\alpha - \beta} + \beta \tag{2.1}$$

The partial derivatives with respect to α and β of (2.1) are given by

$$L_\alpha = \frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2}, \quad L_\beta = \frac{2\alpha(\alpha-\beta)+\beta^2}{(\alpha-\beta)^2},$$
$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}. \tag{2.2}$$

If $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)=0$, then we have $\alpha(1+2b^2)=3\beta$ which leads a contradiction. Thus $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)\neq 0$ and hence Theorem (1.2) can be applied.
Substituting (2.2) into (1.5), we get

$$\begin{aligned}&4\alpha^6\left\{(1+2b^2)s^i_0-2b^is_0\right\}+\alpha^5\left\{(1+2b^2)\gamma^i_{00}-8\beta(2+b^2)s^i_0+2b^i(r_{00}+4\beta s_0)\right\}\\&+\alpha^4\beta\left\{-(5+4b^2)\gamma^i_{00}+2\beta s^i_0(7+2b^2)-4b^ir_{00}-4s_0(\beta b^i-2y^i)\right\}\\&+\alpha^3\left\{-(1+2b^2)\gamma_{000}y^i+6\beta^2\gamma^i_{00}-6\beta^3s^i_0-2\beta y^ir_{00}-8\beta^2y^is_0\right\}\\&+\alpha^2\beta y^i\left\{(5+4b^2)\gamma_{000}+4\beta r_{00}+4\beta^2s_0\right\}-6\alpha\beta^2\gamma_{000}y^i=0.\end{aligned} \tag{2.3}$$

Then the above equation (2.3) can be rewritten as a polynomial of sixth degree in α as follows:

$$(q_4\alpha^4+q_2\alpha^2+q_0)\alpha+(q_5\alpha^6+q_3\alpha^4+q_1\alpha^2)=0,$$

where

$$\begin{aligned}q_5&=4\left\{(1+2b^2)s^i_0-2b^is_0\right\},\\q_4&=(1+2b^2)\gamma^i_{00}-8\beta(2+b^2)s^i_0+2b^i(r_{00}+4\beta s_0),\\q_3&=\beta\left\{-(5+4b^2)\gamma^i_{00}+2\beta s^i_0(7+2b^2)-4b^ir_{00}-4s_0(\beta b^i-2y^i)\right\},\\q_2&=-(1+2b^2)\gamma_{000}y^i+6\beta^2\gamma^i_{00}-6\beta^3s^i_0-2\beta y^ir_{00}-8\beta^2y^is_0,\\q_1&=\beta y^i\left\{(5+4b^2)\gamma_{000}+4\beta r_{00}+4\beta^2s_0\right\},\\q_0&=-6\beta^2\gamma_{000}y^i.\end{aligned}$$

Since $q_5\alpha^6+q_3\alpha^4+q_1\alpha^2$ and $q_4\alpha^4+q_2\alpha^2+q_0$ are rational and α is irrational in y^i, we have

$$q_4\alpha^4+q_2\alpha^2+q_0=0, \tag{2.4}$$

and

$$q_5\alpha^6+q_3\alpha^4+q_1\alpha^2=0. \tag{2.5}$$

The term which does not contain β in (2.5) is $q_5\alpha^6$. Therefore there exists a homogeneous polynomial V_6 of degree six in y^i such that

$$4\left\{(1+2b^2)s^i_0-2b^is_0\right\}\alpha^6=\beta V_6.$$

Since $\alpha^2\not\equiv 0\ (mod\ \beta)$, we must have a function $u^i=u^i(x)$ satisfying

$$4\left\{(1+2b^2)s^i_0-2b^is_0\right\}=u^i\beta. \tag{2.6}$$

Transvecting (2.6) by b_i, we have

$$4s_0=u^i\beta b_i, \tag{2.7}$$

that is,

$$4s_j = u^i b_i b_j.$$

Furthermore transvecting this equation by b^j, we have $u^i b_i b^2 = 0$, that is, $u^i b_i = 0$. Substituting this equation into (2.7), we have $s_0 = 0$. Therefore, from (2.6), we get

$$4(1 + 2b^2)s_{ij} = u_i b_j, \tag{2.8}$$

which implies $u_i b_j + u_j b_i = 0$. Transvecting this equation by b^j, we have $u_i b^2 = 0$ by virtue of $u_j b^j = 0$. Therefore we get $u_i = 0$. Hence, from (2.8), we have $s_{ij} = 0$, provided $(1 + 2b^2) \neq 0$.
On the other hand, from (2.4) we have 1-form $w_0 = w_i(x)y^i$ such that

$$\gamma_{000} = w_0 \alpha^2. \tag{2.9}$$

Substituting $s_0 = 0, s_0^i = 0$ and (2.9) into (2.3), we have

$$(\alpha^2 - 3\alpha\beta + 2b^2\alpha^2)(\gamma_{00}^i - w_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i) = 0 \tag{2.10}$$

by virtue of $\alpha^2(\alpha - 2\beta) \neq 0$. Then the equation (2.10) is written in the form $P\alpha + Q = 0$, where

$$\begin{aligned} P &= -3\beta(\gamma_{00}^i - w_0 y^i), \\ Q &= (\alpha^2 + 2b^2\alpha^2)(\gamma_{00}^i - w_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i). \end{aligned}$$

Since P and Q are rational and α is irrational in y^i, we have $P = 0$ and $Q = 0$.

First, it follows from $P = 0$ that

$$\gamma_{00}^i - w_0 y^i = 0, \tag{2.11}$$

that is,

$$2\gamma_{jk}^i = w_j \delta_k^i + w_k \delta_j^i, \tag{2.12}$$

which shows that the associated Riemannian space (M, α) is projectively flat.

Next, from $Q = 0$ and (2.11) we have

$$r_{00}(\alpha^2 b^i - \beta y^i) = 0. \tag{2.13}$$

Transvecting (2.13) by b_i, we have $r_{00}(\alpha^2 b^2 - \beta^2) = 0$, from which $r_{00} = 0$, that is, $r_{ij} = 0$. Using $s_{ij} = 0$ and $r_{ij} = 0$ in (1.1) we have $b_{i;j} = 0$, by virtue of $\alpha^2 b^2 - \beta^2 \neq 0$.

Conversely, it is easily verified that (2.3) is a consequence of (2.11) and $b_{i;j} = 0$. Thus we have

Theorem 2.1. *A Finsler space F^n with an (α, β)-metric given by (2.1) is projectively flat, if and only if $b_{i;j} = 0$ and we have the associated Riemannian space (M^n, α) is projectively flat.*

3. Finsler Space with the Metric $L = \alpha + \frac{\beta^{n+1}}{\alpha^n}$

Let F^n be a Finsler space with an (α, β)-metric given by

$$L = \alpha + \frac{\beta^{n+1}}{\alpha^n} \tag{3.1}$$

The partial derivatives with respect to α and β of (3.1) are given by

$$\begin{aligned} L_\alpha &= \frac{\alpha^{n+1} - n\beta^{n+1}}{\alpha^{n+1}}, \quad L_\beta &= \frac{(n+1)\beta^n}{\alpha^n}, \\ L_{\alpha\alpha} &= \frac{n(n+1)\beta^{n+1}}{\alpha^{n+2}}, \quad L_{\beta\beta} &= \frac{n(n+1)\beta^{n-1}}{\alpha^n}. \end{aligned} \tag{3.2}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) = 0$, then we have $-(n^2+2n)\beta^{n+2} + \alpha^{n+1}\beta + n(n+1)b^2\alpha^2\beta^n = 0$ which leads a contradiction. Thus $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$ and hence Theorem (1.2) can be applied.
Substituting (3.2) into (1.5), we get

$$\begin{aligned} &\left[-(n^2+2n)\beta^{n+2} + \alpha^{n+1}\beta + n(n+1)b^2\alpha^2\beta^n\right] \\ &\left\{(\alpha^2\gamma^i_{00} - \gamma_{000}y^i)(\alpha^{n+1} - n\beta^{n+1}) + 2\alpha^4\beta^n(n+1)s^i_0\right\} \\ &+ n(n+1)\alpha^2\beta^n(\alpha^2 b^i - \beta y^i)\left\{r_{00}(\alpha^{n+1} - n\beta^{n+1}) - 2(n+1)\alpha^2\beta^n s_0\right\} = 0. \end{aligned} \tag{3.3}$$

Only the term $-n^2(n+2)\beta^{2n+3}\gamma_{000}y^i$ of (3.3) seemingly does not contain α^2 and hence we must have $hp(2n+4)v^i_{2n+4}$ satisfying $-n^2(n+2)\beta^{2n+3}\gamma_{000}y^i = \alpha^2 v^i_{2n+4}$. For the sake of brevity we suppose $\alpha^2 \not\equiv 0$ $(mod\ \beta)$. Then the above is written as

$$\gamma_{000} = v_0\alpha^2, \tag{3.4}$$

where v_0 is $hp(1)$. Substituting from (3.4), the equation (3.3) reduces to

$$\begin{aligned} &(\gamma^i_{00} - v_0 y^i)(\alpha^{n+1} - n\beta^{n+1})\left[-(n^2+2n)\beta^{n+2} + \alpha^{n+1}\beta + n(n+1)b^2\alpha^2\beta^n\right] \\ &+ 2\alpha^2\beta^n(n+1)s^i_0\left[-(n^2+2n)\beta^{n+2} + \alpha^{n+1}\beta + n(n+1)b^2\alpha^2\beta^n\right] \\ &+ n(n+1)\beta^n(\alpha^2 b^i - \beta y^i)\left\{r_{00}(\alpha^{n+1} - n\beta^{n+1}) - 2(n+1)\alpha^2\beta^n s_0\right\} = 0. \end{aligned} \tag{3.5}$$

The terms of (3.5) which seemingly do not contain α^2 are
$n^2\beta^{2n+2}\left\{(n+2)\beta(\gamma^i_{00} - v_0 y^i) + (n+1)y^i r_{00}\right\}$.
Consequently we must have $hp(1)u^i_0$ such that the above is equal to $\alpha^2 n^2\beta^{2n+2}u^i_0$.
Thus we come by

$$(n+2)\beta(\gamma^i_{00} - v_0 y^i) + (n+1)y^i r_{00} = \alpha^2 u^i_0. \tag{3.6}$$

Contraction of (3.6) by $a_{ir}y^r$ leads to

$$(n+1)r_{00} = u^i_0 y_i. \tag{3.7}$$

Substituting (3.7) in (3.6), we obtain

$$\gamma^i_{00} = v_0 y^i, \tag{3.8}$$

which yields

$$2\gamma^i_{jk} = v_k\delta^i_j + v_j\delta^i_k. \tag{3.9}$$

Consequently the equation (3.9) shows that the associated Riemannian space is projectively flat.

Next, substituting (3.8) in (3.5), we have

$$2\alpha^2 s_0^i \left[-(n^2+2n)\beta^{n+2} + \alpha^{n+1}\beta + n(n+1)b^2\alpha^2\beta^n\right] + n(\alpha^2 b^i - \beta y^i)\left\{r_{00}(\alpha^{n+1} - n\beta^{n+1}) - 2(n+1)\alpha^2\beta^n s_0\right\} = 0. \tag{3.10}$$

Transvecting (3.10) by b_i, we have

$$2\alpha^2\beta s_0 + n(\alpha^2 b^2 - \beta^2)r_{00} = 0. \tag{3.11}$$

Since $\alpha^2 b^2 - \beta^2$ of (3.11) does not contain α^2 as a factor, we must have a function $k(x)$ such that

$$nr_{00} = k(x)\alpha^2. \tag{3.12}$$

Substituting form (3.12), the equation (3.11) reduces to $2\beta s_0 + (\alpha^2 b^2 - \beta^2)k(x) = 0$, which gives $(b_i s_j + b_j s_i) + k(x)(b^2 a_{ij} - b_i b_j) = 0$. Transvection by $y^i y^j$ leads to $k(x) = 0$, because $b^2\alpha^2 - \beta^2$ does not vanish. Thus we get

$$r_{00} = 0; \quad r_{ij} = 0 \quad and \quad s_0 = 0; \quad s_i = 0. \tag{3.13}$$

Substituting from (3.13) the equation (3.10) reduces to $s_0^i = 0$, that is, $s_{ij} = 0$.

Summarizing up, we obtain $b_{i;j} = 0$ from $r_{ij} = 0$ and $s_{ij} = 0$.

Conversely, if $b_{i;j} = 0$, then we come by $r_{00} = 0, s_0^i = 0$ and $s_0 = 0$. So (3.5) follows from (3.8).

Thus we have the following

Theorem 3.1. *A Finsler space F^n with an (α, β)-metric $L(\alpha, \beta)$ given by (3.1) is projectively flat, if and only if we have $b_{i;j} = 0$ and the associated Riemannian space (M^n, α) is projectively flat.*

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