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# HYBRID LANGEVIN EQUATIONS INVOLVING $\psi$-HILFER FRACTIONAL-ORDER WITH NONLOCAL BOUNDARY VALUES

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Abstract. In this work, we are interested in studying existence and uniqueness criteria for a class of hybrid $\psi$-Hilfer Langevin equations of fractional order with multipoint boundary conditions. After specifying the conditions and framework, we prove the existence result using Krasnoselskii's fixed point theorem, and under some additional hypotheses, we prove the uniqueness of the solution using Banach's contraction principle theorem. To support our conclusions from a numerical point of view, we formulate two examples that also illustrate the importance of the main results presented.

## 1. Introduction

Fractional calculus is a growing field of mathematics specializing in studying the characteristics of integrals and derivatives of non-integer order, whose first use dates back to the end of the 17th century by G. Leibniz, J. Bernoulli and other mathematicians. It has been recently discovered that fractional differential equations (FDEs) have been successfully used in modeling and describing a wide range of nonlinear phenomena in various branches of science and engineering fields, including mathematical biology, plasma physics, complex systems, quantum mechanics, elasticity, bioengineering, biomathematics, control systems, optics , fluid mechanics, electrochemical reactions, viscoelastic materials, electromagnetic fields, control systems, and other (see [3, 8] and references therein). The success of FDEs models certainly relies on various factors such as reliability, integration of memory effects, accuracy, and flexibility. For modeling physical phenomena, fractional differential equations offer a more flexible framework by giving a more accurate representation of the system behavior, they take into account the history of the system over a certain time interval [18]. Modeling fractional differential equation issues and developing analytical, numerical, and precise solutions to these models is an important area of scientific research. Many mathematical methods have been refined in the literature. Furthermore, fresh techniques to fractional analysis are discovered by developing new definitions of fractional

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derivatives and integrals; among them are Riemann–Liouville [13], Caputo [13], Hadamard [10], Grünwald–Letnikov [16] and Riesz [17].

In 2018, Sousa and al. [21] published a new derivative called the $\psi$-Hilfer fractional derivative with order $\alpha > 0$ and parameter $\beta \in (0,1]$; this derivative is notable because it includes the fractional derivative operators mentioned above, as well as others such as Erdely-Kober [11], $\psi$-Caputo [2], Katugampola [9] and Hilfer [14]. As a result, a single fractional derivative operator can be used to investigate many fractional differential equations (FDEs) that use different fractional derivative operators. For more details, see the works [22, 23, 1, 15].

Recently, Sitho et al. [20] examined and demonstrated existence result for the following equation, supported by specific initial value:

$$\begin{cases} \mathbb{D}^{\alpha}\left[\dfrac{\mathbb{D}^{\beta}\mathtt{y}(\mathfrak{r}) - \sum\limits_{i=1}^{m}\mathbb{I}^{\omega_i}\mathtt{f}_i(\mathfrak{r},\mathtt{y}(\mathfrak{r}))}{\mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))}\right] = \mathtt{h}(\mathfrak{r},\ \mathtt{y}(\mathfrak{r}),\mathbb{I}^{\gamma}\mathtt{y}(\mathfrak{r})), \quad \mathfrak{r}\in[0,1], \\ \mathtt{y}(0) = \mathbb{D}^{\beta}\mathtt{y}(0) = 0, \end{cases}$$

where both $\mathbb{D}^{\alpha}$ and $\mathbb{D}^{\beta}$ are Riemann-Liouville fractional derivatives of orders $\alpha \in (0,1]$ and $\beta \in (0,1]$ respectively.

By making use the technique of measure of noncompactness combined with a generalized version of Darbo's fixed point theorem, Salem et al. [19] proved an existence result for the following hybrid Langevin differential equation of the second type involving two distinct orders of Caputo derivatives:

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha}\left[{}^{C}\mathbb{D}^{\beta}\left[\dfrac{\mathtt{y}(\mathfrak{r})}{\mathtt{f}(\mathfrak{r},\mathtt{y}(\upsilon(\mathfrak{r})))}\right] - \lambda\mathtt{y}(\mathfrak{r})\right] = \mathtt{g}(\mathfrak{r},\mathtt{y}(\mu(\mathfrak{r})),\mathbb{I}^{\gamma}\mathtt{y}(\mu(\mathfrak{r}))), \quad \mathfrak{r}\in[0,1], \\ \mathtt{y}(0) = {}^{C}\mathbb{D}^{\beta}\left[\dfrac{\mathtt{y}(\mathfrak{r})}{\mathtt{f}(\mathfrak{r},\mathtt{y}(\upsilon(\mathfrak{r})))}\right]_{\mathfrak{r}=0} = 0, \\ \mathtt{y}(1) = \eta\mathtt{y}(\eta), \quad \eta\in(0,1),\ \eta = \dfrac{\mathtt{f}(1,\mathtt{y}(\upsilon(1)))}{\mathtt{f}(\eta,\mathtt{y}(\upsilon(\eta)))}, \end{cases}$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$, $\gamma \in (0,1)$ and $\lambda \in \mathbb{R} - \{0\}$ respectively. While, $\mu,\ \upsilon : [0,1] \to [0,1]$ are two continuous functions.

Motivated by the importance of Hybrid FDEs that can treat several dynamical systems as special cases (see [5, 6, 7, 24]) and the fact that $\psi$-Hilfer derivative generalizes the well-known fractional derivatives (Caputo, $\psi$-Riemman-Liouville, Hilfer-Hadamard, Katugampola, ...), we investigate the existence theory for the following nonlocal boundary value problem to the nonlinear fractional hybrid Langevin differential equation,

$$\begin{cases} {}^{H}\mathbb{D}_{\mathtt{a}^+}^{(\alpha_1,\beta_1);\psi}\left[\left({}^{H}\mathbb{D}_{\mathtt{a}^+}^{(\alpha_2,\beta_2);\psi} + \mathtt{k}\right)[\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))]\right] = \mathtt{f}\left(\mathfrak{r},\mathtt{y}(\mathfrak{r})\right), \quad \mathfrak{r}\in\Omega := [\mathtt{a},\mathtt{d}] \\ \mathtt{y}(\mathtt{a}) = 0 \\ \mathtt{y}(\mathtt{d}) = \sum\limits_{i=1}^{p}\lambda_i\mathtt{y}(\tau_i), \end{cases} \tag{1.1}$$

where ${}^{H}\mathbb{D}_{\mathtt{a}^+}^{(\alpha_i,\beta_i);\psi}$ represents the $\psi$-Hilfer fractional derivative with order $\alpha_i \in (0,1]$ and parameter $\beta_i \in (0,1]$ $(i \in \{1,2\})$, $\mathtt{d} > \mathtt{a} \geq 0$, $\mathtt{f},\mathtt{g} : \Omega\times\mathbb{R} \to \mathbb{R}$ are two continuous functions, while $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $\tau_1, \tau_2, \ldots, \tau_p$ are two finite sequence of real numbers such that $\mathtt{a} < \tau_1 < \tau_2 < \ldots < \tau_p < \mathtt{d}$.

We organized this article as follows: After the introduction, we present some definitions and preliminary lemmas that will be used to establish our results in section 2. In section 3, we present and demonstrate the existence of solutions to problem (1.1). In section 4, we present two specific examples in order to concretize the main results and give them credibility. Whereas, the final section is allocated for a brief conclusion on the subject.

## 2. Basic tools

Let $(\mathtt{a}, \mathtt{d}) \in \mathbb{R}^2$ with $0 \leq \mathtt{a} < \mathtt{d}$. The uniform norm of a function $\mathtt{f}$ defined and continuous on the closed interval $[\mathtt{a}, \mathtt{d}]$ with values in $\mathbb{R}$ is defined by $\|\mathtt{f}\| = \sup\limits_{s \in [\mathtt{a}, \mathtt{d}]} |\mathtt{f}(s)|$. Morever, the set off all continuous functions, denoted $(C([\mathtt{a}, \mathtt{d}]), \|.\|)$, is a Banach space.

In this section, we present some lemmas definitions and properties about fractional calculus that we will need in our investigation. We then consider $\psi \in C^1([\mathtt{a}, \mathtt{d}], \mathbb{R})$ an increasing function with $\psi'(\mathfrak{r}) \neq 0$ for all $\mathfrak{r} \in [\mathtt{a}, \mathtt{d}]$.

**Definition 2.1.** [21] Let $\alpha > 0$ and $\mathtt{f} \in L^1([\mathtt{a}, \mathtt{d}], \mathbb{R})$. Then the $\psi$-RL-fractional integral of a function $\mathtt{f}$ with respect to $\psi$ is defined by

$$\mathbb{I}^{\alpha;\psi}\mathtt{f}(\mathfrak{r}) = \frac{1}{\Gamma(\alpha)} \int_{\mathtt{a}}^{\mathfrak{r}} \psi'(s)(\psi(\mathfrak{r}) - \psi(s))^{\alpha-1}\mathtt{f}(s)ds.$$

**Definition 2.2.** [21] Let $n \in \mathbb{N} - \{0\}$ and $\alpha \in (n-1, n]$. If $\mathtt{f} \in C^n([\mathtt{a}, \mathtt{d}], \mathbb{R})$ is a function of class $n$, then the $\psi$-Hilfer fractional derivative (HFD) ${}^H\mathbb{D}^{(\alpha,\beta);\psi}$ of a function $\mathtt{f}$ of order $\alpha$ and type $\beta \in (0, 1]$, is defined by

$${}^H\mathbb{D}^{(\alpha,\beta);\psi}\mathtt{f}(\mathfrak{r}) = \mathbb{I}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(\mathfrak{r})}\frac{d}{dt}\right)^n \mathbb{I}^{(1-\beta)(n-\alpha);\psi}\mathtt{f}(\mathfrak{r}) .$$

**Lemma 2.3.** [21] *If $\mathtt{f} \in C^n[\mathtt{a}, \mathtt{d}]$, $\alpha \in (n-1, n]$ and $\beta \in (0, 1]$, then*

(i) *Formula 1:*

$$\mathbb{I}^{\alpha;\psi}\left({}^H\mathbb{D}^{(\alpha,\beta);\psi}\mathtt{f}(\mathfrak{r})\right) = \mathtt{f}(\mathfrak{r}) - \sum_{k=1}^{n} \frac{(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\gamma-k}}{\Gamma(\varrho - k + 1)} \mathtt{f}_{\psi}^{[n-k]}\mathbb{I}^{(1-\beta)(n-\alpha);\psi}\mathtt{f}(\mathtt{a}), \quad (2.1)$$

*where* $\mathtt{f}_{\psi}^{[n-k]}\mathtt{f}(\mathfrak{r}) = \left(\frac{1}{\psi'(\mathfrak{r})}\frac{d}{dt}\right)^{n-k}\mathtt{f}(\mathfrak{r})$ *and* $\varrho = \alpha + \beta(n-\alpha)$ .

(ii) *Formula 2:*

$${}^H\mathbb{D}^{(\alpha,\beta);\psi}\left(\mathbb{I}^{\alpha;\psi}\mathtt{f}(\mathfrak{r})\right) = \mathtt{f}(\mathfrak{r}). \quad (2.2)$$

Now, we try to formulate a fractional integral equation and prove that it is equivalent to the nonlocal boundary value problem (1.1).

**Lemma 2.4.** [21] *Let $\mathfrak{p} > 0$.*
*The $\psi$-fractional integral of the power function $\cdot \mapsto (\psi(\cdot) - \psi(\mathtt{a}))^{\mathfrak{p}-1}$ are given by*

$$\mathbb{I}_{\mathtt{a}^+}^{\alpha;\psi}(\psi(\cdot) - \psi(\mathtt{a}))^{\mathfrak{p}-1}(\mathfrak{r}) = \frac{\Gamma(\mathfrak{p})}{\Gamma(\alpha + \mathfrak{p})}(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\alpha+\mathfrak{p}-1}, \quad \mathfrak{r} \in [\mathtt{a}, \mathtt{d}] \quad (2.3)$$

**Lemma 2.5.** *A function* $\mathtt{y} \in C([\mathtt{a},\mathtt{d}])$ *is a solution of the nonlocal boundary value problem* (1.1) *if and only if,*

$$\begin{aligned}\mathtt{y}(\mathfrak{r}) &= \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) + \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) - \mathtt{k}\cdot\mathbb{I}^{\alpha_2;\psi}\left(\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))\right)\\ &\quad + \frac{(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q}\cdot\Gamma(\varrho_1+\alpha_2)} \times \sum_{i=1}^{p+1}\lambda_i[\mathtt{g}(\tau_i,\mathtt{y}(\tau_i)) + \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i,\mathtt{y}(\tau_i))\\ &\qquad - \mathtt{k}\cdot\mathbb{I}^{\alpha_2;\psi}\left(\mathtt{y}(\tau_i) - \mathtt{g}(\tau_i,\mathtt{y}(\tau_i))\right)],\end{aligned} \tag{2.4}$$

*where we impose that,*

$$\mathtt{Q} = \sum_{i=1}^{p+1}\lambda_i\frac{(\psi(\tau_i)-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\Gamma(\varrho_1+\alpha_2)} \neq 0,$$

*with* $\lambda_{p+1} = -1$ *and* $\tau_{p+1} = \mathtt{d}$.

*Proof.* If we set $\mathtt{h}(\mathfrak{r}) = \left({}^H\mathbb{D}^{(\alpha_2,\beta_2);\psi} + \mathtt{k}\right)[\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))]$, the first equation of the problem (1.1) is expressed as

$${}^H\mathbb{D}^{(\alpha_1,\beta_1);\psi}\mathtt{h}(\mathfrak{r}) = \mathtt{f}\left(\mathfrak{r},\mathtt{y}(\mathfrak{r})\right). \tag{2.5}$$

Applying the integral operator $\mathbb{I}^{\alpha_1;\psi}_{\mathtt{a}^+}$ to both sides of the equation (2.5), with formula 1 of the Lemma 2.3 we obtain

$$\begin{aligned}\left({}^H\mathbb{D}^{(\alpha_2,\beta_2);\psi} + \mathtt{k}\right)[\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))] &= \mathbb{I}^{\alpha_1;\psi}_{\mathtt{a}}\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))\\ &\quad + \frac{\mathtt{C}_0}{\Gamma(\varrho_1)}(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1-1},\end{aligned} \tag{2.6}$$

here $c_0$ is a real constant and $\varrho_1 = \alpha_1 + \beta_1(1-\alpha_1)$ .
Now, applying the integral operator $\mathbb{I}^{\alpha_2;\psi}_{\mathtt{a}^+}$ to both sides of the equation (2.6), with formula 1 of the Lemma 2.3 and Lemma 2.4 we obtain

$$\begin{aligned}\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) &= \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) - \mathtt{k}\mathbb{I}^{\alpha_2;\psi}[\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))]\\ &\quad + \frac{\mathtt{C}_0}{\Gamma(\varrho_1+\alpha_2)}(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}\\ &\quad + \frac{\mathtt{C}_1}{\Gamma(\varrho_2)}(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_2-1}.\end{aligned} \tag{2.7}$$

Using $\mathtt{y}(\mathtt{a}) = 0$ in (2.7), we obtain $\mathtt{C}_1 = 0$, and hence we get

$$\begin{aligned}\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) &= \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r})) - \mathtt{k}\mathbb{I}^{\alpha_2;\psi}[\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))]\\ &\quad + \frac{\mathtt{C}_0}{\Gamma(\varrho_1+\alpha_2)}(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}.\end{aligned} \tag{2.8}$$

Next, by combining the boundary condition $\mathtt{y}(\mathtt{d}) = \sum_{i=1}^{p}\lambda_i\mathtt{y}(\tau_i)$ with the next equation (2.8), we obtain:

$$\mathtt{C}_0 = \frac{1}{\mathtt{Q}}\sum_{i=1}^{p+1}\lambda_i\left[\mathtt{g}(\tau_i,\mathtt{y}(\tau_i)) + \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i,\mathtt{y}(\tau_i)) - \mathtt{k}\cdot\mathbb{I}^{\alpha_2;\psi}\left(\mathtt{y}(\tau_i) - \mathtt{g}(\tau_i,\mathtt{y}(\tau_i))\right)\right],$$

where $\lambda_{p+1} = -1$ and $\tau_{p+1} = \mathtt{d}$. Substituting $\mathtt{C}_0$ in (2.8), we get the integral equation (2.4).

The converse can be easily proven by introducing the $\psi$-HFD derivative twice and applying formula 2 of the Lemma 2.3. $\square$

Fixed point theorems contribute significantly to the existence theory for solutions of nonlinear equations. We gather two well-known fixed point theorems used in this work .

**Lemma 2.6** (Banach contraction principle [12]). *Let $(\mathtt{M}, \delta)$ be a complete metric space and let $\mathcal{H} : \mathtt{M} \to \mathtt{M}$ be a contraction mapping, with Lipschitz constant $k < 1$. Then, $\mathcal{H}$ has a unique fixed point $\omega \in \mathtt{M}$, and for each $\mathtt{y} \in \mathtt{M}$, we have $\lim_{n\to+\infty} \mathcal{H}^n(\mathtt{y}) = \omega$. Moreover, for each $\mathtt{y} \in \mathtt{M}$,*

$$\delta(\mathcal{H}^n(\mathtt{y}), \omega) \leq \frac{k^n}{1-k} \cdot \delta(\mathtt{y}, \mathcal{H}(\mathtt{y})), \quad n = 1, 2, \ldots .$$

**Lemma 2.7** (Krasnoselskii's fixed point theorem [4]). *Let $\mathtt{S}$ be a closed convex nonempty subset of a Banach space $(\mathtt{E}, \|\cdot\|)$, Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ map $\mathtt{S}$ into $\mathtt{E}$ such that*

(i) *$\mathcal{H}_1\mathtt{y} + \mathcal{H}_2\mathtt{z} \in \mathtt{S}$, for all $(\mathtt{y}, \mathtt{z}) \in \mathtt{S}^2$,*
(ii) *$\mathcal{H}_1$ is continuous and $\mathcal{H}_1(\mathtt{S})$ is contained in a compact set,*
(iii) *$\mathcal{H}_2$ is a contraction with constant $k < 1$.*

*Then there is a $\mathtt{y} \in \mathtt{S}$ with $\mathcal{H}_1\mathtt{y} + \mathcal{H}_2\mathtt{y} = \mathtt{y}$.*

## 3. Main results

We devote this paragraph to presenting the theorems of the existence and uniqueness of solutions to problem (1.1), as well as to the detailed proofs. According to Lemma 2.5, we define an operator $\mathcal{H} : C(\Omega) \to C(\Omega)$ by

$$\begin{aligned}(\mathcal{H}\mathtt{y})(\mathfrak{r}) &= \mathtt{g}(\mathfrak{r}, \mathtt{y}(\mathfrak{r})) + \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r}, \mathtt{y}(\mathfrak{r})) - \mathtt{k}\cdot \mathbb{I}^{\alpha_2;\psi}\left(\mathtt{y}(\mathfrak{r}) - \mathtt{g}(\mathfrak{r}, \mathtt{y}(\mathfrak{r}))\right) \\ &\quad + \frac{(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q}\cdot\Gamma(\varrho_1+\alpha_2)} \times \sum_{i=1}^{p+1} \lambda_i [\mathtt{g}(\tau_i, \mathtt{y}(\tau_i)) + \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i, \mathtt{y}(\tau_i)) \\ &\quad - \mathtt{k}\cdot \mathbb{I}^{\alpha_2;\psi}\left(\mathtt{y}(\tau_i) - \mathtt{g}(\tau_i, \mathtt{y}(\tau_i))\right)].\end{aligned} \tag{3.1}$$

To reduce mathematical expressions, we set

$$\omega_1 = 1 + \frac{(\psi(\mathtt{d}) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)} \sum_{i=1}^{p+1} |\lambda_i|$$

and

$$\omega_2 = 1 + |\mathtt{k}| \cdot \frac{(\psi(\mathtt{d}) - \psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

3.1. **Existence results.** Our first existence result is based on Krasnoselskii's fixed point theorem (Lemma 2.7). Before proceeding with the first theorem, we are obligated to assume the following hypotheses:

($\mathbf{H_1}$)$\triangleright$ $\mathtt{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathtt{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions.
($\mathbf{H_2}$)$\triangleright$ $\mathtt{f}$ and $\mathtt{g}$ are two contraction mapping with respect to the second variable with,

$$|\mathtt{f}(\mathfrak{r}, \mathtt{y}) - \mathtt{f}(\mathfrak{r}, \mathtt{z})| \leq \mathcal{L}_{\mathtt{f}} |\mathtt{y} - \mathtt{z}|, \quad \forall\, (\mathfrak{r}, (\mathtt{y}, \mathtt{z})) \in \Omega \times \mathbb{R}^2,$$

and

$$|\mathtt{g}(\mathfrak{r}, \mathtt{y}) - \mathtt{g}(\mathfrak{r}, \mathtt{z})| \leq \mathcal{L}_{\mathtt{g}} |\mathtt{y} - \mathtt{z}|, \quad \forall\, (\mathfrak{r}, (\mathtt{y}, \mathtt{z})) \in \Omega \times \mathbb{R}^2.$$

($\mathbf{H_3}$)$\triangleright$ Suppose that is $\mathcal{M}_{\mathtt{f}} > 0$ and $\mathcal{M}_{\mathtt{g}} > 0$ such that

$$|\mathtt{f}(\mathfrak{r}, 0)| \leq \mathcal{M}_{\mathtt{f}} \text{ and } |\mathtt{g}(\mathfrak{r}, 0)| \leq \mathcal{M}_{\mathtt{g}}, \quad \forall \mathfrak{r} \in \Omega.$$

**Theorem 3.1.** *Assume that the hypotheses* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *are verified. If the previously defined expressions $\omega_1$ and $\omega_2$ satisfy the inequality*

$$\omega_1 \cdot \omega_2 \left(|\mathtt{k}| + \mathcal{L}_{\mathtt{f}} + \mathcal{L}_{\mathtt{g}}\right) < 1. \tag{3.2}$$

*The nonlocal boundary value problem (1.1) have at least one solution $\mathtt{y} \in C(\Omega)$.*

*Proof.* In order to show that the operator $\mathcal{H}$ defined by (3.1) satisfies the conditions required by Krasnoselskii's fixed point theorem, we proceed by dividing $\mathcal{H}$ into two adequate operators $\mathcal{H}_1$ and $\mathcal{H}_2$ defined on the closed ball $\mathtt{B}_{\mathtt{R}} = \{\mathtt{y} \in C(\Omega) \;:\; \|\mathtt{y}\| \leq \mathtt{R}\}$ with $\mathtt{R} \geq \dfrac{\omega_1 \cdot \omega_2(\mathcal{M}_{\mathtt{f}} + \mathcal{M}_{\mathtt{g}})}{1 - \omega_1 \cdot \omega_2(|\mathtt{k}| + \mathcal{L}_{\mathtt{f}} + \mathcal{L}_{\mathtt{g}})}$, as:

$$\begin{aligned}(\mathcal{H}_1\mathtt{y})(\mathfrak{r}) &= \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r}, \mathtt{y}(\mathfrak{r})) - \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\mathfrak{r}) \\ &\quad + \frac{(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q} \cdot \Gamma(\varrho_1 + \alpha_2)} \sum_{i=1}^{p+1} \lambda_i \left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i, \mathtt{y}(\tau_i)) - \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\tau_i)\right], \quad \mathfrak{r} \in \Omega,\end{aligned}$$

and

$$\begin{aligned}(\mathcal{H}_2\mathtt{y})(\mathfrak{r}) &= \mathtt{g}\left(\mathfrak{r}, \mathtt{y}(\mathfrak{r})\right) + \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{g}\left(\mathfrak{r}, \mathtt{y}(\mathfrak{r})\right) \\ &\quad + \frac{(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q} \cdot \Gamma(\varrho_1 + \alpha_2)} \sum_{i=1}^{p+1} \lambda_i \left[\mathtt{g}(\tau_i, \mathtt{y}(\tau_i)) + \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{g}(\tau_i, \mathtt{y}(\tau_i))\right], \quad \mathfrak{r} \in \Omega.\end{aligned}$$

For any $\mathtt{y}, \mathtt{z} \in \mathtt{B_R}$, referring to hypotheses $(\mathbf{H_2})$ and $(\mathbf{H_3})$, we get:

$$
\begin{aligned}
|(\mathcal{H}_1\mathtt{y})(\mathfrak{r})| \leq{}& \mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))-\mathtt{f}(\mathfrak{r},0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\mathfrak{r})| + \frac{(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q}\cdot\Gamma(\varrho_1+\alpha_2)}\times \\
& \sum_{i=1}^{p+1}|\lambda_i|\left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{f}(\tau_i,0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\tau_i)|\right] \\
& + \mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\mathfrak{r},0)| + \frac{(\psi(\mathfrak{r})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q}\cdot\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\cdot\mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\tau_i,0)| \\
& \leq \omega_1\cdot\left(1+|\mathtt{k}|\cdot\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\cdot\left(\mathcal{M}_\mathtt{f}+[|\mathtt{k}|+\mathcal{L}_\mathtt{f}]\,\|\mathtt{y}\|\right)
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
|(\mathcal{H}_2\mathtt{z})(\mathfrak{r})| \leq{}& |\mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))-\mathtt{g}(\mathfrak{r},0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\left(|\mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))-\mathtt{g}(\mathfrak{r},0)|\right) \\
& + \frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\times\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\times \\
& \left[|\mathtt{g}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{g}(\tau_i,0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\left(|\mathtt{g}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{g}(\tau_i,0)|\right)\right] \\
& + |\mathtt{g}(\mathfrak{r},0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\left(|\mathtt{g}(\mathfrak{r},0)|\right) \\
& + \frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\times\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\left[\mathtt{g}(\tau_i,0)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\left(|\mathtt{g}(\tau_i,0)|\right)\right] \\
& \leq \omega_1\cdot\left(1+|\mathtt{k}|\cdot\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\cdot\left(\mathcal{M}_\mathtt{g}+\mathcal{L}_\mathtt{g}\|\mathtt{z}\|\right)
\end{aligned}
$$

By triangle inequality formula, we obtain

$$
\begin{aligned}
\sup_{\mathfrak{r}\in\Omega}|(\mathcal{H}_1\mathtt{y})(\mathfrak{r})+(\mathcal{H}_2\mathtt{z})(\mathfrak{r})| &\leq \omega_1\cdot\omega_2\left[\mathcal{M}_\mathtt{f}+\mathcal{M}_\mathtt{g}+\left(|\mathtt{k}|+\mathcal{L}_\mathtt{f}\right)\|\mathtt{y}\|+\mathcal{L}_\mathtt{g}\|\mathtt{z}\|\right] \\
&\leq \omega_1\cdot\omega_2\left[\mathcal{M}_\mathtt{f}+\mathcal{M}_\mathtt{g}\right]+\omega_1\cdot\omega_2\left(|\mathtt{k}|+\mathcal{L}_\mathtt{f}+\mathcal{L}_\mathtt{g}\right)\mathtt{R} \\
&\leq \mathtt{R}.
\end{aligned}
$$

Hence $\|\mathcal{H}_1\mathtt{y}+\mathcal{H}_2\mathtt{z}\| \leq \mathtt{R}$, which implies that $\mathcal{H}_1\mathtt{y}+\mathcal{H}_2\mathtt{z}\in\mathtt{B_R}$.
For $\mathtt{y},\mathtt{z}\in\mathtt{B_R}$, we have

$$
\begin{aligned}
|(\mathcal{H}_2\mathtt{y})(\mathfrak{r})-(\mathcal{H}_2\mathtt{z})(\mathfrak{r})| \leq{}& |\mathtt{g}\left(\mathfrak{r},\mathtt{y}(\mathfrak{r})\right)-\mathtt{g}\left(\mathfrak{r},\mathtt{y}(\mathfrak{r})\right)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{g}\left(\mathfrak{r},\mathtt{y}(\mathfrak{r})\right)-\mathtt{g}\left(\mathfrak{r},\mathtt{z}(\mathfrak{r})\right)| \\
& + \frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{\mathtt{Q}\cdot\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\times \\
& \left[|\mathtt{g}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{g}\left(\tau_i,\mathtt{z}(\tau_i)\right)| + |\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{g}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{g}(\tau_i,\mathtt{z}(\tau_i))|\right] \\
& \leq \mathcal{L}_\mathtt{g}\cdot\left[1+\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\times\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\right]\times \\
& \qquad\left[1+|\mathtt{k}|\cdot\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]\cdot\|\mathtt{y}-\mathtt{z}\| \\
& \leq \mathcal{L}_\mathtt{g}\omega_1\omega_2\cdot\|\mathtt{y}-\mathtt{z}\|.
\end{aligned}
$$

According to (3.2), we obtain $\mathcal{L}_{\mathtt{g}}\omega_1\omega_2 < 1$. Hence, the operator $\mathcal{H}_2$ is a contraction mapping. The operator $\mathcal{H}_1$ is continuous. Indeed, let $\mathtt{y} \in \mathtt{B}_{\mathtt{R}}$ and $\{\mathtt{y}_n\}_0^\infty \subset \mathtt{B}_{\mathtt{R}}$ with $\lim\limits_{n\to+\infty} \|\mathtt{y}_n - \mathtt{y}\| = 0$. According to the hypothesis $(\mathbf{H_2})$, for each $t \in \Omega$,

$$\begin{aligned}
|(\mathcal{H}_1 y_n)(\mathfrak{r}) - (\mathcal{H}_1 \mathtt{y})(\mathfrak{r})| &\leq \mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}\left(\mathfrak{r}, y_n(\mathfrak{r})\right) - \mathtt{f}\left(\mathfrak{r}, \mathtt{y}(\mathfrak{r})\right)| + |\mathtt{k}| \cdot \mathbb{I}^{\alpha_2;\psi}|\mathtt{y}_n - \mathtt{y}(\mathfrak{r})| \\
&\quad + \frac{(\psi(\mathfrak{r}) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}| \cdot \times\Gamma(\varrho_1+\alpha_2)} \sum_{i=1}^{p+1} |\lambda_i| \times \\
&\quad \left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}\left(\mathfrak{r}, y_n(\mathfrak{r})\right) - \mathtt{f}\left(\mathfrak{r}, \mathtt{y}(\mathfrak{r})\right)| + |\mathtt{k}| \cdot \mathbb{I}^{\alpha_2;\psi}|\mathtt{y}_n - \mathtt{y}(\mathfrak{r})|\right] \\
&\leq \mathcal{L}_{\mathtt{f}} \cdot \omega_1 \left[1 + |\mathtt{k}| \cdot \frac{(\psi(\mathtt{d}) - \psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right] \|\mathtt{y}_n - \mathtt{y}\| \\
&\leq \mathcal{L}_{\mathtt{f}} \cdot \omega_1\omega_2 \|\mathtt{y}_n - \mathtt{y}\|.
\end{aligned}$$

Thus, $\|(\mathcal{H}_1\mathtt{y}_n) - (\mathcal{H}_1\mathtt{y})\| \leq \mathcal{L}_{\mathtt{f}} \cdot \omega_1\omega_2\|\mathtt{y}_n - \mathtt{y}\|$, which means that the opertator $\mathcal{H}_1$ is continuous on $\mathtt{B}_{\mathtt{R}}$.

From the inequality (3.3) and for all $\mathtt{y} \in \mathtt{B}_{\mathtt{R}}$, we have

$$\begin{aligned}
\sup_{\mathfrak{r}\in\Omega} |(\mathcal{H}_1\mathtt{y})(\mathfrak{r})| &\leq \omega_1 \cdot \omega_2 \cdot \left(\mathcal{M}_{\mathtt{f}} + [|\mathtt{k}| + \mathcal{L}_{\mathtt{f}}]\, \|\mathtt{y}\|\right) \\
&\leq \omega_1 \cdot \omega_2 \cdot \left(\mathcal{M}_{\mathtt{f}} + [|\mathtt{k}| + \mathcal{L}_{\mathtt{f}}]\, \mathtt{R}\right) < \infty.
\end{aligned}$$

Thus, $\mathcal{H}_1$ is uniformly bounded on $\mathtt{B}_{\mathtt{R}}$.

Now, we must prove that the operator $\mathcal{H}_1$ is compact. For $\mathfrak{r} \in \Omega$, we have:

$$\begin{aligned}
|(\mathcal{H}_1\mathtt{y})(\mathfrak{r}_2) - (\mathcal{H}_1\mathtt{y})(\mathfrak{r}_1)| &\leq \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r}_2, \mathtt{y}(\mathfrak{r}_2)) - \mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\mathfrak{r}_1, \mathtt{y}(\mathfrak{r}_1)) \\
&\quad - \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\mathfrak{r}_2) + k \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\mathfrak{r}_1) \\
&\quad + \frac{(\psi(\mathfrak{r}_2) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1} - (\psi(\mathfrak{r}_1) - \psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}| \cdot \Gamma(\varrho_1+\alpha_2)} \times \\
&\quad \sum_{i=1}^{p+1} |\lambda_i| \left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i, \mathtt{y}(\tau_i)) - \mathtt{k} \cdot \mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\tau_i)\right]
\end{aligned}$$

According to hypothesis $(\mathbf{H_1})$, the function $\mathtt{f}$ is continuous on the compact set $\Omega \times \mathtt{B}_{\mathtt{R}} \subset \mathbb{R}^2$, we denote $\|\mathtt{f}\|^{\Omega\times\mathtt{B}_{\mathtt{R}}} := \sup\limits_{\substack{\mathfrak{r}\in\Omega \\ \mathtt{y}\in\mathtt{B}_{\mathtt{R}}}} |\mathtt{f}(\mathfrak{r}, \mathtt{y})| < \infty$ the supremum of $\mathtt{f}$. Then,

$$
\begin{aligned}
|(\mathcal{H}_1\mathtt{y})(\mathfrak{r}_2)-(\mathcal{H}_1\mathtt{y})(\mathfrak{r}_1)| \leq\ & \frac{\|f\|^{\Omega\times\mathtt{B}_\mathtt{R}}}{\Gamma(\alpha_1+\alpha_2)}\left|\int_{\mathtt{a}}^{\mathfrak{r}_1}[\psi'(s)(\psi(\mathfrak{r}_2)-\psi(s))^{\alpha_1+\alpha_2-1}\right.\\
& -\psi'(s)(\psi(\mathfrak{r}_1)-\psi(s))^{\alpha_1+\alpha_2-1}]ds\\
& \left.+\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\psi'(s)(\psi(\mathfrak{r}_2)-\psi(s))^{\alpha_1+\alpha_2-1}ds\right|\\
& +\frac{\|\mathtt{y}\|}{\Gamma(\alpha_2)}\left|\int_{\mathtt{a}}^{\mathfrak{r}_1}[\psi'(s)(\psi(\mathfrak{r}_2)-\psi(s))^{\alpha_2-1}\right.\\
& \quad -\psi'(s)(\psi(\mathfrak{r}_1)-\psi(s))^{\alpha_2-1}]ds\\
& \left.+\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\psi'(s)(\psi(\mathfrak{r}_2)-\psi(s))^{\alpha_1+\alpha_2-1}ds\right|\\
& +\frac{(\psi(\mathfrak{r}_2)-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}-(\psi(\mathfrak{r}_1)-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)}\times\\
& \sum_{i=1}^{p+1}|\lambda_i|\left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{k}\cdot\mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\tau_i)\right]\\
& \leq\frac{\|f\|^{\Omega\times\mathtt{B}_\mathtt{R}}}{\Gamma(\alpha_1+\alpha_2+1)}\left[2(\psi(\mathfrak{r}_2)-\psi(\mathfrak{r}_1))^{\alpha_1+\alpha_2}\right.\\
& \left.+|(\psi(\mathfrak{r}_2)-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}-(\psi(\mathfrak{r}_1)-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}|\right]\\
& +\frac{\|\mathtt{y}\|}{\Gamma(\alpha_2+1)}[2(\psi(\mathfrak{r}_2)-\psi(\mathfrak{r}_1))^{\alpha_2}\\
& +|(\psi(\mathfrak{r}_2)-\psi(\mathtt{a}))^{\alpha_2}-(\psi(\mathfrak{r}_1)-\psi(\mathtt{a}))^{\alpha_2}|]\\
& +\frac{(\psi(\mathfrak{r}_2)-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}-(\psi(\mathfrak{r}_1)-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)}\times\\
& \sum_{i=1}^{p+1}|\lambda_i|\left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathtt{f}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{k}\cdot\mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\tau_i)\right]
\end{aligned}
$$

For arbitrary $\mathtt{y}\in\mathtt{B}_\mathtt{R}$, all terms on the left of the previous inequality tend to 0 as $\mathfrak{r}_2-\mathfrak{r}_1\to 0$. Thus, $\mathcal{H}_1$ is equicontinuous and therefore relatively compact on $\mathtt{B}_\mathtt{R}$. By Arzelà Ascoli theorem, the operator $\mathcal{H}_1$ is compact on $\mathtt{B}_\mathtt{R}$, it follows by Krasnoselskii's fixed point theorem (Lemma 2.7), that the problem (1.1) has at least one solution $\mathtt{y}\in C(\Omega)$. $\square$

3.2. **Uniqueness result.** In this section, we simply weaken the framework under which the first result was reached, by adding another condition to ensure uniqueness. Banach's contraction principle theorem constitutes the key to our investigation.

**Theorem 3.2.** *Assume that only hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are verified. If the previously defined expressions* $\omega_1$ *and* $\omega_2$ *satisfy the inequality*

$$\omega_1\cdot\omega_2\left(|\mathtt{k}|+\mathcal{L}_\mathtt{f}+\mathcal{L}_\mathtt{g}\right)<1. \tag{3.4}$$

*The nonlocal boundary value problem* (1.1) *have a unique solution* $\mathtt{y}^* \in C(\Omega)$, *provided that*

$$\omega_1\left[\mathcal{L}_{\mathtt{f}}\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+(\omega_2-1)\cdot(\mathcal{L}_{\mathtt{g}}+1)\right]<1. \tag{3.5}$$

*Proof.* Set $\mathtt{B}_{\mathtt{R}}=\{\mathtt{y}\in C(\Omega)\ :\ \|\mathtt{y}\|\leq \mathtt{R}\}$ the ball of radius $\mathtt{R}$, with

$$\mathtt{R}\geq\frac{\omega_1\cdot\omega_2(\mathcal{M}_{\mathtt{f}}+\mathcal{M}_{\mathtt{g}})}{1-\omega_1\cdot\omega_2(|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}+\mathcal{L}_{\mathtt{g}})}$$

First, let show that $\mathcal{H}(\mathtt{B}_{\mathtt{R}})\subset\mathtt{B}_{\mathtt{R}}$. For any $\mathtt{y}\in\mathtt{B}_{\mathtt{R}}$, by referring to (3.3) and (3.1) we obtain

$$\begin{aligned}|(\mathcal{H}\mathtt{y})(\mathfrak{r})|&\leq\omega_1\omega_2\left(\mathcal{M}_{\mathtt{f}}+[|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}]\,\|\mathtt{y}\|\right)+\omega_1\cdot\omega_2\left(\mathcal{M}_{\mathtt{g}}+\mathcal{L}_{\mathtt{g}}\|\mathtt{y}\|\right)\\&\leq\omega_1\cdot\omega_2\left[\mathcal{M}_{\mathtt{f}}+\mathcal{M}_{\mathtt{g}}+(|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}+\mathcal{L}_{\mathtt{g}})\,\|\mathtt{y}\|\right]\\&\leq\omega_1\cdot\omega_2(\mathcal{M}_{\mathtt{f}}+\mathcal{M}_{\mathtt{g}})+\omega_1\cdot\omega_2\left(|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}+\mathcal{L}_{\mathtt{g}}\right)\mathtt{R}\\&\leq\mathtt{R}.\end{aligned}$$

Therefore, $\|\mathcal{H}\mathtt{y}\|\leq\mathtt{R}$, which proves that $\mathcal{H}(\mathtt{B}_{\mathtt{R}})\subset\mathtt{B}_{\mathtt{R}}$.
Now, for $\mathtt{y},\mathtt{z}\in C(\Omega)$ we have:

$$\begin{aligned}|(\mathcal{H}\mathtt{y})(\mathfrak{r})-(\mathcal{H}\mathtt{z})(\mathfrak{r})|&\leq\mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))-\mathtt{f}(\mathfrak{r},\mathtt{z}(\mathfrak{r}))|+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\mathfrak{r})-\mathtt{z}(\mathfrak{r})|\\&\quad+|\mathtt{k}|\mathbb{I}^{\alpha_2;\psi}|\mathtt{g}(\mathfrak{r},\mathtt{y}(\mathfrak{r}))-\mathtt{g}(\mathfrak{r},\mathtt{z}(\mathfrak{r}))|\\&\quad+\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\times\\&\quad\Big[\mathbb{I}^{\alpha_1+\alpha_2;\psi}|\mathtt{f}(\tau_i,\mathtt{y}(\tau_i))-\mathtt{f}(\tau_i,\mathtt{z}(\tau_i))|\\&\quad+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\tau_i)-\mathtt{z}(\tau_i)|\\&\quad+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\mathtt{y}(\tau_i)-\mathtt{g}(\tau_i,\mathtt{y}(\tau_i))\Big]\\&\leq\mathbb{I}^{\alpha_1+\alpha_2;\psi}\left(\mathcal{L}_{\mathtt{f}}|\mathtt{y}(\mathfrak{r})-\mathtt{z}(\mathfrak{r})|\right)+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\mathfrak{r})-\mathtt{z}(\mathfrak{r})|\\&\quad+|\mathtt{k}|\mathbb{I}^{\alpha_2;\psi}\left(\mathcal{L}_{\mathtt{g}}|\mathtt{y}(\mathfrak{r})-\mathtt{z}(\mathfrak{r})|\right)+\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\\&\quad\times\left[\mathbb{I}^{\alpha_1+\alpha_2;\psi}\mathcal{L}_{\mathtt{f}}\left(|\mathtt{y}(\tau_i)-\mathtt{z}(\tau_i)|\right)+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}|\mathtt{y}(\tau_i)-\mathtt{z}(\tau_i)|\right.\\&\quad\left.+|\mathtt{k}|\cdot\mathbb{I}^{\alpha_2;\psi}\mathcal{L}_{\mathtt{g}}\left(|\mathtt{y}(\tau_i)-\mathtt{z}(\tau_i)|\right)\right]\\&\leq\left[\mathcal{L}_{\mathtt{f}}\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+|\mathtt{k}|\cdot(\mathcal{L}_{\mathtt{g}}+1)\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]\\&\quad\times\left[1+\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\varrho_1+\alpha_2-1}}{|\mathtt{Q}|\cdot\Gamma(\varrho_1+\alpha_2)}\sum_{i=1}^{p+1}|\lambda_i|\right]\|\mathtt{y}-\mathtt{z}\|\\&\leq\left[\mathcal{L}_{\mathtt{f}}\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+|\mathtt{k}|\cdot(\mathcal{L}_{\mathtt{g}}+1)\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]\\&\quad\times\omega_1\cdot\|\mathtt{y}-\mathtt{z}\|\end{aligned}$$

Therefore,

$$\|(\mathcal{H}\mathtt{y})-(\mathcal{H}\mathtt{z})\| \leq \omega_1 \left[\mathcal{L}_{\mathtt{f}} \frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + (\omega_2-1)\cdot(\mathcal{L}_{\mathtt{g}}+1)\right] \|\mathtt{y}-\mathtt{z}\|.$$

According to (3.5), the operator $\mathcal{H}$ is a contraction map and the ball $\mathtt{B}_{\mathtt{R}}$ is compact. By the Banach theorem (Lemma 2.3), we conclude that $\mathcal{H}$ have a unique a fixed point, which is the unique solution of the nonlocal boundary value problem (1.1). □

## 4. Numerical examples

**Example 4.1.** Consider the hybrid Langevin equation with the $\psi$-Hilfer fractional derivative and multipoint boundary values:

$$\begin{cases} {}^{H}\mathbb{D}^{\left(\frac{3}{4},\frac{1}{5}\right);\sqrt{\mathfrak{r}}} \left[\left({}^{H}\mathbb{D}^{\left(\frac{1}{5},\frac{2}{3}\right);\sqrt{\mathfrak{r}}} + \frac{1}{25}\right)\left(\mathtt{y}(\mathfrak{r}) - \frac{1}{e^{\mathfrak{r}}+14}\sqrt{1+|\mathtt{y}(\mathfrak{r})|}\right)\right] = \frac{\ln(2-\mathfrak{r})}{e^{\mathfrak{r}}+20} \times \\ \qquad\qquad \frac{y^2(\mathfrak{r})+2|\mathtt{y}(\mathfrak{r})|}{1+|\mathtt{y}(\mathfrak{r})|} + \frac{1}{3}, \ \mathfrak{r}\in[0,1], \\ \mathtt{y}(0) = 0, \\ \mathtt{y}(1) = \frac{3}{8}\mathtt{y}\left(\frac{1}{4}\right) + \frac{5}{8}\mathtt{y}\left(\frac{1}{2}\right) + \frac{7}{8}\mathtt{y}\left(\frac{3}{4}\right). \end{cases} \tag{4.1}$$

The problem (4.1) is a case similar to the boundary value problem (1.1) with $\Omega = [0,1]$, $(\alpha_1,\beta_1) = (\frac{3}{4},\frac{1}{5})$, $(\alpha_2,\beta_2) = \left(\frac{1}{5},\frac{2}{3}\right)$, $(\lambda_1,\tau_1) = (\frac{3}{8},\frac{1}{4})$, $(\lambda_2,\tau_2) = (\frac{5}{8},\frac{1}{2})$, $(\lambda_3,\tau_3) = (\frac{7}{8},\frac{3}{4})$, $(\lambda_4,\tau_4) = (-1,1)$ and $\psi(\mathfrak{r}) = \sqrt{\mathfrak{r}}$.

Note that $\mathtt{f}$ and $\mathtt{g}$ are continuous functions on $\Omega\times\mathbb{R}$ and lipschitz with respect to the second variable with constants $\mathcal{L}_{\mathtt{g}} = \frac{1}{30}$ and $\mathcal{L}_{\mathtt{f}} = \frac{\ln 2}{10}$ as

$$\begin{aligned} |\mathtt{f}(\mathfrak{r},\mathtt{y}) - \mathtt{f}(\mathfrak{r},\mathtt{z})| &\leq \sup_{\mathfrak{r}\in\Omega}\left|\frac{\ln(2-\mathfrak{r})}{e^{\mathfrak{r}}+19}\right|\left(\frac{\mathtt{y}^2+2|\mathtt{y}|}{1+|\mathtt{y}|} - \frac{\mathtt{z}^2+2|\mathtt{z}|}{1+|\mathtt{z}|}\right) \\ &\leq \frac{2\ln 2}{20}\left(1+\frac{1}{(1+|\mathtt{y}|)(1+|\mathtt{z}|)}\right)|\mathtt{y}-\mathtt{z}| \\ &\leq \frac{\ln 2}{10}|\mathtt{y}-\mathtt{z}|, \quad \text{for any } \mathtt{y},\mathtt{z}\in\mathbb{R}. \end{aligned}$$

In addition, we have immediately $\mathcal{M}_{\mathtt{f}} := \sup_{\mathfrak{r}\in\Omega}|\mathtt{f}(\mathfrak{r},0)| \leq \frac{1}{3}$ and $\mathcal{M}_{\mathtt{g}} := \sup_{\mathfrak{r}\in\Omega}|\mathtt{g}(\mathfrak{r},0)| \leq \frac{1}{\pi^2}$. According to the above data, we find: $\alpha_1+\alpha_2 = \frac{19}{20}$, $\varrho_1 = \frac{4}{5}$, $\mathtt{Q} = \frac{7}{8}$, $\omega_1 = 1+\frac{8}{7}\times\frac{23}{8} = \frac{30}{7}$ and $\omega_2 = 1+\frac{1}{25}\times\frac{1}{0.92} \approx 1.4$. Hence,

$$\begin{aligned} \omega_1\cdot\omega_2\left(|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}+\mathcal{L}_{\mathtt{g}}\right) &= 6\times\left(\frac{1}{25}+\frac{1}{30}+\frac{\ln 2}{10}\right) \\ &\approx \frac{43}{50} < 1. \end{aligned} \tag{4.2}$$

All the conditions of Theorem 3.1 are satisfied, then the nonlocal boundary value problem (4.1) has at least one solution defined and continuous on $[0,1]$.

**Example 4.2.** Consider the hybrid Langevin equation with the $\psi$-Hilfer fractional derivative and multipoint boundary values:

$$\begin{cases} {}^{H}\mathbb{D}^{\left(\frac{1}{2},\frac{3}{5}\right);\ln(\mathfrak{r})}\left[\left({}^{H}\mathbb{D}^{\left(\frac{7}{10},\frac{1}{2}\right);\ln(\mathfrak{r})}+\frac{1}{25}\right)\left(\mathtt{y}(\mathfrak{r})-\frac{\tanh(\mathtt{y}(\mathfrak{r}))}{\pi^2}\right)\right]=\frac{e^{-\mathfrak{r}}}{3+\ln(\mathfrak{r})}\times \\ \qquad\qquad\qquad\qquad\left(\frac{|\mathtt{y}(\mathfrak{r})|}{5+|\mathtt{y}(\mathfrak{r})|}\right),\ \mathfrak{r}\in[1,e], \\ \mathtt{y}(1)=0, \\ \mathtt{y}(e)=\frac{1}{4}\cdot\mathtt{y}\left(\frac{3}{2}\right)+\frac{1}{5}\cdot\mathtt{y}(2)-\frac{5}{4}\cdot\mathtt{y}\left(\frac{5}{2}\right). \end{cases} \tag{4.3}$$

The problem (4.3) is a case similar to the boundary value problem (1.1) with $\Omega=[1,e]$, $(\alpha_1,\beta_1)=(\frac{1}{2},\frac{1}{3})$, $(\alpha_2,\beta_2)=\left(\frac{3}{4},\frac{1}{2}\right)$, $(\lambda_1,\tau_1)=(\frac{1}{4},\frac{3}{2})$, $(\lambda_2,\tau_2)=(\frac{1}{5},2)$, $(\lambda_3,\tau_3)=(-\frac{5}{4},\frac{5}{2})$, $(\lambda_4,\tau_4)=(-1,e)$ and $\psi(\mathfrak{r})=\ln(\mathfrak{r})$.
Using the given data, we get $\varrho_1=\frac{4}{5}$, $\varrho_1+\alpha_2=1.5$, then

$$\begin{aligned}\mathtt{Q}&=\frac{1}{\Gamma(1.5)}\left(\frac{1}{4}\times\sqrt{\ln\left(\frac{3}{2}\right)}+\frac{1}{5}\times\sqrt{\ln(2)}-\frac{5}{4}\times\sqrt{\ln\left(\frac{5}{2}\right)}-\sqrt{\ln(e)}\right)\\&\approx-2.1\end{aligned}$$

and,

$$\begin{aligned}\omega_1&\approx1+\frac{1}{2.1\times0.89}\times2.7\\&\approx2.44\end{aligned}\qquad\qquad\begin{aligned}\omega_2&\approx1+\frac{1}{25}\times\frac{1}{\Gamma(1.7)}\\&\approx1.04\end{aligned}$$

The function $\mathtt{g}:(\mathfrak{r},\mathtt{y})\longmapsto\dfrac{\tanh(\mathtt{y}(\mathfrak{r}))}{\pi^2}$ is continuous on $\Omega\times\mathbb{R}\to\mathbb{R}$ with,

$$\sup_{\mathfrak{r}\in\Omega}|\mathtt{g}(\mathfrak{r},\mathtt{y})-\mathtt{g}(\mathfrak{r},\mathtt{z})|\le\frac{1}{\pi^2}|\mathtt{y}-\mathtt{z}|,\quad\forall(\mathtt{y},\mathtt{z})\in\mathbb{R}\times\mathbb{R}.$$

In addition, the function $\mathtt{f}:(\mathfrak{r},\mathtt{y})\longmapsto\dfrac{e^{-\mathfrak{r}}}{3+\ln\mathfrak{r}}\cdot\left(\dfrac{|\mathtt{y}|}{15+|\mathtt{y}|}\right)$ is continuous on $\Omega\times\mathbb{R}\to\mathbb{R}$ with,

$$\sup_{\mathfrak{r}\in\Omega}|\mathtt{f}(\mathfrak{r},\mathtt{y})-\mathtt{f}(\mathfrak{r},\mathtt{z})|\le\frac{1}{15e}|\mathtt{y}-\mathtt{z}|,\quad\forall(\mathtt{y},\mathtt{z})\in\mathbb{R}\times\mathbb{R}.$$

Consequently, the hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$ holds with $\mathcal{L}_{\mathtt{f}}=\frac{1}{15e}$ and $\mathcal{L}_{\mathtt{g}}=\frac{1}{\pi^2}$. Morever,

$$\begin{aligned}\omega_1\cdot\omega_2\left(|\mathtt{k}|+\mathcal{L}_{\mathtt{f}}+\mathcal{L}_{\mathtt{g}}\right)&\approx2.44\times1.04\left(\frac{1}{25}+\frac{1}{15e}+\frac{1}{\pi^2}\right)\\&\approx0.42<1,\end{aligned}\tag{4.4}$$

and,

$$\begin{aligned}\omega_1\left[\mathcal{L}_{\mathtt{f}}\frac{(\psi(\mathtt{d})-\psi(\mathtt{a}))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+(\omega_2-1)\cdot(\mathcal{L}_{\mathtt{g}}+1)\right] &\approx 2.44\times\\ &\left[\frac{1}{15e}\times\frac{1}{\Gamma\,(2.2)}+0.04\times\left(1+\frac{1}{\pi^2}\right)\right]\\ &\approx 0.97<1.\end{aligned}$$

All the conditions of Theorem 3.2 are satisfied, then the problem (4.3) has a unique solution continuous and defined on $[1, e]$.

## Conclusion

In this work we provide the existence proof of solutions to a class of hybrid Langevin equations of $\psi$-Hilfer fractional order derivative, which are strengthened by certain nonlocal value conditions. Using Krasnoselskii's fixed point theorem, it was proved the availability of solutions for the specified problem. Under additional conditions a unique solution is also guaranteed by Banach's principle theorem. This study is unique because it encompasses works based on classical fractional derivatives such as Caputo, $\psi$-Caputo, Katugampola, Erdely-Kober, Hilfer and others. In our future work, we intend to conduct a qualitative investigation on the stability and behavior of the solutions, we still plan to solve the same problem using another type of fractional derivative.

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