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ON GENERALIZED WEAKLY GP-CONTRACTIVE MAPPINGS IN ORDERED GP-METRIC SPACES

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ABSTRACT. The aim of this paper is to present some fixed and common fixed point results for generalized weakly G-contractive mappings in the context of ordered GP-metric spaces. We also provide examples to illustrate the results presented herein. As an application of our results, periodic points of weakly G-contractive mappings are obtained.

1. Introduction and mathematical preliminaries

The concept of a partial metric space was introduced by Matthews (see [19]) as follows:

Definition 1.1. Let X be a nonempty set and let $p : X \times X \to \mathbb{R}^+$ satisfies:
(P1) $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$, for all $x, y \in X$;
(P2) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;
(P3) $p(x, y) = p(y, x)$, for all $x, y \in X$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for all $x, y, z \in X$.
Then, p is called a partial metric on X and the pair (X, p) is called a partial metric space.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [20]. In recent years, many authors have obtained different fixed point theorems for mappings satisfying various contractive conditions on G-metric spaces.

Definition 1.2. ([20]) Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:
(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

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Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

For a survey of fixed point theory, its applications, comparison of different contractive conditions and related results in both partial metric spaces and G-metric spaces we refer to, e.g., ([2, 4, 5, 7, 10, 11, 15, 16, 18, 21, 23, 24, 28] and references mentioned therein.

Recently, Zand and Nezhad [29] introduced a new generalized metric space, a GP-metric space, as a generalization both of partial metric spaces and G-metric spaces.

We will use the following definition of a GP-metric space.

Definition 1.3. Let X be a nonempty set. Suppose that a mapping $G_p : X \times X \times X \to R^+$ satisfies:

$(GP1)$ $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;

$(GP2)$ $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

$(GP3)$ $G_p(x, y, z) = G_p(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all three variables);

$(GP4)$ $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G_p is called a GP-metric and (X, G_p) is called a GP-metric space.
The GP-metric G_p is called symmetric if

$$G_p(x, x, y) = G_p(x, y, y) \tag{1.1}$$

holds for all $x, y \in X$. Otherwise, G_p is an asymmetric GP-metric.

Remark 1.4. In [29] (see also [6]), instead of $(GP2)$, the following condition was used:

$(GP2')$ $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$.

However, with this assumption, it is very easy to obtain that (1.1) holds for all $x, y \in X$, i.e., the respective space is symmetric. On the other hand, there are a lot of examples of asymmetric G-metric spaces. Hence, the conclusion stated in [6, 29] that each G-metric space is a GP-metric space (satisfying $(GP2')$) does not hold. With our assumption $(GP2)$, this conclusion holds true.

The following are some easy examples of GP-metric spaces.

Example 1.5. Let $X = [0, +\infty)$ and let $G_p : X^3 \to \mathbb{R}^+$ be given by $G_p(x, y, z) = \max\{x, y, z\}$. Obviously, (X, G_p) is a symmetric GP-metric space which is not a G-metric space.

Example 1.6. Let $X = \{0, 1, 2, 3\}$. Let

$$\begin{aligned} A = \{&(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),(0,0,2),(3,0,0),(0,3,0),(0,0,3),\\ &(1,2,2),(2,1,2),(2,2,1),(2,3,3),(3,2,3),(3,3,2)\},\\ B = \{&(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),(2,2,0),(0,3,3),(3,0,3),(3,3,0)\}\\ &(2,1,1),(1,2,1),(1,1,2),(3,2,2),(2,3,2),(2,2,3)\}. \end{aligned}$$

Define $G_p : X^3 \to \mathbb{R}^+$ by

$$G(x,y,z) = \begin{cases} 1, & \text{if } x = y = z \neq 2, \\ 0, & \text{if } x = y = z = 2, \\ 2, & \text{if } (x,y,z) \in A, \\ \frac{5}{2}, & \text{if } (x,y,z) \in B, \\ 3, & \text{if } x \neq y \neq z \neq x. \end{cases}$$

It is easy to see that (X, G_p) is an asymmetric GP-metric space.

Proposition 1.7. *([29]) Every GP-metric space (X, G_p) defines a metric space (X, d_{G_p}), where,*

$$d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y),$$

for all $x, y \in X$.

Proposition 1.8. *([29]) Let X be a GP-metric space. Then for each $x, y, z, a \in X$ it follows that:*

(1) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;
(2) $G_p(x,y,z) \leq G(x,x,y) + G(x,x,z) - G_p(x,x,x)$;
(3) $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x)$;
(4) $G_p(x,y,z) \leq G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$, $a \neq z$.

Definition 1.9. ([29]) Let (X, G_p) be a GP-metric space. Let $\{x_n\}$ be a sequence of points of X.

1. A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x,x,x)$.
2. $\{x_n\}$ is said to be a GP-Cauchy sequence, if $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite).
3. (X, G_p) is said to be GP-complete if every GP-Cauchy sequence in X is GP-convergent to an $x \in X$.

Using the above definitions, one can easily prove the following proposition.

Proposition 1.10. *([29]) Let (X, G_p) be a GP-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$ the following are equivalent:*

(1) $\{x_n\}$ is GP-convergent to x.
(2) $G_p(x_n, x_n, x) \to G_p(x,x,x)$, as $n \to \infty$.
(3) $G_p(x_n, x, x) \to G_p(x,x,x)$, as $n \to \infty$.

Lemma 1.11. *If G_p is a GP-metric on X, then the mappings $d_{G_p}, d'_{G_p} : X \times X \to R^+$ given by*

$$d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y)$$

and

$$d'_{G_p}(x,y) = max\{G_p(x,y,y) - G_p(x,x,x), G_p(y,x,x) - G_p(y,y,y)\}$$

define equivalent metrics on X.

Proof. $\frac{a+b}{2} \leq \max\{a,b\} \leq a+b$, for all nonnegative real numbers a, b. ☐

Based on Lemma 2.2 of [24], we prove the following essential lemma.

Lemma 1.12. *(1) A sequence $\{x_n\}$ is a GP-Cauchy sequence in a GP-metric space (X, G_p) if and only if it is a Cauchy sequence in the metric space (X, d_{G_p}).*

(2) A GP-metric space (X, G_p) is GP-complete if and only if the metric space (X, d_{G_p}) is complete. Moreover, $\lim_{n\to\infty} d_{G_p}(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} G_p(x, x_n, x_n) = \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m)$$
$$= \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x).$$

Proof. First, we show that every GP-Cauchy sequence in (X, G_p) is a Cauchy sequence in (X, d_{G_p}). Let $\{x_n\}$ be a GP-Cauchy sequence in (X, G_p). Then, there exists $\alpha \in \mathbb{R}$ such that, for arbitrary $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with

$$|G_p(x_n, x_m, x_m) - \alpha| < \frac{\varepsilon}{4},$$

for all $n, m \geq n_\varepsilon$. Hence,

$$\begin{aligned}
&|d_{G_p}(x_n, x_m)| \\
&\quad = G_p(x_n, x_m, x_m) + G_p(x_m, x_n, x_n) - G_p(x_n, x_n, x_n) - G_p(x_m, x_m, x_m) \\
&\quad = |G_p(x_n, x_m, x_m) - \alpha + \alpha - G_p(x_n, x_n, x_n) + G_p(x_m, x_m, x_n) \\
&\qquad - \alpha + \alpha - G_p(x_m, x_m, x_m)| \\
&\quad \leq |G_p(x_n, x_m, x_m) - \alpha| + |\alpha - G_p(x_n, x_n, x_n)| + |G_p(x_m, x_m, x_n) - \alpha| \\
&\qquad + |\alpha - G_p(x_m, x_m, x_m)| \\
&\quad < \varepsilon,
\end{aligned}$$

for all $n, m \geq n_\varepsilon$. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d_{G_p}).

Next, we prove that completeness of (X, d_{G_p}) implies GP-completeness of (X, G_p). Indeed, if $\{x_n\}$ is a GP-Cauchy sequence in (X, G_p) then it is also a Cauchy sequence in (X, d_{G_p}). Since the metric space (X, d_{G_p}) is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} d_{G_P}(y, x_n) = 0$. Hence,

$$\lim_{n\to\infty} [G_p(x_n, y, y) - G_p(y, y, y) + G_p(y, x_n, x_n) - G_p(x_n, x_n, x_n)] = 0,$$

therefore, $\lim_{n\to\infty} [G_p(x_n, y, y) - G_p(y, y, y)] = 0$.

On the other hand,

$$\begin{aligned}
\lim_{n,m\to\infty} G_p(x_n, x_m, y) &\leq \lim_{n,m\to\infty} G_p(x_n, y, y) + \lim_{n,m\to\infty} G_p(x_m, y, y) - G_p(y, y, y) \\
&= G_p(y, y, y).
\end{aligned}$$

Also, from (GP2),

$$G_p(y, y, y) \leq \lim_{n,m\to\infty} G_p(x_n, x_m, y).$$

Hence, we obtain that $\{x_n\}$ is a GP-convergent sequence in (X, G_p).

Now, we prove that every Cauchy sequence $\{x_n\}$ in (X, d_{G_p}) is a GP-Cauchy sequence in (X, G_p). Let $\varepsilon = \frac{1}{2}$. Then, there exists $n_0 \in \mathbb{N}$ such that $d_{G_p}(x_n, x_m) <$

$\frac{1}{2}$, for all $n, m \geq n_0$. Since,

$$G_p(x_n, x_{n_0}, x_{n_0}) - G_p(x_{n_0}, x_{n_0}, x_{n_0}) \leq d_{G_p}(x_n, x_{n_0}) < \frac{1}{2},$$

hence,

$$G_p(x_n, x_n, x_n) \leq d_{G_p}(x_n, x_{n_0}) + G_p(x_n, x_{n_0}, x_{n_0}) < \frac{1}{2} + G_p(x_{n_0}, x_{n_0}, x_{n_0}).$$

Consequently, the sequence $\{G_p(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$, and so there exists $a \in \mathbb{R}$ such that a subsequence $\{G_p(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to a, i.e.

$$\lim_{k\to\infty} G_p(x_{n_k}, x_{n_k}, x_{n_k}) = a.$$

Now, we prove that $\{G_p(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in (X, d_{G_p}), for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_{G_p}(x_n, x_m) < \varepsilon$, for all $n, m \geq n_\varepsilon$. Thus, for all $n, m \geq n_\varepsilon$,

$$\begin{aligned} G_p(x_n, x_n, x_n) - G_p(x_m, x_m, x_m) &\leq G_p(x_n, x_m, x_m) - G_p(x_m, x_m, x_m) \\ &\leq d_{G_p}(x_m, x_n) < \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} G_p(x_n, x_n, x_n) = a$.

On the other hand,

$$\begin{aligned} |G_p(x_n, x_m, x_m) - a| &= |G_p(x_n, x_m, x_m) - G_p(x_n, x_n, x_n) + G_p(x_n, x_n, x_n) - a| \\ &\leq d_{G_p}(x_m, x_n) + |G_p(x_n, x_n, x_n) - a|, \end{aligned}$$

for all $n, m \geq n_\varepsilon$. Hence, $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = a$, and consequently, $\{x_n\}$ is a GP-Cauchy sequence in (X, G_p).

Conversely, let $\{x_n\}$ be a Cauchy sequence in (X, d_{G_p}). Then, $\{x_n\}$ is a GP-Cauchy sequence in (X, G_p), and so it is convergent to a point $y \in X$ with

$$\lim_{n\to\infty} G_p(x, x_n, x_n) = \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x, x_m, x_n) = G_p(x, x, x).$$

Then, for given $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that

$$G_p(x, x_n, x_n) - G_p(x, x, x) < \frac{\varepsilon}{4},$$

and

$$G_p(x_n, x, x) - G_p(x, x, x) < \frac{\varepsilon}{4}.$$

Therefore,

$$\begin{aligned} |d_{G_p}(x_n, x)| &= |G_p(x_n, x, x) - G_p(x_n, x_n, x_n) + G_p(x_n, x_n, x) - G_p(x, x, x)| \\ &\leq |G_p(x_n, x, x) - G_p(x, x, x)| + |G_p(x, x, x) - G_p(x_n, x_n, x_n)| \\ &\quad + |G_p(x_n, x_n, x) - G_p(x, x, x)| < \varepsilon, \end{aligned}$$

whenever $n \geq n_\varepsilon$. Therefore, (X, d_{G_p}) is complete.

Finally, let $\lim_{n\to\infty} d_{G_p}(x_n, x) = 0$. So,

$$\lim_{n\to\infty} [G_p(x_n, x, x) - G_p(x_n, x_n, x_n)] + \lim_{n\to\infty} [G_p(x_n, x_n, x) - G_p(x, x, x)] = 0$$

and

$$\lim_{n\to\infty} [G_p(x_n, x, x) - G_p(x, x, x)] + \lim_{n\to\infty} [G_p(x_n, x_n, x) - G_p(x_n, x_n, x_n)] = 0.$$

On the other hand,

$$\begin{aligned}
&\lim_{n,m\to\infty}[G_p(x_n,x_m,x_m)-G_p(x,x,x)] \\
&\qquad \le \lim_{n\to\infty}[G_p(x_n,x,x)+G_p(x,x_m,x_m)-G_p(x,x,x)] \\
&\qquad = G_p(x,x,x).
\end{aligned}$$

□

Lemma 1.13. *Assume that $x_n \to x$ as $n \to \infty$ in a GP-metric space (X, G_p) such that $G_p(x,x,x) = 0$. Then, for every $y \in X$,*
(i) $\lim_{n\to\infty} G_p(x_n,y,y) = G_p(x,y,y)$,
(ii) $\lim_{n\to\infty} G_p(x_n,x_n,y) = G_p(x,x,y)$,

Proof. First note that $\lim_{n\to\infty} G_p(x_n,x,x) = G_p(x,x,x) = 0 = \lim_{n\to\infty} G_p(x_n,x_n,x)$. By the rectangle inequality we have,

$$\begin{aligned}
G_p(x_n,y,y) &\le G_p(x_n,x,x)+G_p(x,y,y)-G_p(x,x,x) \\
&= G_p(x_n,x,x)+G_p(x,y,y) \\
&\le 2G_p(x_n,x,x)+G_p(x,y,y)
\end{aligned}$$

and

$$\begin{aligned}
G_p(x,y,y) &\le G_p(x,x_n,x_n)+G_p(x_n,y,y)-G_p(x_n,x_n,x_n) \\
&\le G_p(x,x_n,x_n)+G_p(x_n,y,y) \\
&\le 2G_p(x_n,x,x)+G_p(x_n,y,y).
\end{aligned}$$

Hence,

$$0 \le |G_p(x_n,y,y)-G_p(x,y,y)| \le 2G_p(x_n,x,x).$$

Letting $n \to \infty$ we conclude our claim.

Analogously, the part (ii) can be proved. □

Lemma 1.14. *([6]) Let (X, G_p) be a GP-metric space. Then,*
(A) If $G_p(x,y,z) = 0$, then $x = y = z$.
(B) If $x \neq y$, then $G_p(x,y,y) > 0$.

Definition 1.15. ([29]) Let (X_1, G_1) and (X_2, G_2) be two GP-metric spaces and let $f : (X_1, G_1) \to (X_2, G_2)$ be a mapping. Then f is said to be GP-continuous at a point $a \in X_1$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X_1$ and $G_1(a,x,y) < \delta + G_1(a,a,a)$ imply that $G_2(f(a), f(x), f(y)) < \varepsilon + G_2(f(a), f(a), f(a))$. The mapping f is GP-continuous on X_1 if it is GP-continuous at all $a \in X_1$.

Proposition 1.16. *([29]) Let (X_1, G_1) and $(X_2, , G_2)$ be two GP-metric spaces. Then a mapping $f : X_1 \to X_2$ is GP-continuous at a point $x \in X_1$ if and only if it is GP-sequentially continuous at x; that is, whenever $\{x_n\}$ is GP-convergent to x, $\{f(x_n)\}$ is GP-convergent to $f(x)$.*

The concept of an altering distance function was introduced by Khan et al. [17] as follows.

Definition 1.17. The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.

1. ψ is continuous and nondecreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

In [7], Aydi et al. established some common fixed point results for two self-mappings f and g on a generalized metric space X. The following is their definition adjusted for a GP-metric space.

Definition 1.18. ([7]) Let (X, G_p) be a GP-metric space and $f, g : X \to X$ be two mappings. We say that f is a generalized weakly GP-contractive mapping of type A (resp. of type B) with respect to g if for all $x, y, z \in X$, the following inequality holds:

$$\begin{aligned}\psi\big(G_p(fx, fy, fz)\big) \leq \psi\Big(&\frac{G_p(gx, fy, fy) + G_p(gy, fz, fz) + G_p(gz, fx, fx)}{3}\Big)\\ &- \varphi\big(G_p(gx, fy, fy), G_p(gy, fz, fz), G_p(gz, fx, fx)\big),\end{aligned}$$

(resp.

$$\begin{aligned}\psi\big(G_p(fx, fy, fz)\big) \leq \psi\Big(&\frac{G_p(gx, gx, fy) + G_p(gy, gy, fz) + G_p(gz, gz, fx)}{3}\Big)\\ &- \varphi\big(G_p(gx, gx, fy), G_p(gy, gy, fz), G_p(gz, gz, fx)\big)\big),\end{aligned}$$

where
(1) ψ is an altering distance function;
(2) $\varphi : [0, \infty)^3 \to [0, \infty)$ is a continuous function with $\varphi(t, s, u) = 0$ if and only if $t = s = u = 0$.

Note that the concept of a generalized weakly G-contraction is the extension of the concept of weakly C-contraction which has been defined by Choudhury in [8]. For more details on weakly C-contractive mappings we refer the reader to [12] and [27].

Also, in [25], Razani and Parvaneh proved some coincidence and common fixed point theorems for nonlinear weakly G-contractive mappings in partially ordered G-metric spaces. In fact, their results are extensions of the results of Aydi et al. [7] from the context of G-metric spaces to the setup of ordered G-metric spaces.

Definition 1.19. ([1]) Let $(X, \preceq)$ be a partially ordered set. A mapping $f : X \to X$ is called dominating if $x \preceq fx$ for each x in X.

Example 1.20. ([1]) Let $X = [0, \infty)$ be endowed with the usual ordering. Let $f : X \to X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0, 1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any $n \in \mathbb{N}$. Then, for all $x \in X$, $x \leq fx$; that is, f is a dominating map.

A subset W of a partially ordered set X is said to be well ordered if every two elements of W be comparable.

Let X be a non-empty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 1.21. ([22]) Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ be given mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. We say that f and g are weakly increasing with respect to h if for all $x \in X$, we have:

$$fx \preceq gy, \forall y \in h^{-1}(fx),$$

and

$$gx \preceq fy, \forall y \in h^{-1}(gx).$$

If $f = g$, we say that f is weakly increasing with respect to h.

If $h = I$ (the identity mapping on X), then the above definition reduces to those of weakly increasing mapping [5] (see also [22]).

Recall the following well known notion introduced by Jungck.

Definition 1.22. ([14]) Let X be a nonempty set and $f, g : X \to X$ be two mappings. They are called weakly compatible if $fgx = gfx$ whenever $fx = gx$.

The aim of this paper is to prove some coincidence and common fixed point theorems for nonlinear weakly GP-contractive mappings in partially ordered GP-metric spaces.

2. Main results

From now on, we assume:

$$\Phi = \{\varphi \mid \varphi : [0, \infty)^3 \to [0, \infty) \text{ is a continuous function such that } \varphi(x, y, z) = 0 \iff x = y = z = 0\}.$$

Definition 2.1. Let $(X, \preceq, G_p)$ be a partially ordered GP-metric space. We say that X is regular if the following hypotheses hold:

(i) If $\{x_n\}$ is a nondecreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

(ii) If $\{x_n\}$ is a nonincreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \succeq x$, for all $n \in \mathbb{N}$.

Our first result is the following.

Theorem 2.2. *Let $(X, \preceq, G_p)$ be a partially ordered GP-metric space. Let $f, g : X \to X$ be two mappings such that $f(X) \subseteq g(X)$, f is weakly increasing with respect to g and*

$$\psi\big(G_p(fx, fy, fz)\big) \leq \psi\left(\frac{G_p(gx, fy, fy) + G_p(gy, fz, fz) + G_p(gz, fx, fx)}{3}\right) - \varphi\big(G_p(gx, fy, fy), G_p(gy, fz, fz), G_p(gz, fx, fx)\big), \tag{2.1}$$

for every $x, y, z \in X$ such that $gx \preceq gy \preceq gz$, where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function and $\varphi \in \Phi$. Then, f and g have a coincidence point in X if X is regular and $g(X)$ is a GP-complete subset of (X, G_p).

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subseteq g(X)$, we can construct a sequence $\{z_n\}$ defined by $z_n = gx_n = fx_{n-1}$, for all $n \geq 0$.

Now, since $x_1 \in g^{-1}(fx_0)$ and $x_2 \in g^{-1}(fx_1)$, as f is weakly increasing with respect to g, we obtain:

$$gx_1 = fx_0 \preceq fx_1 = gx_2 \preceq fx_2 = gx_3.$$

Continuing this process, we get:

$$gx_1 \preceq gx_2 \preceq gx_3 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots .$$

We will complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} G(z_n, z_{n+1}, z_{n+1}) = 0$.

Since $gx_{n-1} \preceq gx_n$, using (2.1) we obtain that

$$\begin{aligned}
&\psi\big(G_p(z_n, z_{n+1}, z_{n+1})\big) = \psi\big(G_p(fx_{n-1}, fx_n, fx_n)\big)\\
&\quad\leq \psi\Big(\frac{G_p(gx_{n-1}, fx_n, fx_n) + G_p(gx_n, fx_n, fx_n) + G_p(gx_n, fx_{n-1}, fx_{n-1})}{3}\Big)\\
&\qquad - \varphi\big(G_p(gx_{n-1}, fx_n, fx_n), G_p(gx_n, fx_n, fx_n), G_p(gx_n, fx_{n-1}, fx_{n-1})\big)\\
&\quad= \psi\Big(\frac{G_p(z_{n-1}, z_{n+1}, z_{n+1}) + G_p(z_n, z_{n+1}, z_{n+1}) + G_p(z_n, z_n, z_n)}{3}\Big)\\
&\qquad - \varphi\big(G_p(z_{n-1}, z_{n+1}, z_{n+1}), G_p(z_n, z_{n+1}, z_{n+1}), G_p(z_n, z_n, z_n)\big)\\
&\quad\leq \psi\Big(\frac{G_p(z_{n-1}, z_n, z_n) + 2G_p(z_n, z_{n+1}, z_{n+1})}{3}\Big)\\
&\qquad - \varphi\big(G_p(z_{n-1}, z_{n+1}, z_{n+1}), G_p(z_n, z_{n+1}, z_{n+1}), G_p(z_n, z_n, z_n)\big)\\
&\quad\leq \psi\Big(\frac{G_p(z_{n-1}, z_n, z_n) + 2G_p(z_n, z_{n+1}, z_{n+1})}{3}\Big).
\end{aligned} \tag{2.2}$$

Since ψ is a nondecreasing function, from (2.2), we have

$$\begin{aligned}
G_p(z_n, z_{n+1}, z_{n+1}) &\leq \frac{G_p(z_{n-1}, z_{n+1}, z_{n+1}) + G_p(z_n, z_{n+1}, z_{n+1}) + G_p(z_n, z_n, z_n)}{3}\\
&\leq \frac{G_p(z_{n-1}, z_n, z_n) + 2G_p(z_n, z_{n+1}, z_{n+1})}{3}.
\end{aligned} \tag{2.3}$$

Hence, we conclude that $\{G_p(z_n, z_{n+1}, z_{n+1})\}$ is a nondecreasing sequence of non-negative real numbers. Thus, there is an $r \geq 0$ such that

$$\lim_{n\to\infty} G_p(z_n, z_{n+1}, z_{n+1}) = r. \tag{2.4}$$

Letting $n \to \infty$ in (2.3), we get that

$$r \leq \frac{\lim_{n\to\infty} G_p(z_{n-1}, z_{n+1}, z_{n+1}) + r + \lim_{n\to\infty} G_p(z_n, z_n, z_n)}{3} \leq r,$$

That is,

$$\lim_{n\to\infty} G_p(z_{n-1}, z_{n+1}, z_{n+1}) + \lim_{n\to\infty} G_p(z_n, z_n, z_n) = 2r. \tag{2.5}$$

Again, from (2.2) we have

$$\begin{aligned}
\psi\big(G_p(z_n, z_{n+1}, z_{n+1})\big) \leq \psi\Big(&\frac{G_p(z_{n-1}, z_{n+1}, z_{n+1}) + G_p(z_n, z_{n+1}, z_{n+1}) + G_p(z_n, z_n, z_n)}{3}\Big)\\
&- \varphi\big(G_p(z_{n-1}, z_{n+1}, z_{n+1}), G_p(z_n, z_{n+1}, z_{n+1}), G_p(z_n, z_n, z_n)\big).
\end{aligned}$$

Letting $n \to \infty$ and using (2.4), (2.5) and the continuity of ψ and φ, we get

$$\psi(r) \leq \psi\left(\frac{2r+r}{3}\right) - \varphi\big(\lim_{n\to\infty} G_p(z_{n-1}, z_{n+1}, z_{n+1}), r, \lim_{n\to\infty} G_p(z_n, z_n, z_n)\big),$$

and hence $\varphi\big(\lim_{n\to\infty} G_p(z_{n-1}, z_{n+1}, z_{n+1}), r, \lim_{n\to\infty} G_p(z_n, z_n, z_n)\big) = 0$. This gives us that

$$\lim_{n\to\infty} G_p(z_n, z_{n+1}, z_{n+1}) = \lim_{n\to\infty} G_p(z_{n-1}, z_{n+1}, z_{n+1}) = \lim_{n\to\infty} G_p(z_n, z_n, z_n) = 0, \tag{2.6}$$

from our assumptions about φ.

Step II. We will show that $\{z_n\}$ is a GP-Cauchy sequence in X. That is, we will show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n > m \geq k$,

$$G_p(z_m, z_n, z_n) < \varepsilon;$$

i.e., we prove that $\lim_{m,n\to\infty} G_p(z_m, z_n, z_n) = 0$.

Suppose the above statement is false. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ such that $n(k) > m(k) > k$ and

$$G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) \geq \varepsilon, \tag{2.7}$$

where $n(k)$ is the smallest index with this property, i.e.,

$$G_p(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) < \varepsilon. \tag{2.8}$$

From the rectangle inequality,

$$G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) \leq G_p(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) + G_p(z_{n(k)-1}, z_{n(k)}, z_{n(k)}). \tag{2.9}$$

Letting $k \to \infty$ in (2.9), from (2.6), (2.7) and (2.8) we conclude that

$$\lim_{k\to\infty} G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) = \varepsilon. \tag{2.10}$$

Again, from the rectangle inequality,

$$\begin{aligned} G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) &\leq G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G_p(z_{n(k)}, z_{n(k)}, z_{n(k)+1}) \\ &\leq G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) + 2G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}), \end{aligned} \tag{2.11}$$

and

$$G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) \leq G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}). \tag{2.12}$$

Hence, if $k \to \infty$ in (2.11) and (2.12), using (2.6) and (2.10), we have

$$\lim_{k\to\infty} G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{2.13}$$

On the other hand,

$$G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) \leq G_p(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}), \tag{2.14}$$

and

$$G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \leq G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}). \tag{2.15}$$

Hence, if $k \to \infty$ in (2.14) and (2.15), from (2.6), (2.10) and (2.13) we have

$$\lim_{k\to\infty} G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{2.16}$$

In a similar way, we have

$$\begin{aligned} G_p(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) &\le G_p(z_{m(k)+1}, z_{m(k)}, z_{m(k)}) + G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \\ &\le 2G_p(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}), \end{aligned} \tag{2.17}$$

and

$$G_p(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \le G_p(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G_p(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}), \tag{2.18}$$

and therefore, from (2.17) and (2.18) by taking the limit when $k \to \infty$, using (2.6) and (2.13), we get that

$$\lim_{k\to\infty} G_p(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{2.19}$$

Also,

$$G_p(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \le G_p(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)}), \tag{2.20}$$

and

$$G_p(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) \le G_p(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)+1}, z_{n(k)+1}, z_{n(k)}). \tag{2.21}$$

Hence, from (2.6), (2.19), (2.20) and (2.21), we have

$$\lim_{k\to\infty} G_p(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{2.22}$$

Finally,

$$G_p(z_{n(k)}, z_{m(k)+1}, z_{m(k)+1}) \le G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)+1}, z_{m(k)+1}, z_{m(k)+1}), \tag{2.23}$$

and

$$\begin{aligned} G_p(z_{n(k)+1}, z_{m(k)+1}, z_{m(k)+1}) &\le G_p(z_{n(k)+1}, z_{n(k)}, z_{n(k)}) + G_p(z_{n(k)}, z_{m(k)+1}, z_{m(k)+1}) \\ &\le 2G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)}, z_{m(k)+1}, z_{m(k)+1}). \end{aligned} \tag{2.24}$$

Hence, if $k \to \infty$ in (2.23) and (2.24) and using (2.6) and (2.22), we have

$$\lim_{k\to\infty} G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{2.25}$$

Since $gx_{m(k)} \preceq gx_{n(k)} \preceq gx_{n(k)}$, putting $x = x_{m(k)}$, $y = x_{n(k)}$ and $z = x_{n(k)}$ in (2.1), for arbitrary $k \ge 0$, we have

$$\begin{aligned} &\psi\big(G_p(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1})\big) \\ &= \psi\big(G_p(fx_{m(k)}, fx_{n(k)}, fx_{n(k)})\big) \\ &\le \psi\left(\frac{G_p(gx_{m(k)}, fx_{n(k)}, fx_{n(k)}) + G_p(gx_{n(k)}, fx_{n(k)}, fx_{n(k)}) + G_p(gx_{n(k)}, fx_{m(k)}, fx_{m(k)})}{3}\right) \\ &\quad - \varphi\big(G_p(gx_{m(k)}, fx_{n(k)}, fx_{n(k)}), G_p(gx_{n(k)}, fx_{n(k)}, fx_{n(k)}), G_p(gx_{n(k)}, fx_{m(k)}, fx_{m(k)})\big) \\ &= \psi\Big(\frac{G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) + G_p(z_{n(k)}, z_{m(k)+1}, z_{m(k)+1})}{3}\Big) \\ &\quad - \varphi\big(G_p(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}), G_p(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}), G_p(z_{n(k)}, z_{m(k)+1}, z_{m(k)+1})\big). \end{aligned} \tag{2.26}$$

Now, if $k \to \infty$ in (2.26), from (2.6), (2.16), (2.22) and (2.25), we have

$$\psi(\varepsilon) \leq \psi\left(\frac{2\varepsilon}{3}\right) - \varphi(\varepsilon, 0, \varepsilon).$$

Hence, $\varepsilon = 0$, which is a contradiction. Therefore, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n > m \geq k$, $G(z_m, z_n, z_n) < \varepsilon$. Consequently,

$$\lim_{m,n\to\infty} G_p(z_m, z_n, z_n) = 0, \tag{2.27}$$

and hence $\{z_n\}$ is a GP-Cauchy sequence.

Step III. We will show that f and g have a coincidence point.

Since $\{z_n\} = \{gx_n\}$ is a GP-Cauchy sequence and $g(X)$ is GP-complete, from Lemma 1.12 and (2.27) there exists $z \in g(X)$ such that:

$$\lim_{n\to\infty} G_p(z_n, z_n, z) = \lim_{n\to\infty} G_p(gx_n, gx_n, z) = \lim_{n\to\infty} G_p(fx_n, fx_n, z) = G_p(z, z, z) = 0. \tag{2.28}$$

There exists $u \in X$ such that $z = gu$ and

$$\begin{aligned}\lim_{n\to\infty} G_p(z_n, z_n, gu) = \lim_{n\to\infty} G_p(gx_n, gx_n, gu) &= \lim_{n\to\infty} G_p(fx_n, fx_n, gu)\\ &= G_p(gu, gu, gu) = 0.\end{aligned} \tag{2.29}$$

We will prove that u is a coincidence point of f and g.

We know that $\{gx_n\}$ is a non-decreasing sequence in X. Regularity of X and (2.29) yields that $gx_n \preceq z = gu$. Hence, from (2.1) we have

$$\begin{aligned}\psi\big(G_p(z_{n+1}, z_{n+1}, fu)\big) &= \psi\big(G_p(fx_n, fx_n, fu)\big)\\ &\leq \psi\Big(\frac{G_p(gx_n, fx_n, fx_n) + G_p(gx_n, fu, fu) + G_p(gu, fx_n, fx_n)}{3}\Big)\\ &\quad - \varphi\big(G_p(gx_n, fx_n, fx_n), G_p(gx_n, fu, fu), G_p(gu, fx_n, fx_n)\big)\\ &= \psi\Big(\frac{G_p(z_n, z_{n+1}, z_{n+1}) + G_p(z_n, fu, fu) + G_p(gu, z_{n+1}, z_{n+1})}{3}\Big)\\ &\quad - \varphi\big(G_p(z_n, z_{n+1}, z_{n+1}), G_p(z_n, fu, fu), G_p(gu, z_{n+1}, z_{n+1})\big).\end{aligned} \tag{2.30}$$

Letting $n \to \infty$ in (2.30), from the continuity of ψ and φ and Lemma 1.13, we get

$$\psi\big(G_p(z, z, fu)\big) \leq \psi\left(\frac{G_p(z, fu, fu)}{3}\right) - \varphi\big(0, G_p(z, fu, fu), 0\big). \tag{2.31}$$

As $G_p(z, fu, fu) \leq 2G_p(z, z, fu)$, we have

$$\psi\big(G_p(z, z, fu)\big) \leq \psi\left(\frac{2G_p(z, z, fu)}{3}\right) - \varphi\big(0, G_p(z, fu, fu), 0\big).$$

Hence, $\varphi\big(0, G_p(z, fu, fu), 0\big) \leq \psi\big(\frac{2G_p(z,z,fu)}{3}\big) - \psi\big(G_p(z, z, fu)\big) \leq 0$. So, $G_p(z, fu, fu) = 0$ and hence, $gu = z = fu$. This means that g and f have a coincidence point. □

Taking $g = I_X$ (the identity mapping on X) and $\psi = I_{[0,\infty)}$ in the above theorem, we obtain the first part of the following fixed point result:

Corollary 2.3. *Let $(X, \preceq, G_p)$ be a complete partially ordered GP-metric space. Let $f : X \to X$ be a mapping such that $fx \preceq f(fx)$, for all $x \in X$ and*

$$\begin{aligned} G_p(fx, fy, fz) \leq \frac{G_p(x, fy, fy) + G_p(y, fz, fz) + G_p(z, fx, fx)}{3} \\ - \varphi\big(G_p(x, fy, fy), G_p(y, fz, fz), G_p(z, fx, fx)\big), \end{aligned} \tag{2.32}$$

for every $x, y, z \in X$ such that $x \preceq y \preceq z$, where $\varphi \in \Phi$. Then, f has a fixed point in X provided that one of the following two conditions is satisfied:
(a) X is regular, or,
(b) f is continuous.

Proof. It remains to prove the part (b).
Following the proof of Theorem 2.2, from (2.28) we have that

$$\lim_{n\to\infty} G_p(fx_n, fx_n, z) = G_p(z, z, z) = 0.$$

The continuity of f implies that

$$\lim_{n\to\infty} G_p(x_{n+1}, x_{n+1}, fz) = \lim_{n\to\infty} G_p(fx_n, fx_n, fz) = G_p(fz, fz, fz).$$

On the other hand, from the rectangular inequality it follows that

$$G_p(z, fz, fz) \leq G_p(z, x_{n+1}, x_{n+1}) + G_p(fx_n, fz, fz),$$

and passing to the limit as $n \to \infty$ and using Lemma 1.13, we get that $G_p(z, fz, fz) \leq G_p(fz, fz, fz)$. Since the opposite inequality always holds, we get that

$$G_p(z, fz, fz) = G_p(fz, fz, fz).$$

Since $z \preceq z \preceq z$, applying (2.32) we get that

$$\begin{aligned} G_p(z, fz, fz) &= G_p(fz, fz, fz) \\ &\leq G_p(z, fz, fz) - \varphi\big(G_p(z, fz, fz), G_p(z, fz, fz), G_p(z, fz, fz)\big) \\ &= G_p(fz, fz, fz) - \varphi\big(G_p(z, fz, fz), G_p(z, fz, fz), G_p(z, fz, fz)\big). \end{aligned}$$

By the properties of function φ it follows that $G_p(z, fz, fz) = 0$ and hence $fz = z$. Thus, z is a fixed point of f. □

Taking $\varphi(x, y, z) = (\frac{1}{3} - \alpha)(x + y + z)$, where $\alpha \in [0, \frac{1}{3})$, in the above corollary, we obtain the following result:

Corollary 2.4. *Let $(X, \preceq, G_p)$ be a complete partially ordered GP-metric space. Let $f : X \to X$ be a mapping such that $fx \preceq f(fx)$, for all $x \in X$ and*

$$G_p(fx, fy, fz) \leq \alpha\big(G_p(x, fx, fx) + G_p(y, fy, fy) + G_p(z, fz, fz)\big),$$

for every $x, y, z \in X$ such that $x \preceq y \preceq z$, where $\alpha \in [0, \frac{1}{3})$. Then, f has a fixed point in X if one of the following two conditions is satisfied:
(a) X is regular, or,
(b) f is continuous,.

Theorem 2.5. *Under the hypotheses of Theorem 2.2, f and g have a common fixed point in X provided that f and g are weakly compatible and g is a nondecreasing dominating map.*

Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Following the proof of Theorem 2.2 we obtain that the sequence $\{z_n\}$ is GP-convergent to $z = fu = gu$. Since f and g are weakly compatible, we have $fgu = gfu$, i.e., $fz = gz$. Denote $w = gz = fz$. As g is a nondecreasing dominating map,

$$u \preceq gu \preceq ggu = gz.$$

Hence, applying (2.1) we have

$$\begin{aligned}
\psi\big(G_p(fu,fu,fz)\big) &\leq \psi\left(\frac{G_p(gu,fu,fu)+G_p(gu,fz,fz)+G_p(gz,fu,fu)}{3}\right)\\
&\quad - \varphi\big(G_p(gu,fu,fu),G_p(gu,fz,fz),G_p(gz,fu,fu)\big)\\
&\leq \psi\left(\frac{G_p(fu,fu,fu)+G_p(fu,fz,fz)+G_p(fz,fu,fu)}{3}\right)\\
&\quad - \varphi\big(G_p(fu,fu,fu),G_p(fu,fz,fz),G_p(fz,fu,fu)\big)\\
&\leq \psi\left(\frac{2G_p(fu,fu,fz)+G_p(fu,fu,fz)}{3}\right)\\
&\quad - \varphi\big(G_p(fu,fu,fu),G_p(fu,fz,fz),G_p(fz,fu,fu)\big).
\end{aligned}$$

Therefore, $\varphi\big(G_p(fu,fu,fu),G_p(fu,fz,fz),G_p(fz,fu,fu)\big) = 0$ So, $fu = fz$. Now, since $z = gu = fu$ and $fz = gz$, we have $z = gz = fz$ and z is a common fixed point of f and g.

Suppose that the set of common fixed points of f and g is well ordered. We claim that f and g have a unique common fixed point. Assume on contrary that, $fu = gu = u$ and $fv = gv = v$, and $u \neq v$. Without any loss of generality, we may assume that $gu = u \preceq v = gv$. Using (2.1), we obtain

$$\begin{aligned}
\psi\big(G_p(u,u,v)\big) &= \psi\big(G_p(fu,fu,fv)\big)\\
&\leq \psi\left(\frac{G_p(gu,fu,fu)+G_p(gu,fv,fv)+G_p(gv,fu,fu)}{3}\right)\\
&\quad - \varphi\big(G_p(gu,fu,fu),G_p(gu,fv,fv),G_p(gv,fu,fu)\big)\\
&\leq \psi\left(\frac{2G_p(v,u,u)+G_p(v,u,u)}{3}\right)\\
&\quad - \varphi\big(G_p(u,u,u),G_p(u,v,v),G_p(v,u,u)\big).
\end{aligned}$$

Therefore, $u = v$, a contradiction. Conversely, if f and g have only one common fixed point then, clearly, the set of common fixed points of f and g is well ordered. □

Remark 2.6. Following arguments similar to those given in the proof of Theorem 2.2, we obtain the previous results for a generalized weakly GP-contractive mapping of type B.

Denote by Λ the set of all functions $\mu : [0,+\infty) \to [0,+\infty)$ verifying the following conditions:
(I) μ is a positive Lebesgue integrable mapping on each compact subset of $[0,+\infty)$.
(II) For all $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)\,dt > 0$.

By the standard argument we obtain the following result for mappings satisfying a contraction of integral type.

Corollary 2.7. *Replace the contractive condition (2.1) of Theorem 2.2 by the following condition:*
There exists a $\mu \in \Lambda$ such that

$$\int_0^{\psi(G_p(fx,fy,fz))} \mu(t)\,dt \le \int_0^{\psi\left(\frac{G_p(gx,fy,fy)+G_p(gy,fz,fz)+G_p(gz,fx,fx)}{3}\right)} \mu(t)\,dt - \int_0^{\varphi(G_p(gx,fy,fy),G_p(gy,fz,fz),G_p(gz,fx,fx))} \mu(t)\,dt.$$

Then, f and g have a coincidence point, if the other conditions of Theorem 2.2 are satisfied.

Similar to [22], let $N \in \mathbb{N}$ be fixed. Let $\{\mu_i\}_{1\le i\le N}$ be a family of N functions which belong to Λ. For all $t \ge 0$, we define:

$$\begin{aligned}
I_1(t) &= \int_0^{t} \mu_1(s)\,ds,\\
I_2(t) &= \int_0^{I_1 t} \mu_2(s)\,ds = \int_0^{\int_0^t \mu_1(s)\,ds} \mu_2(s)\,ds,\\
I_3(t) &= \int_0^{I_2 t} \mu_3(s)\,ds = \int_0^{\int_0^{\int_0^t \mu_1(s)\,ds} \mu_2(s)\,ds} \mu_3(s)\,ds,\\
&\cdots,\\
I_N(t) &= \int_0^{I_{(N-1)} t} \mu_N(s)\,ds.
\end{aligned}$$

We have the following result.

Corollary 2.8. *Replace the inequality (2.1) of Theorem 2.2 by the following condition:*

$$I_N\Big(\psi\big(G_p(fx,fy,fz)\big)\Big) \le I_N\Big(\psi\big(\frac{G_p(gx,fy,fy)+G_p(gy,fz,fz)+G_p(gz,fx,fx)}{3}\big)\Big) - I_N\Big(\varphi\big(G_p(gx,fy,fy),G_p(gy,fz,fz),G_p(gz,fx,fx)\big)\Big).$$

Then, f and g have a coincidence point, if the other conditions of Theorem 2.2 are satisfied.

Proof. Consider $\hat{\Psi} = I_N \circ \psi$ and $\hat{\Phi} = I_N \circ \varphi$. □

We conclude this section by presenting some examples that illustrate our results.

Example 2.9. Consider the GP-metric space from Example 1.5, i.e., $X = [0,+\infty)$ and $G_p(x,y,z) = \max\{x,y,z\}$. Define an order $\preceq$ on X by

$$x \preceq y \iff x = y \vee (x, y \in [0,1] \wedge x \geq y).$$

Obviously, $(X, \preceq, G_p)$ is a complete ordered GP-metric space. Let $f : X \to X$ be given by

$$fx = \begin{cases} \frac{x^2}{3(1+x)}, & x \in [0,1], \\ \frac{x}{6}, & x > 1. \end{cases}$$

Finally, take $\varphi \in \Phi$ given by $\varphi(r,s,t) = \frac{1}{6}(r+s+t)$ for all $r,s,t \geq 0$. In order to check the conditions of Corollary 2.3, take $x, y, z \in X$ such that $x \preceq y \preceq z$ and consider the following two possible cases.

1° $x \leq 1$. Then, obviously also $y, z \leq 1$ and $x \leq y \leq z$. It is easy to check that

$$\begin{aligned} G_p(fx,fy,fz) &= \max\left\{\frac{x^2}{3(1+x)}, \frac{y^2}{3(1+y)}, \frac{z^2}{3(1+z)}\right\} \\ &= \frac{z^2}{3(1+z)} = \frac{z}{3(1+z)} \cdot z \leq \frac{z}{6} \\ &\leq \frac{1}{6}\left(G_p\left(x, \frac{y^2}{3(1+y)}, \frac{y^2}{3(1+y)}\right) + G_p\left(y, \frac{z^2}{3(1+z)}, \frac{z^2}{3(1+z)}\right)\right. \\ &\qquad \left. + G_p\left(z, \frac{x^2}{3(1+x)}, \frac{x^2}{3(1+x)}\right)\right) \\ &= \frac{G_p(x,fy,fy) + G_p(y,fz,fz) + G_p(z,fx,fx)}{3} \\ &\qquad - \varphi\big(G_p(x,fy,fy), G_p(y,fz,fz), G_p(z,fx,fx)\big). \end{aligned}$$

1° $z > 1$. Then $x = y = z > 1$ and

$$\begin{aligned} G_p(fx,fy,fz) &= \max\left\{\frac{x}{6}, \frac{y}{6}, \frac{z}{6}\right\} = \frac{z}{6} \\ &= \frac{1}{6} G_p\left(z, \frac{z}{6}, \frac{z}{6}\right) \\ &\leq \frac{1}{2} G_p\left(z, \frac{z}{6}, \frac{z}{6}\right) \\ &= \frac{G_p(x,fy,fy) + G_p(y,fz,fz) + G_p(z,fx,fx)}{3} \\ &\qquad - \varphi\big(G_p(x,fy,fy), G_p(y,fz,fz), G_p(z,fx,fx)\big). \end{aligned}$$

Hence, all the conditions of Corollary 2.3 are satisfied and f has a fixed point (which is $z = 0$).

Example 2.10. Consider the subset $Y = \{0, 1, 2\}$ of the GP-metric space given in Example 2.10, i.e., let

$$\begin{gathered} G_p(x,x,x) = 1, \text{ for } x \neq 2, \text{ and } G_p(2,2,2) = 0, \\ G_p(0,0,1) = G_p(0,0,2) = G_p(1,1,2) = 2, \\ G_p(0,1,1) = G_p(0,2,2) = G_p(1,2,2) = \tfrac{5}{2}, \\ G_p(0,1,2) = 3, \end{gathered}$$

with symmetry in all the variables. Introduce an order $\preceq$ on Y by

$$\preceq := \{(0,0),(1,1),(1,2),(2,2)\}.$$

Finally, let

$$f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \qquad \psi(t) = t, \quad \varphi(r,s,t) = \frac{r+s+t}{6}.$$

It is easy to see that f is weakly increasing with respect to g and that $f(Y) \subseteq g(Y)$.

It remains to check the condition (2.1) for $x, y, z \in Y$ satisfying $x \preceq y \preceq z$.

If $x = y = z = 0$, then $G_p(fx, fy, fz) = G_p(1,1,1) = 1$, $G_p(x, fy, fy) = G_p(y, fz, fz) = G_p(z, fx, fx) = \frac{5}{2}$. Hence, (2.1) reduces to $1 \leq \frac{5}{4}$ and holds true. In all other possible cases ($(x,y,z) \in \{(1,1,1),(1,1,2),(1,2,2),(2,2,2)\}$), it is $G_p(fx, fy, fz) = G_p(2,2,2) = 0$ and (2.1) holds trivially.

Thus, all the conditions of Theorem 2.2 are fulfilled and f and g have a coincidence point ($2 = f2 = g2$). This is also their common fixed point.

3. Periodic point results

Let $F(f) = \{x \in X \mid fx = x\}$, the fixed point set of f.

If $F(f) = F(f^n)$ for every $n \in \mathbb{N}$, then f is said to have property P. For more details, we refer the reader to [3, 9, 13, 18] and the references mentioned therein.

Theorem 3.1. *Let X and f be as in Corollary 2.3. Then f has property P.*

Proof. From Corollary 2.3, $F(f) \neq \emptyset$. Let $u \in F(f^n)$ for some $n > 1$. We will show that $u = fu$. Since f is weakly increasing on X, we have $f^{n-1}u \preceq f^n u$. Using (2.1), we obtain that

$$\begin{aligned} G_p(u, fu, fu) &= G_p(f^n u, f^{n+1}u, f^{n+1}u) \\ &= G_p(ff^{n-1}u, ff^n u, ff^n u) \\ &\leq \frac{1}{3}(G_p(f^{n-1}u, f^{n+1}u, f^{n+1}u) + G_p(f^n u, f^{n+1}u, f^{n+1}u) + G_p(f^n u, f^n u, f^n u)) \\ &\quad - \varphi\big(G_p(f^{n-1}u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^n u, f^n u)\big) \\ &\leq \frac{1}{3}(G_p(f^{n-1}u, f^n u, f^n u) + 2G_p(f^n u, f^{n+1}u, f^{n+1}u)) \\ &\quad - \varphi\big(G_p(f^{n-1}u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^n u, f^n u)\big); \end{aligned}$$

that is,

$$\begin{aligned}G_p(u,fu,fu) &= G_p(f^n u, f^{n+1}u, f^{n+1}u) \le G_p(f^{n-1}u, f^n u, f^n u)\\ &\quad - 3\varphi\big(G_p(f^{n-1}u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^{n+1}u, f^{n+1}u), G_p(f^n u, f^n u, f^n u)\big).\end{aligned}$$

Repeating the above process, we get

$$\begin{aligned}G_p(f^{n-(i)}u, f^{n-(i-1)}u, f^{n-(i-1)}u) &\le G_p(f^{n-(i+1)}u, f^{n-(i)}u, f^{n-(i)}u)\\ &\quad - 3\varphi\big(G_p(f^{n-(i+1)}u, f^{n-(i-1)}u, f^{n-(i-1)}u), G_p(f^{n-(i)}u, f^{n-(i-1)}u, f^{n-(i-1)}u),\\ &\qquad G_p(f^{n-(i)}u, f^{n-(i)}u, f^{n-(i)}u)\big).\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}G_p(u,fu,fu) &\le G(u,fu,fu)\\ &\quad - 3\sum_{i=0}^{n-1}\varphi\big(G_p(f^{n-(i+1)}u, f^{n-(i-1)}u, f^{n-(i-1)}u), G_p(f^{n-(i)}u, f^{n-(i-1)}u, f^{n-(i-1)}u),\\ &\qquad G_p(f^{n-(i)}u, f^{n-(i)}u, f^{n-(i)}u)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{i=0}^{n-1}\varphi\big(G_p(f^{n-(i+1)}u, f^{n-(i-1)}u, f^{n-(i-1)}u), G_p(f^{n-(i)}u, f^{n-(i-1)}u, f^{n-(i-1)}u),\\ &\qquad G_p(f^{n-(i)}u, f^{n-(i)}u, f^{n-(i)}u)\big) = 0,\end{aligned}$$

which from our assumptions about φ implies that

$$G_p(f^{n-(i+1)}u, f^{n-(i-1)}u, f^{n-(i-1)}u) = G_p(f^{n-(i)}u, f^{n-(i-1)}u, f^{n-(i-1)}u) = 0$$

for all $0 \le i \le n-1$. Now, taking $i = n-1$, we have $u = fu$. $\square$

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