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BOUNDEDNESS OF MAXIMAL FUNCTIONS IN L^p SPACES

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Abstract. In this paper, we derive suitable L^p bounds for a specific class of maximal operators defined along the surface of revolution, where the kernels belong to $L^q(\mathbb{S}^{n-1})$ with $q > 1$. Using these bounds in conjunction with extrapolation techniques, we establish the L^p boundedness of the maximal operator when its kernels lie in the block space $B_q^{(0,s-1)}(\mathbb{S}^{n-1})$ or in $(L \log L)^s(\mathbb{S}^{n-1})$.

1. Introduction and preliminaries

For $n \geq 2$, let $\mathbb{R}^n$ be the n-dimensional Euclidean space. In $\mathbb{R}^n$, let $\mathbb{S}^{n-1}$ represent the unit sphere with the normalized Lebesgue measure $d\sigma$. Use $x' = \frac{x}{|x|}$ to define $x \in \mathbb{R}^n \setminus \{0\}$. Consider the following real-valued polynomial function $P : \mathbb{R}^n \to \mathbb{R}$. Considering $1 \leq \alpha < \infty$, let $\mathfrak{L}^\alpha$ represent the collection of all measurable functions $k : \mathbb{R}^+ \to \mathbb{R}$ such that

$$||k||_{L^\alpha(\mathbb{R}^+, r^{-1}dr)} = \left(\int_0^\infty |k(r)|^\alpha r^{-1} dr \right)^{1/\alpha} \leq 1,$$

and define $\mathfrak{L}^\infty(\mathbb{R}^+)$ as $L^\infty(\mathbb{R}^+, r^{-1}dr)$.

Furthermore, let ω be an integrable function over $\mathbb{S}^{n-1}$ that is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies the cancellation property:

$$\int_{\mathbb{S}^{n-1}} \omega(x') d\sigma(x') = 0. \tag{1.1}$$

For a suitable map $\theta : (0, \infty) \to (-\infty, \infty)$, we define the maximal function $\mu^{(\alpha)}_{P,\omega,\theta}$ associated with functions $g \in \mathcal{S}(\mathbb{R}^{n+1})$ as follows:

$$\mu^{(\alpha)}_{P,\omega,\theta}(g)(x, x_{n+1}) = \sup_{h \in \mathfrak{L}^\alpha(\mathbb{R}^+)} \left| \int_{\mathbb{R}^n} e^{iP(\xi)} g(x - \xi, x_{n+1} - \theta(|\xi|)) \omega(\xi) \frac{k(|\xi|)}{|\xi|^n} \, d\xi \right|. \tag{1.2}$$

In the case where $P(\xi) = 0$, we denote $\mu^{(\alpha)}_{P,\omega,\theta}$ simply as $\mu^{(\alpha)}_{\omega,\theta}$. Additionally, when $\theta(t) = t$ and $\alpha = 2$, we refer to $\mu^{(\alpha)}_{\omega,\theta}$ as μ_ω, which corresponds to the classical maximal operator introduced by Chen and Lin in [10]. They established the

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boundedness of μ_ω on $L^p(\mathbb{R}^n)$ for $p > \frac{2n}{2n-1}$ and $\omega \in C(\mathbb{S}^{n-1})$. Al-Salman further enhanced this result in [6], showing that the boundedness on $L^p(\mathbb{R}^n)$ holds for $p \geq 2$ provided $\omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$. Moreover, he demonstrated that μ_ω may fail to be bounded on L^2 if ω is replaced by some $\omega \in L(\log L)^r(\mathbb{S}^{n-1})$ for $r < 1/2$.

In [4], Al-Qassem generalized this finding by confirming the $L^p(\mathbb{R}^{n+1})$ boundedness of $\mu^{(\alpha)}_{\omega,\theta}$ for $\omega \in L(\log L)^{1/\alpha'}(\mathbb{S}^{n-1})$ for some $1 < \alpha \leq 2$, where θ is a convex and increasing function in $C^2([0,\infty))$. Al-Salman, in [8], also showed the L^p boundedness of maximal functions when their kernels belong to $L(\log L)^{1/2}(\mathbb{S}^{n-1})$ or to the block space $B_q^{(0,-1/2)}(\mathbb{S}^{n-1})$ for some $q > 1$. Furthermore, it was established that the condition $\omega \in B_q^{(0,-1/2)}(\mathbb{S}^{n-1})$ is optimal. In the same paper, Al-Salman provided L^p estimates for a certain class of maximal functions with kernels in $L^q(\mathbb{S}^{n-1})$.

For additional related results, readers may refer to [8, 12, 15, 16, 17, 18, 19, 3, 20, 2, 5, 11] and the references therein.

Subsequently, the authors of [1] established the L^p boundedness of the maximal operator $\mu^{(\alpha)}_{P,\omega,\theta}$ whenever the kernels lie in $L(\log L)^{1/\alpha'}(\mathbf{S}^{n-1})$ or in the block space $B_q^{(0,-1/\alpha)}(\mathbb{S}^{n-1})$ for some $1 < \alpha \leq 2$ under certain conditions on θ. In this article, we expand this result by introducing new conditions on θ that ensure the boundedness of the maximal operator $\mu^{(\alpha)}_{P,\omega,\theta}$ on $L^p(\mathbb{R}^{n+1})$. Specifically, by defining a class of functions θ that satisfies either hypothesis I or hypothesis D, we provide a more general framework for the boundedness of the maximal operator.

A non-negative C^1 function θ is said to satisfy hypothesis I if it is strictly increasing, $\theta(2x) > \lambda\theta(x)$ for some $\lambda > 1$, and $\theta'(x)$ is decreasing on $(0,\infty)$. Conversely, θ satisfies hypothesis D if it is strictly decreasing, $\theta(x) > \lambda\theta(2x)$ for some $\lambda > 1$, and $\theta'(x)$ is increasing on $(0,\infty)$. These hypotheses (I, and D) are closely aligned with those in [13].

The maximal functions defined by equation (1.2) have been extensively studied by many researchers, especially when the function θ adheres to either hypothesis I or hypothesis D. For a thorough understanding of the background and related findings regarding these functions, we suggest that readers consult [14, 9, 7].

Our results are inspired by the research on the parametric maximal operator presented in [1]. By modifying the conditions on θ, we seek to ascertain whether we can reach the same conclusions as the authors in [1]. We will address this question in the following theorems.

Now we are presenting and proving several auxiliary lemmas that are essential for establishing our main results.

Lemma 1.1. [1] *Let $\omega \in L^q(\mathbb{S}^{n-1})$, $q > 1$ satisfy the cancelation property* (1.1) *with $||\omega||_{L^1(\mathbb{S}^{n-1})} \leq 1$. Assume that $P = \sum\limits_{\alpha \leq d} a_\alpha x^\alpha$ is a polynomial of degree $n \geq 1$ such that $|x|^d$ is not one of its terms and $\sum\limits_{|\alpha|=d} |a_\alpha| = 1$. Let θ be an arbitrary function on $\mathbb{R}^+$. For $k \in \mathbb{Z}$, define $\Gamma_{k,\omega,\theta} : \mathbb{R}^{n+1} \to \mathbb{R}$ by*

$$\Gamma_{k,\omega,\theta}(\zeta,\eta) = \int_1^{2^{2(1+\nu_\omega)}} \Big| \int_{\mathbb{S}^{n-1}} \omega(v) \chi_{k,\omega,\theta}(r, v, \zeta.v, \eta) d\sigma(v) \Big|^2 r^{-1} dr, \tag{1.3}$$

where

$$\chi_{k,\omega,\theta}(r,v,\zeta.v,\eta) = e^{-i[p(2^{-(k+1)\nu_\omega}rv)+(2^{-(k+1)\nu_\omega})rv.\zeta+\theta(2^{-(k+1)\nu_\omega}r)\eta]} \tag{1.4}$$

and

$$\nu_\omega = log(e + ||\omega||_{L^q(\mathbb{S}^{n-1})})$$

Then, there exists a positive constant β such that

$$\sup_{(\zeta,\eta)\in\mathbb{R}^n\times R} \Gamma_{k,\omega,\theta} \leq \beta\nu_\omega 2^{(k+1)/4q'}.$$

Lemma 1.2. *Let $\omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero and satisfy the cancelation property. Suppose that $\theta : \mathbb{R}^+ \to \mathbb{R}$ is in $C^2[0,\infty)$, that satisfies hypothesis I or hypothesis D. Define the maximal function by*

$$M_{\omega,\theta}g(x,x_{n+1}) = \sup_{j\in\mathbb{Z}} \int_{2^j\nu_\omega\leq|\alpha|\leq 2^{j+1}\nu_\omega} \left|g(x-\xi, x_{n+1}-\theta(|\xi|))\right| \frac{|\omega(\xi)|}{|\xi|^n} d\xi. \tag{1.5}$$

Then, for $1 < p < \infty$, there exists a positive number β_p such that

$$||M_{\omega,\theta}(g)||_p^2 \leq \beta_p(1+\nu_\omega)||g||_p^2||\omega||^2_{L^1(\mathbb{S}^{n-1})} \tag{1.6}$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

We obtain the following result by employing arguments similar to those in ([4], Theorem 1.6).

Lemma 1.3. *Let $\omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ satisfy the cancellation property, and assume $\|\omega\|_{L^1(\mathbb{S}^{n-1})} \leq 1$. Suppose $\theta : \mathbb{R}^+ \to \mathbb{R}$ is in $C^2[0,\infty)$ and meets either hypothesis I or hypothesis D. Then, for $2 \leq p < \infty$, there exists a positive constant $\beta_{p,q}$ such that*

$$\|\mu^{(2)}_{0,\omega,\theta}(g)\|_p^2 \leq \beta_{p,q}(1+\nu_\omega)\|g\|_p^2 \tag{1.7}$$

for every $g \in L^p(\mathbb{R}^{n+1})$.

Proof. since $L^q(\mathbb{S}^{n-1}) \subset L^2(\mathbb{S}^{n-1})$ for $q \geq 2$, it suffices to prove this lemma for $1 < q \leq 2$. We begin by noting that

$$\mu^{(2)}_{0,\omega,\theta}g(x,x_{n+1}) \leq \left(\int_0^\infty \left|\int_{\mathbb{S}^{n-1}} g(x-rv, x_{n+1}-\theta(r))\omega(v)d\sigma(v)\right|^2 r^{-1}dr\right)^{1/2}. \tag{1.8}$$

Next, let $\{\Lambda\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $I_{k,\nu_\omega} = [2^{-(k+1)\nu_\omega}, 2^{-(k-1)\nu_\omega}]$. Specifically,

$$\sum_{k\in\mathbb{Z}} \Lambda_k = 1, \quad 0 \leq \Lambda_k \leq 1, \quad \Lambda_k \in C^\infty,$$

$$\mathrm{supp}(\Lambda_k) \subset I_{k,\nu_\omega}, \quad \text{and} \quad \left|\frac{d^j\Lambda_k(r)}{dr^j}\right| \leq \frac{C_j}{r^j},$$

where C_j is independent of the lacunary sequence I_{k,ν_ω}. Define $\widehat{(\Psi_k g)}(\zeta,\eta) = \Lambda_k(|\zeta|)\widehat{g}(\zeta,\eta)$ for $(\zeta,\eta) \in \mathbb{R}^n \times \mathbb{R}$. Then for $g \in \mathcal{S}(\mathbb{R}^{n+1})$, we have

$$\mu^{(2)}_{0,\omega,\theta}g(x,x_{n+1})\leq\sum_{j\in\mathbf{Z}}\left(\int_0^\infty\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{S}^{n-1}}(\Psi_{k+j}g)(x-rv,x_{n+1}-\theta(r))\chi_{I_{k,\nu_\omega}}\omega(v)d\sigma(v)\right|^2r^{-1}dr\right)^{1/2}.$$

Applying Minkowski's inequality, Plancherel's theorem, Fubini's theorem, and Lemma 1.2, we obtain

$$\|\mu^{(2)}_{0,\omega,\theta}g(x,x_{n+1})\|_p\leq\int_{\mathbf{R}^{n+1}}\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\left(\int_{\nu_\omega^k}^{\nu_\omega^{k+1}}\left|\int_{\mathbb{S}^{n-1}}\omega(v)e^{-i(rx\zeta+\eta\theta(r))}d\sigma(v)\right|^2r^{-1}dr|\widehat{g}(\zeta,\eta)|^2d\zeta d\eta\right)^{1/2}\right)$$

This leads to

$$\leq\sum_{j\in\mathbb{Z}}\beta_{p,q}(1+\nu_\omega)^{1/2}2^{-\beta|j|}\|g\|_p$$

for some $\beta\in(0,1)$. Additionally, by interpolating between the above inequality and the inequality

$$\|\mu^{(2)}_{0,\omega,\theta}(g)\|_p^2\leq\beta_{p,q}(1+\nu_\omega)\|g\|_p^2$$

for $p>2$, we can directly conclude that

$$\|\mu^{(2)}_{0,\omega,\theta}(g)\|_p^2\leq\beta_{p,q}(1+\nu_\omega)\|g\|_p^2$$

for all $p\geq 2$. □

2. Main results

Theorem 2.1. *Let $\theta:\mathbb{R}^+\to\mathbb{R}$ be a function in C^2 that satisfies either hypothesis I or hypothesis D. Suppose $\omega\in L^q(\mathbb{S}^{n-1})$ for $q>1$ and fulfills the cancellation property (1.1), with $\|\omega\|_{L^1(\mathbb{S}^{n-1})}\leq 1$. Consider $P:\mathbb{R}^n\to\mathbb{R}$ as a real polynomial. Let $\mu^{(\alpha)}_{P,\omega,\theta}$ be defined as in (1.2). Then, there exists a constant $\beta_{p,q}>0$ such that*

$$\|\mu^{(\alpha)}_{P,\omega,\theta}(g)\|_p\leq\beta_{p,q}\left(1+\log(e+\|\omega\|_{L^q(\mathbb{S}^{n-1})})\right)^{1/\alpha'}\|g\|_p \tag{2.1}$$

for all $\alpha'\leq p<\infty$ and for all $\alpha\in(1,2]$. Additionally, we have

$$\|\mu^{(1)}_{P,\omega,\theta}(g)\|_{L^\infty(\mathbb{R}^{n+1})}\leq\beta\|g\|_{L^\infty(\mathbb{R}^{n+1})}, \tag{2.2}$$

where $\beta_{p,q}=\frac{2^{1/q'}}{2^{1/q'}-1}\beta_p$ and β_p is a positive constant that does not depend on ω, θ, or q.

To prove theorem 2.1, we will follow the same argument used in ([1], Theorem 1.1), and ([8], Theorem 1.1). We will consider three cases: $\alpha=1$, $\alpha=2$, and $1<\alpha<2$.

Case 1: For $1<\alpha<2$. By duality, we have

$$\mu^{(\alpha)}_{P,\omega,\theta}(g)(x,x_{n+1})=\left(\int_0^\infty\left|\int_{\mathbb{S}^{n-1}}e^{iP(rv)}g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\right|^{\alpha'}r^{-1}dr\right)^{1/\alpha'}. \tag{2.3}$$

Assuming that Theorem 2.1 holds for $\alpha = 1$ and $\alpha = 2$, we obtain:

$$||\mu^{(2)}_{P,\omega,\theta}(g)(x, x_{n+1})||_{L^p(\mathbb{R}^{n+1})} \le \beta_{p,q}(1 + \nu_\omega)^{1/2}||g||_p, \tag{2.4}$$

for $2 \le p < \infty$, and

$$||\mu^{(1)}_{P,\omega,\theta}(g)(x, x_{n+1})||_{L^\infty(\mathbb{R}^{n+1})} \le \beta||g||_{L^\infty(\mathbb{R}^{n+1})}.$$

Thus, by applying interpolation to the previous inequalities, we conclude that

$$||\mu^{(\alpha)}_{P,\omega,\theta}(g)(x, x_{n+1})||_p \le \beta_{p,q}(1 + \nu_\omega)^{1/\alpha'}||g||_p$$

for $\alpha' \le p < \infty$ with $1 < \alpha \le 2$, and

$$||\mu^{(1)}_{P,\omega,\theta}(g)(x, x_{n+1})||_{L^\infty(\mathbb{R}^{n+1})} \le \beta||g||_{L^\infty(\mathbb{R}^{n+1})}.$$

Therefore, it suffices to prove our theorem for the cases $\alpha = 1$ and $\alpha = 2$.

Case 2: For $\alpha = 1$. Let $k \in L^1(\mathbb{R}^+, r^{-1}dr)$ and let $g \in L^\infty(\mathbb{R}^{n+1})$. Then for all $(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, we have

$$\begin{aligned}
&\left|\int_0^\infty k(r)\int_{\mathbb{S}^{n-1}} e^{iP(rv)}g(x - rv, x_{n+1} - \theta(r))\omega(v)d\sigma(v)\right|r^{-1}dr \\
&\le \int_0^\infty |k(r)|\left|\int_{\mathbb{S}^{n-1}} e^{iP(rv)}g(x - rv, x_{n+1} - \theta(r))\omega(v)d\sigma(v)\right|r^{-1}dr \\
&\le \beta||k||_{L^1(\mathbb{R}^+,r^{-1}dr)}||g||_{L^\infty(\mathbb{R}^{n+1})}.
\end{aligned}$$

Now, by taking the supremum on both sides for all h with $||k||_{L^1(\mathbb{R}^+,r^{-1}dr)} \le 1$, we obtain that

$$\mu^{(1)}_{P,\omega,\theta}g(x, x_{n+1}) \le \beta||g||_{L^\infty(\mathbb{R}^{n+1})}$$

for almost every $(x, x_{n+1}) \in \mathbb{R}^{n+1}$. Therefore,

$$||\mu^{(1)}_{P,\omega,\theta}g(x, x_{n+1})||_{L^\infty(\mathbb{R}^{n+1})} \le \beta||g||_{L^\infty(\mathbb{R}^{n+1})}.$$

Case 3: For $\alpha = 2$. We proceed by induction on the degree of the polynomial P. If $\deg(P) = 0$, the result follows directly from Lemma 1.3 and we have:

$$||\mu^{(2)}_{P,\omega,\theta}g(x, x_{n+1})||_p \le \beta_{p,q}(1 + \nu_\omega)^{1/2}||g||_p$$

for all $p \ge 2$. Now assume that the following inequality:

$$||\mu^{(\alpha)}_{P,\omega,\theta}(g)||_p \le \beta_{p,q}(1 + \nu_\omega)^{1/\alpha'}||g||_p \tag{2.5}$$

holds for all polynomials P with $\deg(P) \le m$ where $m \ge 1$. We aim to show that this inequality also holds for polynomials of degree $m + 1$. Let

$$P(x) = \sum_{|\alpha| \le m+1} a_\alpha x^\alpha$$

be a polynomial of degree $m+1$, which does not contain $|x|^{m+1}$ as one of its terms. As in Lemma 1.3, we let $\{\Lambda_k\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $I_{k,\nu_\omega} = [2^{-(k+1)\nu_\omega}, 2^{-(k-1)\nu_\omega}]$. Define

$$\widehat{(\Psi_k g)}(\zeta,\eta) = \Lambda_k(|\zeta|)\widehat{g}(\zeta,\eta) \quad \text{for} \quad (\zeta,\eta)\in\mathbb{R}^n\times\mathbb{R} \text{ for } g\in\mathcal{S}(\mathbb{R}^{n+1}).$$

Set

$$F_\infty(r) = \sum_{k=-\infty}^{0}\Lambda_k(r) \quad \text{and} \quad F_0(r) = \sum_{k=1}^{\infty}\Lambda_k(r).$$

Hence, we have

$$\mu^{(2)}_{P,\omega,\theta}g(x,x_{n+1}) \le \mu^{(2)}_{P,\omega,\theta,\infty}g(x,x_{n+1}) + \mu^{(2)}_{P,\omega,\theta,0}g(x,x_{n+1}), \tag{2.6}$$

where

$$||\mu^{(2)}_{P,\omega,\theta,\infty}g(x,x_{n+1})||^2 = \int_{2^{-\nu_\omega}}^{\infty}\left|F_\infty(r)\int_{\mathbb{S}^{n-1}}e^{iP(rv)}g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\right|^2 r^{-1}dr$$

and

$$||\mu^{(2)}_{P,\omega,\theta,0}g(x,x_{n+1})||^2 = \int_0^1\left|F_0(r)\int_{\mathbb{S}^{n-1}}e^{iP(rv)}g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\right|^2 r^{-1}dr.$$

Now we estimate the L^p norm of $\mu^{(2)}_{P,\omega,\theta,\infty}g(x,x_{n+1})$. Let

$$||\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1})||^2 = \int_{2^{-(k+1)\nu_\omega}}^{2^{-(k-1)\nu_\omega}}\left|\int_{\mathbb{S}^{n-1}}e^{iP(rv)}g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\right|^2 r^{-1}dr.$$

By applying the generalized Minkowski's inequality, we obtain

$$\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1}) \le \sum_{k=-\infty}^{0}\mu^{(2)}_{P,\omega,\theta,\infty}g(x,x_{n+1}). \tag{2.7}$$

Next, we apply Plancherel's theorem, Fubini's theorem along with Lemma 1.2 to estimate the L^2 norm of each term $\mu^{(2)}_{P,\omega,\theta,\infty,k}$. Plancherel's theorem ensures that the L^2 norm of a function is preserved under Fourier transforms, Fubini's theorem allows us to interchange the order of the integration, and we obtain that

$$\begin{aligned}||\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1})||^2_{L^2(\mathbb{R}^{n+1})} &= \int_{\mathbb{R}^{n+1}}|\hat{g}|^2\int_1^{2^{2(1+\nu_\omega)}}\left|\int_{\mathbb{S}^{n-1}}\omega(v)\chi_{k,\omega,\theta}(r,v,\zeta.v,\eta)d\sigma(v)\right|^2 r^{-1}dr\\ &\le \beta 2^{\frac{(k+1)}{4q'}}(1+\nu_\omega)||g||^2_{L^2(\mathbb{R}^{n+1})}. \end{aligned}\tag{2.8}$$

For $p>2$, there exists $\varkappa\in L^{(p/2)'}(\mathbb{R}^{n+1})$ such that $||\varkappa||_{L^{(p/2)'}(\mathbb{R}^{n+1})}=1$. Thus, we have

$$
\begin{aligned}
||\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1})||_p^2 &= \int_{\mathbb{R}^{n+1}}\int_1^{2^{2(1+\nu_\omega)}}\Big|\int_{\mathbb{S}^{n-1}} e^{iP(rv)}g(x-2^{-(k+1)\nu_\omega}rv,x_{n+1}-\theta(2^{-(k+1)\nu_\omega}r))\\
&\times \quad \chi_{k,\omega,P}(r,v,0,0)d\sigma(v)\Big|^2 r^{-1}dr|\varkappa(x,x_{n+1})|\,dx\,dx_{n+1}.
\end{aligned}
$$

Now, applying Lemma 1.2 and Hölder's inequality, we get

$$
\begin{aligned}
||\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1})||_p^2 \leq &\int_{\mathbb{R}^{n+1}}\left|g(x-2^{-(k+1)\nu_\omega}rv,x_{n+1}-\theta(2^{-(k+1)\nu_\omega}r))\right|^2\\
&\times\int_1^{2^{2(1+\nu_\omega)}}\Big|\int_{\mathbb{S}^{n-1}}\chi_{k,\omega,P}(r,v,0,0)d\sigma(v)\Big|^2|\varkappa(x,x_{n+1})|r^{-1}dr\,dx\,dx_{n+1}\\
&\leq \beta(1+\nu_\omega)|||g|^2||_{(p/2)}||M_{\omega,\theta}(\tilde{\varkappa})||_{(p/2)'}\\
&\leq \beta(1+\nu_\omega)||g||_p^2. \qquad (2.9)
\end{aligned}
$$

combined this inequality (2.9) with (2.8) we obtain

$$
||\mu^{(2)}_{P,\omega,\theta,\infty,k}g(x,x_{n+1})||_p^2 \leq \beta 2^{\frac{(k+1)}{4q'}}(1+\nu_\omega)||g||_p^2 \qquad (2.10)
$$

for all $2\leq p<\infty$, and for $q^{'}\in(1,\infty)$ which when combined with (2.7) gives

$$
||\mu^{(2)}_{P,\omega,\theta,\infty}(g)||_p^2 \leq C_{p,q}(1+\nu_\omega)||g||_p^2 \qquad (2.11)
$$

Next, we estimate the L^p norm of $\mu^{(2)}_{P,\omega,\theta,0}(g)$. Let $H(x)=\sum_{\alpha\leq m}a_\alpha x^\alpha$. Define

$$
\mu^{(2)}_{H,\omega,\theta,0}g(x,x_{n+1}) = \Big(\int_0^1\Big|\int_{\mathbb{S}^{n-1}}e^{iH(rv)}g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\Big|^2 r^{-1}dr\Big)^{1/2}
$$

and

$$
\mu^{(2)}_{P,H,\omega,\theta,0}g(x,x_{n+1}) = \Big(\int_0^1\Big|\int_{\mathbb{S}^{n-1}}(e^{iP(rv)}-e^{iH(rv)})g(x-rv,x_{n+1}-\theta(r))\omega(v)d\sigma(v)\Big|^2 r^{-1}dr\Big)^{1/2}.
$$

It is evident that Minkowski's inequality leads to

$$
\mu^{(2)}_{P,\omega,\theta,0}g(x,x_{n+1}) \leq \mu^{(2)}_{H,\omega,\theta,0}g(x,x_{n+1}) + \mu^{(2)}_{P,H,\omega,\theta,0}g(x,x_{n+1}). \qquad (2.12)
$$

Since $\deg(H)\leq m$, by our assumption regarding the polynomial H, we obtain

$$
||\mu^{(2)}_{H,\omega,\theta,0}g(x,x_{n+1})||_p^2 \leq \beta_{p,q}(1+\nu_\omega)||g||_p^2 \qquad (2.13)
$$

for $p\geq 2$.
Note that

$$
|e^{iP(rv)}-e^{iH(rv)}| \leq |iP(rv)|+|iH(rv)| \leq |r^{m+1}\sum_{\alpha=m+1}a_\alpha v^\alpha| \leq r^{m+1}.
$$

By applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mu^{(2)}_{P,H,\omega,\theta,0}g(x,x_{n+1}) &\leq \Big(\int_0^1 \Big|\int_{\mathbb{S}^{n-1}} r^{2(m+1)}|g(x-rv,x_{n+1}-\theta(r))|^2|\omega(v)|d\sigma(v)r^{-1}dr\Big|\Big)^{1/2}\\ &\leq \Big(\sum_{k=1}^{\infty} 2^{-k(2(m+1))}\int_{2^{-k}}^{2^{-k+1}}\int_{\mathbb{S}^{n-1}} r^{2(m+1)}|g(x-rv,x_{n+1}-\theta(r))|^2|\omega(v)|d\sigma(v)r^{-1}dr\\ &\leq \beta\left(\mu_{\omega,\theta}(|g|^2)\right)^{1/2}\end{aligned}$$

Thus, by applying Lemma 1.2, we get

$$\begin{aligned}||\mu^{(2)}_{P,H,\omega,\theta,0}g(x,x_{n+1})||_p^2 &\leq \beta_p|||g|^2||_{L^{p/2}(\mathbb{R}^{n+1})}||\omega||^2_{L^1(\mathbb{S}^{n-1})}\\ &\leq \beta_p(1+\nu_\omega)||g||_p^2. \end{aligned} \tag{2.14}$$

This, along with (2.13), yields

$$||\mu^{(2)}_{P,\omega,\theta,0}g(x,x_{n+1})||_p^2 \leq \beta_p(1+\nu_\omega)||g||_p^2 \tag{2.15}$$

for all $2 \leq p < \infty$. Consequently, from (2.6), (2.11), and (2.15), we complete the proof of Theorem 2.1.

We derive the following theorem by utilizing the conclusions from the previous theorem and employing Yano's extrapolation arguments.

Theorem 2.2. [21] *Let ω, θ and P be given as in Theorem 2.1. Suppose that $\mu^{(\alpha)}_{P,\omega,\theta}$ is given by (1.2). If $\omega \in L(logL)^{1/\alpha'} \cup B_q^{(0,-1/\alpha)}(\mathbb{S}^{n-1})$, then $\mu^{(\alpha)}_{P,\omega,\theta}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for all $\alpha' \leq p < \infty$ and for $\alpha \in (1,2]$.*

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