Gulf Journal of Mathematics ISSN: 2309-4966
Vol 18, Issue 1 (2024) 197-203 https://doi.org/10.56947/gjom.v18i1.2416

NORMED JULIA SET OF BOUNDED LINEAR OPERATORS

SANOOJ B[1*] and VINODKUMAR P B[2]

Abstract. It is proved that normed Julia set of a bounded linear operator T satisfying $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$ cannot be the whole space. It is also proved that there exists at least one bounded linear operator on $\ell^p, 1 \leq p < \infty$ such that the normed Julia set is the whole space.

1. Introduction and preliminaries

The Julia set is a central concept in the study of complex dynamics, particularly in the context of meromorphic functions and polynomials. For a meromorphic function $g : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, the filled Julia set is defined as $L(g) = \{z \in \hat{\mathbb{C}} \mid \ g^n(z) \nrightarrow \infty\}$, where g^n denotes the n-th iterate of g (see [1], [4], [5], and [6]). The Julia set $J(g)$ of g is the boundary of $L(g)$, i.e., $J(g) = \partial L(g)$. Julia sets of quadratic polynomials, in particular, have been the subject of extensive study in works such as [1] and [4]. In this paper, we extend the concept of Julia sets to bounded linear operators on Banach spaces, introducing and analyzing the Normed Julia set. In Section 2, we formally define the Normed Julia set of a bounded linear operator and provide illustrative examples. In Section 3, we prove the main theorems.

Here, we consider X to be a Banach space, either real or complex, and let $B(X)$ denote the space of all bounded linear operators on X.

2. Definitions And Examples

Definition 2.1. The Normed Julia set of T is defined as

$$NJ(T) = \partial \{x \in X \mid \ \| T^n x \| \nrightarrow \infty\}.$$

Here T is a bounded linear operator and ∂ is the boundary of a set.

Example 2.2. Let $X = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the operator defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$. Then, $NJ(T) = \phi$.

Date: Received: Nov 5, 2024; Accepted: Nov 30, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 47A16; Secondary 47B01.

Key words and phrases. Filled Julia set, Julia set, Normed Julia set.

Consider

$$\mathrm{T}^n(x) = T^n(x_1, x_2) = \begin{cases} 2^{\frac{n}{2}} x & \text{if n is even,} \\ 2^{\frac{n-1}{2}} T(x) & \text{if n is odd.} \end{cases}$$

Therefore, $\|T^n(x)\| \to \infty$ for all $x \in \mathbb{R}^2$. Hence, $NJ(T) = \phi$.

Example 2.3. Let $X = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the operator defined by $T(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}\right)$. Then, $NJ(T) = \mathbb{R}^2$.

Consider

$$T^n(x) = T^n(x_1, x_2) = \frac{1}{2^n} x.$$

Therefore, $\|T^n(x)\| = \frac{1}{2^n}\|x\|$. Since $\|x\|$ is finite, the term $\frac{1}{2^n}$ tends to 0 as $n \to \infty$. Thus, $\|T^n(x)\|$ converges to 0 for all $x \in \mathbb{R}^2$. Hence, $NJ(T) = \mathbb{R}^2$.

Example 2.4. Let $X = \ell^2(\mathbb{N})$ and $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the operator defined by $T\left((y_n)_{n\in\mathbb{N}}\right) = (y_{n+1})_{n\in\mathbb{N}}$. Then, $NJ(T) = \ell^2(\mathbb{N})$.

Consider

$$T^n\left((y_n)_{n\in\mathbb{N}}\right) = (y_{n+1}, y_{n+2}, y_{n+3}, \ldots).$$

Therefore, $\|T^n\left((y_n)_{n\in\mathbb{N}}\right)\| = \sqrt{\sum\limits_{i=n+1}^{\infty} |y_i|^2} \to 0$ for all $(y_n) \in \ell^2(\mathbb{N})$ as $n \to \infty$. Hence, $NJ(T) = \ell^2(\mathbb{N})$.

Example 2.5. Fix $r > 1$. Let $X = \ell^2(\mathbb{N})$ and $f : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the operator defined by $f(x_1, x_2, \cdots) = (rx_2^2, rx_3^2, rx_4^2, \cdots)$. Then, $NJ(f) = \ell^2(\mathbb{N})$.

Consider

$$f^n(x) = f^n(x_1, x_2, \cdots) = \left(r^{2^n-1} x_{n+1}^{2^n},\ \ r^{2^n-1} x_{n+2}^{2^n}, \cdots\right).$$

And:

$$\|f^n\| = r^{2^n-1}, \quad \text{for all } n \geq 1.$$

There is no $y \in \ell^2(\mathbb{N})$ such that $\|f^n(y)\| \to \infty$.
Moreover,

$$\|f^n(y)\| \to 0 \quad \text{for all } y \in \ell^2(\mathbb{N}).$$

Let $y \in \ell^2(\mathbb{N})$ and $n_0 \geq 1$ be such that $|y_j| < \dfrac{1}{r}$, for all $j \geq n_0$.
Then, for $n \geq n_0$,

$$\begin{aligned}\|f^n(y)\|^2 &= \sum_{j=n+1}^{\infty} \left|r^{2^n-1} y_j^{2^n}\right|^2 \\ &= \sum_{j=n+1}^{\infty} \left(|ry_j|^{2^n-1}|y_j|\right)^2 \\ &\leq \sum_{j=n+1}^{\infty} |y_j|^2.\end{aligned}$$

Thus, $\|f^n(y)\| \to 0$. Hence, $NJ(f) = \ell^2(\mathbb{N})$.

Example 2.6. Let $X = \mathbb{C}^{m+1}$ with supremum norm, and $C : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be the Cesaro operator defined by

$$C(\Gamma_0, \Gamma_1, \cdots, \Gamma_m) = \left(\Gamma_0, \frac{\Gamma_0 + \Gamma_1}{2}, \frac{\Gamma_0 + \Gamma_1 + \Gamma_2}{3}, \cdots, \frac{\Gamma_0 + \Gamma_1 + \cdots + \Gamma_m}{m+1}\right).$$

Then, $NJ(C) = \mathbb{C}^{m+1}$.

Consider

$$M = \{(x_0, y_1, y_2, \cdots, y_m) \mid y_1, y_2, \cdots, y_m \in \mathbb{C}\}.$$

Then, we have $C(M) \subset M$. Now, let

$$\hat{p} = (x_0, y_1, y_2, \cdots, y_m) \text{ and } \hat{q} = (x_0, z_1, z_2, \cdots, z_m)$$

be elements of M. Then,

$$\left\| \frac{x_0 + y_1 + \cdots + y_k}{k+1} - \frac{x_0 + z_1 + \cdots + z_k}{k+1} \right\| \leq \frac{k}{k+1} \|\hat{p} - \hat{q}\|, \quad 2 \leq k \leq m.$$

This shows that

$$\|C(\hat{p}) - C(\hat{q})\| \leq L\, \|\hat{p} - \hat{q}\| \text{ for all } \hat{p}, \hat{q} \in M, \text{ with } L = \left(1 - \frac{1}{m+1}\right) < 1.$$

Thus, C is a contraction mapping on M, and consequently, C has a unique fixed point, given by $(x_0, x_0, \cdots, x_0)$.
Hence,

$$\|C^n(x)\| \to |x_0| \quad \text{for all}\, n \in \mathbb{N}.$$

Therefore, $NJ(C) = \mathbb{C}^{m+1}$.

Remark 2.7. In Example 2.6, $C^n(x)$ converges to $(y, y, \cdots, y) \in \mathbb{C}^{m+1}$ depending upon the chosen point x. We refer to $(y, y, \cdots, y)$ as a diagonal point. Additionally, if $\mathrm{diag}(\mathbb{C}^{m+1}) = \{(y, y, \cdots, y) \mid y \in \mathbb{C}\}$, then

$$\textstyle\bigcup_{x \in \mathbb{C}^{m+1}} \lim_{n\to\infty} C^n(x) = \mathrm{diag}(\mathbb{C}^{m+1}).$$

3. Main Theorems

The following notations and lemmas are used to establish the main theorems:

$$R_1(X) = \left\{ T \in B(X) \mid \sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty \right\},$$
$$F(X) = \{T \in B(X) \mid NJ(T) = X\}.$$

Lemma 3.1. [2] *Let Y be a real or complex Banach space, and let $g_1, g_2, \cdots$ be unit functionals in the dual space Y^*. For each $m \in \mathbb{N}$, suppose $\beta_m \geq 0$ satisfies the condition $\sum_{m=1}^{\infty} \beta_m < 1$. Then, there exists an element $y \in Y$ with $\|y\| = 1$ such that $|\langle y, g_m \rangle| \geq \beta_m$ for all m.*

Lemma 3.2. [3] *Let Z be a complex Hilbert space, and let $h_1, h_2, \cdots$ be unit functionals in the dual space Z^*. For each $k \in \mathbb{N}$, suppose $\eta_k \geq 0$ satisfies the condition $\sum_{k=1}^{\infty} \eta_k^2 < 1$. Then, there exists an element $z \in Z$ such that $\|z\| = 1$ and $|\langle z, h_k \rangle| \geq \eta_k$ for all k.*

Theorem 3.3. *(First Main Theorem) $T \in R_1(X)$ implies $T \notin F(X)$.*

Proof. Let γ_n be a sequence of positive real numbers satisfying the condition that $\gamma_n \to \infty$ as $n \to \infty$, and $\sum_{n=1}^{\infty} \frac{\gamma_n}{\|T^n\|} < \infty$.
Define

$$S = \sum_{n=1}^{\infty} \frac{\gamma_n}{\|T^n\|}.$$

Now, consider the sequence of coefficients

$$\lambda_n = \frac{\gamma_n}{(S+1)\|T^n\|},$$

which satisfies

$$\sum_{n=1}^{\infty} \lambda_n < 1 \quad \text{and} \quad \lambda_n \|T^n\| \to \infty.$$

Let $T^\star \in B(X^\star)$. For every $n \in \mathbb{N}$, choose $h_n \in X^\star$ such that

$$\|h_n\| < 1 \quad \text{and} \quad \|(T^n)^\star h_n\| \geq \frac{1}{2}\|(T^n)^\star\| = \frac{1}{2}\|T^n\|.$$

Define $p_n \in X^*$ by

$$p_n = \frac{(T^n)^\star h_n}{\|(T^n)^\star h_n\|}.$$

By Lemma 3.1, there exists an element $x \in X$ with norm equal to one such that $|\langle x, p_n \rangle| \geq \lambda_n$ for all n.

Therefore,

$$\begin{aligned}\|T^n x\| &\geq |\langle T^n x, h_n\rangle| \\ &= |\langle x, (T^n)^\star h_n\rangle| \\ &= |\langle x, p_n\rangle| \|(T^n)^\star h_n\| \\ &\geq \frac{\lambda_n}{2}\|T^n\| \to \infty \text{ as } n \to \infty.\end{aligned}$$

It follows that $x \in X \setminus NJ(T)$. Hence, $T \notin F(X)$. □

Corollary 3.4. *If X is a complex Hilbert space and*

$$R_2(X) = \left\{ T \in B(X) \mid \sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty \right\},$$

then $T \in R_2(X)$ implies $T \notin F(X)$.

Proof. The proof follows a similar approach to Theorem 3.3 by defining $S = \sum_{n=1}^{\infty} \frac{\gamma_n}{\|T^n\|^2}$, ensuring that $\sum_{n=1}^{\infty} \lambda_n^2 < 1$. Lemma 3.2 then ensures the existence of an element $x \in X$ with norm equal to one such that $|\langle x, p_n\rangle| \geq \lambda_n$ for all n. The divergence of $\|T^n x\|$ can be shown analogously to Theorem 3.3. □

Theorem 3.5. *(Second Main Theorem) If $X = \ell^p, 1 \leq p < \infty$, then $F(X) \neq \phi$.*

Proof. Take $(r_k)_{k=1}^{\infty}$ as the standard basis of X and define T as follows:

$$\mathrm{T}(r_k) = \begin{cases} \left(\dfrac{k}{k-1}\right)^{1/p} r_{k-1}, & \text{if } k > 1, \\ \qquad 0, & \text{if } k = 1. \end{cases}$$

Then,

$$\begin{aligned}\|T^n\| &= \prod_{k>1}^{n+1} \left(\frac{k}{k-1}\right)^{1/p} \\ &= (n+1)^{1/p}, \text{ for all } n.\end{aligned}$$

Suppose $\exists$ an $x = \sum_{k=1}^{\infty} a_k r_k \in \ell^p$ such that $\|x\| = \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \leq 1$, but $\|T^n x\| \to \infty$ as $n \to \infty$.
Thus,

$$\frac{1}{n} \sum_{j=n}^{2n-1} \|T^j x\|^p \to \infty \text{ as } n \to \infty.$$

It follows that we have

$$\begin{aligned}\|T^j x\|^p &= \left\| \sum_{k>j}^{\infty} \left(\frac{k}{k-j} \right)^{1/p} a_k r_{k-j} \right\|^p \\ &\leq \sum_{k>j}^{k=2j} |a_k|^p \frac{k}{k-j} + \sum_{k>2j}^{\infty} |a_k|^p \frac{k}{k-j}.\end{aligned}$$

The subsequent term in the above expression can be bounded by $2\|x\|^p \leq 2$, as $\dfrac{k}{k-j} < 2$ holds for $k > 2j$.
Therefore,

$$\begin{aligned}\sum_{j=n}^{2n-1} \|T^j x\|^p &\leq 2n + \sum_{j=n}^{2n-1} \sum_{k=j+1}^{2j} |a_k|^p \frac{k}{k-j} \\ &\leq 2n + \sum_{k=n+1}^{4n} |a_k|^p \sum_{i=1}^{k} \frac{k}{i} \\ &\leq 2n + \sum_{k=n+1}^{4n} |a_k|^p 4n(1+\log 4n)\end{aligned}$$

so that

$$2 + 4(1+\log 4n) \sum_{k>n}^{4n} |a_k|^p \geq \frac{1}{n} \sum_{j=n}^{2n-1} \|T^j x\|^p \to \infty.$$

Thus, for all sufficiently large n, the L.H.S. exceeds 6.
i.e., if

$$Z_n = \sum_{k=n+1}^{4n} |a_k|^p, \text{ then } Z_n \geq \frac{1}{1+\log 4n}.$$

This results in a contradiction since, for n of this type, we have

$$\begin{aligned}1 = \sum_{k\geq 1}^{\infty} |a_k|^p &\geq Z_n + Z_{4.4n} + Z_{4.4.4n} + \cdots \\ &\geq \sum_{j=1}^{\infty} \frac{1}{1+\log 4^j n} \\ &= \sum_{j=1}^{\infty} \frac{1}{1+\log n + j\log 4} \\ &= \infty.\end{aligned}$$

This implies that there exists an operator $T \in B(X)$ for which $NJ(T) = X$, and hence $F(X) \neq \phi$. $\square$

Acknowledgement. The authors sincerely thank APJ Abdul Kalam Technological University, Government Polytechnic College, Ezhukone, Kollam, Muthoot Institute of Technology and Science and Rajagiri School of Engineering and Technology, Kochi, for providing the necessary facilities to support this research work.

References

1. John Milnor (2006). Dynamics in One Complex Variable, 3rd edition, Princeton University Press, Princeton and Oxford.
2. Keith Ball (1991). The Plank Problem for Symmetric bodies, Invent. math. 104, 535-543. https://doi.org/10.1007/BF01245089
3. Keith M. Ball (2001). The Complex Plank Problem, Bull. London. Math. Soc 33, 433-442. https://doi.org/10.1017/S002460930100813X
4. Lennart Carleson (1993). Theodore W. Gamelin, Complex Dynamics, Springer-Verlag New York, Inc.
5. Paul Blanchard (1984). Complex Analytic Dynamics on the Riemann Sphere, Bulletin of the American Mathematical Society, Volume 11, Number 1. https://doi.org/10.1090/S0273-0979-1984-15240-6
6. R. L. Devaney (1989). An introduction to Chaotic Dynamical Systems, 2nd edition, Addison-Wesley, Redwood City, CA.
7. R. L. Devaney (1992). A first course in chaotic dynamical systems - theory and experiment, Perseus Books Publishing, L. L. C.

[1] Government Polytechnic College, Ezhukone, Kollam, Rajagiri School of Engineering and Technology, Kochi, APJ Abdul Kalam Technological University, Kerala, India

Email address: sanoojb.123@gmail.com

[2] Muthoot Institute of Technology and Science, APJ Abdul Kalam Technological University, Kerala, India

Email address: vinodkumarpb@mgits.ac.in