

MATRIX EQUATIONS WITH DIAGONALIZABLE COEFFICIENTS

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ABSTRACT. Let A and $\tilde{A}$ be diagonalizable matrices. The paper deals with the equation

$$\sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} A^j X \tilde{A}^k = M,$$

where X should be found, c_{jk} are numbers and M is a given matrix. A solution of this equation is represented as a function of two non-commuting matrices. We establish a norm estimate for a function of A and $\tilde{A}$ defined on their spectra. That estimate gives us a norm estimate for X . In addition, we suggest a new bound for the similarity condition number of a diagonalizable matrix.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $\mathbb{C}^n$ be a Euclidean space with the scalar product $(\cdot, \cdot)$, the Euclidean norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$ and the unit matrix I . For an $n \times n$ -matrix A , $\sigma(A)$ denotes the spectrum of A , A^* is the adjoint to A , $\|A\|$ is the operator (spectral) norm; $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A : $N_2^2(A) = \text{Trace } A^* A$.

In the sequel A and $\tilde{A}$ are diagonalizable $n \times n$ -matrices whose eigenvalues are λ_k and $\tilde{\lambda}_k$, $k = 1, \dots, n$, respectively. So there are biorthogonal sequences $\{u_k\}$ and $\{v_k\}$: $(v_j, u_k) = 0$ ($j \neq k$), $(v_j, u_j) = 1$ ($j, k = 1, \dots, n$), such that

$$A = \sum_{k=1}^n \lambda_k Q_k = \sum_{k=1}^n \lambda_k (\cdot, u_k) v_k, \quad (1.1)$$

where $Q_k = (\cdot, u_k) v_k$ ($k = 1, \dots, n$) are the eigenprojections. Similarly,

$$\tilde{A} = \sum_{k=1}^n \tilde{\lambda}_k \tilde{Q}_k,$$

where $\tilde{Q}_k$ are the eigenprojections of $\tilde{A}$. Our main object in this paper is the equation

$$\sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} A^j X \tilde{A}^k = M, \quad (1.2)$$

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where X should be found, M is a given $n \times n$ -matrix and c_{jk} are numbers. These equations arose in numerous applications, [7, 10]. Recently matrix linear and nonlinear equations attracted the attention of many mathematicians, [6, 8, 9, 11, 12, 13]. For equation (1.2) mainly numerical methods for the solutions of were developed, [6, 9, 13]. In the paper [1], reflexive and anti-reflexive solutions of a linear matrix equation were explored. No estimates were established for solutions of equation (1.2). In the present paper we suggest a norm estimate for X .

For simplicity we assume that all the eigenvalues are different:

$$\delta(A) := \min_{j,k=1,\dots,n; k \neq j} \frac{|\lambda_j - \lambda_k|}{2} > 0 \text{ and } \delta(\tilde{A}) = \min_{j,k=1,\dots,n; k \neq j} \frac{|\tilde{\lambda}_j - \tilde{\lambda}_k|}{2} > 0. \quad (1.3)$$

The general case of diagonalizable matrices can be considered similarly. The following quantity plays an essential role hereafter:

$$g(A) := [N_2^2(A) - \sum_{k=1}^n |\lambda_k|^2]^{1/2}.$$

$g(A)$ enjoys the following properties: $g^2(A) \leq N_2^2(A) - |\text{Trace} A^2|$,

$$g^2(A) \leq 2N_2^2(A_I) \quad (A_I = (A - A^*)/2i) \text{ and } g(A) = g(Ae^{it} + zI) \quad (z \in \mathbb{C}, t \in \mathbb{R}), \quad (1.4)$$

[3, Section 2.1]. If A is normal, then $g(A) = 0$. If A_1 and A_2 are commuting, then $g(A_1 + A_2) \leq g(A_1) + g(A_2)$. By the inequality between the geometric and arithmetic mean values,

$$\left(\frac{1}{n} \sum_{k=1}^n |\lambda_k|^2\right)^n \geq \left(\prod_{k=1}^n |\lambda_k|\right)^2. \text{ So } g^2(A) \leq N_2^2(A) - n(\det A)^{2/n}.$$

Put

$$p(z, w) = \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} z^j w^k \quad (z, w \in \mathbb{C}) \text{ and } \eta(A) := \sum_{k=1}^{n-1} \frac{g^k(A)}{\delta^k(A) \sqrt{k!}}.$$

Now we are in a position to formulate our main result.

Theorem 1.1. *Let the conditions (1.3) and*

$$p(\lambda_k, \tilde{\lambda}_j) \neq 0 \quad (j, k = 1, \dots, n) \quad (1.5)$$

hold. Then equation (1.2) has a unique solution X which satisfies the inequality

$$N_2(X) \leq \frac{N_2(M) \gamma(A) \gamma(\tilde{A})}{\min_{j,k} |p(\lambda_j, \tilde{\lambda}_k)|}, \quad (1.6)$$

where

$$\gamma(A) := \begin{cases} (1 + \eta(A))n & \text{if } \eta(A) \geq 1, \\ (1 + \eta(A)) \min \{n, \left(1 + \frac{2\sqrt{n}\eta(A)}{1-\eta(A)}\right)^2\} & \text{if } \eta(A) < 1 \end{cases}.$$

This theorem is proved in the next section. It is sharp: if A and $\tilde{A}$ are normal matrices, then $\eta(A) = \eta(\tilde{A}) = 0$ and $\gamma(A) = \gamma(\tilde{A}) = 1$. One can check that in this case (1.6) is attained. Note that in [5], a norm estimate for solutions of (1.2) with

arbitrary matrices A and $\tilde{A}$ is derived, but that estimate requires that $p(z, w)$ does not have zeros on the convex hull of the spectra of A and $\tilde{A}$.

2. PROOF OF THEOREM 1.1

Since A and $\tilde{A}$ are diagonalizable, there are normal operators

$$D = \sum_{k=1}^n \lambda_k P_k, \tilde{D} = \sum_{k=1}^n \tilde{\lambda}_k \tilde{P}_k$$

and invertible operators T and $\tilde{T}$, such that

$$A = TDT^{-1} \quad (2.1)$$

and $\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$. Here $P_k, \tilde{P}_k$ are the orthogonal eigenprojections of D and $\tilde{D}$, respectively. Put $\kappa(A, T) = \|T\|\|T^{-1}\|$. Below we suggest a bound for $\kappa(A, T)$. Our main tool is the matrix function

$$F(f, A, M, \tilde{A}) = \sum_{j,k=1}^n f(\lambda_k, \tilde{\lambda}_k) Q_k M \tilde{Q}_j.$$

Clearly, (1.2) can be rewritten as $F(p, A, X, \tilde{A}) = M$.

Lemma 2.1. *Suppose A and $\tilde{A}$ are diagonalizable and condition (1.5) holds. Then equation (1.2) has a unique solution X which can be represented as*

$$X = F\left(\frac{1}{p}, A, M, \tilde{A}\right).$$

Proof. From (1.2) for all integers $r, l = 1, \dots, n$ it follows that

$$Q_r \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} A^j X \tilde{A}^k \tilde{Q}_l = Q_r M \tilde{Q}_l;$$

but $Q_r A^j X \tilde{A}^k \tilde{Q}_l = \lambda_r^j Q_r X \tilde{\lambda}_l^k \tilde{Q}_l$. Thus

$$\sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} \lambda_r^j \tilde{\lambda}_l^k Q_r X \tilde{Q}_l = Q_r M \tilde{Q}_l,$$

or $p(\lambda_r, \tilde{\lambda}_l) Q_r X \tilde{Q}_l = Q_r M \tilde{Q}_l$. Hence $Q_r X \tilde{Q}_l = \frac{1}{p(\lambda_r, \tilde{\lambda}_l)} Q_r M \tilde{Q}_l$. So

$$\sum_{r,l=1}^n Q_r X \tilde{Q}_l = \sum_{r,l=1}^n \frac{1}{p(\lambda_r, \tilde{\lambda}_l)} Q_r M \tilde{Q}_l.$$

But $\sum_{r=1}^n Q_r = I, \sum_{l=1}^n \tilde{Q}_l = I$ and therefore,

$$X = \sum_{r,l=1}^n \frac{1}{p(\lambda_r, \tilde{\lambda}_l)} Q_r M \tilde{Q}_l,$$

as claimed. $\square$

We need the following two simple lemma.

Lemma 2.2. *Let $\{e_k\}$ and $\{\tilde{e}_k\}$ be orthogonal normal bases in $\mathbb{C}^n$ and K be a linear operator in $\mathbb{C}^n$. Then*

$$N_2^2(K) = \sum_{j,k=1}^n |(Ke_j, \tilde{e}_k)|^2.$$

Proof. We have

$$Ke_j = \sum_{k=1}^n (Ke_j, \tilde{e}_k) \tilde{e}_k, \quad \|Ke_j\|^2 = \sum_{k=1}^n |(Ke_j, \tilde{e}_k)|^2$$

and

$$N_2^2(K) = \sum_{j=1}^n \|Ke_j\|^2 = \sum_{j,k=1}^n |(Ke_j, \tilde{e}_k)|^2,$$

as claimed. $\square$

Lemma 2.3. *Let $\{e_k\}$ and $\{\tilde{e}_k\}$ be orthogonal normal bases in $\mathbb{C}^n$, $P_k = (., e_k)e_k$, $\tilde{P}_k = (., \tilde{e}_k)\tilde{e}_k$ and*

$$L = \sum_{j,k=1}^n a_{jk} P_j K \tilde{P}_k$$

where a_{jk} are numbers and K is a linear operator in $\mathbb{C}^n$. Then $N_2(L) \leq \max_{j,k} |a_{jk}| N_2(K)$.

Proof. We have

$$(L\tilde{e}_l, e_s) = \sum_{j,k=1}^n a_{jk} (P_j K \tilde{P}_k \tilde{e}_l, e_s) = a_{ls} (K\tilde{e}_l, e_s).$$

So

$$\sum_{s,l=1}^n |(L\tilde{e}_l, e_s)|^2 = \sum_{s,l=1}^n |a_{ls} (K\tilde{e}_l, e_s)|^2 \leq \max_{j,k} |a_{jk}|^2 \sum_{s,l=1}^n |(K\tilde{e}_l, e_s)|^2.$$

Now the previous lemma completes the proof. $\square$

According to (1.1) and (2.1), we have $Q_j = TP_jT^{-1}$, $\tilde{Q}_j = \tilde{T}\tilde{P}_j\tilde{T}^{-1}$ and thus,

$$F(f, A, M, \tilde{A}) = T \sum_{j,k=1}^n f(\lambda_j, \tilde{\lambda}_k) P_j T^{-1} M \tilde{T} \tilde{P}_k \tilde{T}^{-1}.$$

Hence

$$F(f, A, M, \tilde{A}) = TF(f, D, \hat{M}, \tilde{D})T^{-1} \quad (2.2)$$

where $\hat{M} = T^{-1}M\tilde{T}$. Since $N_2(\hat{M}) = N_2(T^{-1}M\tilde{T}) \leq \|T^{-1}\| \|\tilde{T}\| N_2(M)$, the previous lemma yields

$$N_2(F(f, D, \hat{M}, \tilde{D})) \leq \max_{j,k} |f(\lambda_j, \tilde{\lambda}_k)| N_2(M) \|T^{-1}\| \|\tilde{T}\|.$$

Now (2.2) implies

Lemma 2.4. *Let A and $\tilde{A}$ be diagonalizable. Then*

$$N_2(F(f, A, M, \tilde{A})) \leq N_2(M) \kappa(A, T) \kappa(\tilde{A}, \tilde{T}) \max_{j,k} |f(\lambda_j, \tilde{\lambda}_k)|.$$

The previous lemma and Lemma 2.1 imply

Corollary 2.5. *Let A and $\tilde{A}$ be diagonalizable and (1.5) hold. Then*

$$N_2(X) \leq \frac{N_2(M) \kappa(A, T) \kappa(\tilde{A}, \tilde{T})}{\min_{j,k} |p(\lambda_j, \tilde{\lambda}_k)|}.$$

Now we are going to estimate $\kappa(A, T)$. In [4, [4, Corollary 3.6], for T defined below by (2.5), the inequality

$$\kappa(A, T) \leq n(\eta(A) + 1) \quad (2.3)$$

has been derived. That inequality is not sharp: if A is a normal matrix, then it gives $\kappa(A, T) \leq n$, but $\kappa(A, T) = 1$ in this case. To improve (2.3) we need the following lemma [5, Lemma 3.4].

Lemma 2.6. *Let A be defined by (1.1) and*

$$D = \sum_{k=1}^n \lambda_k(\cdot, e_k) e_k, \quad (2.4)$$

where $\{e_k\}$ is an orthonormal basis. Then the operator

$$T = \sum_{k=1}^n (\cdot, e_k) v_k \quad (2.5)$$

has the inverse one defined by

$$T^{-1} = \sum_{k=1}^n (\cdot, u_k) e_k, \quad (2.6)$$

and (2.1) holds.

Without loss of generality, take

$$\|u_k\| = \|v_k\|. \quad (2.7)$$

This leads to the equality $\|T^{-1}\| = \|T\|$. In the rest of this section $\{e_k\}_{k=1}^n$ is the Schur orthonormal basis of the triangular representation of A . So in that basis $A = D + V$, where V is an upper triangular nilpotent matrix and D is a diagonal matrix. Assume that

$$\eta(A) < 1 \quad (2.8)$$

and put

$$\hat{\gamma}(A) := (\eta(A) + 1) \left[\frac{2\sqrt{n}\eta(A)}{1 - \eta(A)} + 1 \right]^2.$$

Lemma 2.7. *Let A and T be defined by (1.1) and (2.5), respectively, and conditions (1.3) and (2.8) hold. Then $\kappa(A, T) \leq \hat{\gamma}(A)$.*

To prove Lemma 2.7 we need the following result proved in [2].

Lemma 2.8. *Let L_1 and L_2 be projections satisfying the condition $r := \|L_1 - L_2\| < 1$. Then for any eigenvector f_1 of L_1 with $\|f_1\| = 1$ and $L_1 f_1 = f_1$, there exists an eigenvector f_2 of L_2 with $\|f_2\| = 1$ and $L_2 f_2 = f_2$, such that*

$$\|f_1 - f_2\| \leq \frac{2r}{1-r}.$$

For a fixed index $j \leq n$, let P_j be the eigenprojection of D , and Q_j the eigenprojection of A corresponding to the same eigenvalue λ_j . The following inequality is proved in [4, inequality (3.3)]:

$$\|Q_j - P_j\| \leq \eta(A) \quad (j = 1, \dots, n). \quad (2.9)$$

Inequality (2.9) and the previous lemma yield.

Corollary 2.9. *Assume that relations (1.1), (1.3) and (2.8) hold. Then*

$$\left\| \frac{v_k}{\|v_k\|} - e_j \right\| \leq \psi(A) \text{ and } \left\| \frac{u_k}{\|u_k\|} - e_k \right\| \leq \psi(A) \quad (k = 1, \dots, n), \quad (2.10)$$

where

$$\psi(A) = \frac{2\eta(A)}{1 - \eta(A)}.$$

Proof of Lemma 2.7: By (2.6) we have

$$\begin{aligned} \|T^{-1}x\| &= \left[\sum_{k=1}^n |(x, u_k)|^2 \right]^{1/2} = \left[\sum_{k=1}^n |(x, \|u_k\|e_k) + (x, u_k - \|u_k\|e_k)|^2 \right]^{1/2} \\ &\leq \left[\sum_{k=1}^n \|u_k\|^2 \left| \left(x, \frac{u_k}{\|u_k\|} - e_k \right) \right|^2 \right]^{1/2} + \left[\sum_{k=1}^n \|u_k\|^2 |(x, e_k)|^2 \right]^{1/2} \quad (x \in \mathbb{C}^n). \end{aligned}$$

Hence, due to (2.10), $\|T^{-1}x\| \leq \max_k \|u_k\| \|x\| (\psi(A)\sqrt{n} + 1)$ ($x \in \mathbb{C}^n$). Thanks to Corollary 3.3 from [4] [4] we have $\max_k \|u_k\|^2 \leq 1 + \eta(A)$. Thus, $\|T^{-1}\|^2 \leq \hat{\gamma}(A)$. Similarly, according to (2.7), $\|T\|^2 \leq \hat{\gamma}(A)$. We thus get the required result. $\square$

Combining Lemma 2.7 with (2.3), we get

Corollary 2.10. *Let A and T be defined by (1.1) and (2.5), respectively, and condition (1.3) holds. Then*

$$\kappa(A, T) \leq \gamma(A). \quad (2.11)$$

Inequality (2.11) is sharp: if A is normal, then $g(A) = 0$. Therefore $\eta(A) = 0$ and $\gamma(A) = 1$. Thus we obtain the equality $\kappa(A, T) = 1$.

Proof of Theorem 1.1: The required result is due (2.11) and Corollary 2.5. $\square$

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