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ELLIPTIC CURVES OVER IMAGINARY BIQUADRATIC NUMBER FIELDS OF CLASS NUMBER ONE

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Abstract. This research investigates elliptic curves within an imaginary biquadratic number field $\aleph$ of class number one. Our study focuses on the existence of torsion subgroups isomorphic to G_{11}, G_{13}, G_{14}, G_{15}, G_{16}, G_{18}, $G_2 \oplus G_{10}$, and $G_2 \oplus G_{12}$ over $\aleph$.

1. Introduction and preliminaries

Let $\aleph$ be a number field field of degree d over $\mathbb{Q}$. Unless otherwise specified, E represents an EC (Elliptic curve). In this article, we denote by G_m a cyclic subgroup of order m, by $X_1(N, M)$ a modular curve and J_1 the Jacobian variant of X_1. $E(\aleph)$ the group of $\aleph$-rational points on E and $E(\aleph)_{tors}$ the set of torsion subgroups of E on $\aleph$. The field of ECs [3, 5, 6, 7, 21], with its deep links to algebraic geometry and number theory, has long captivated mathematicians, providing a rich partition of mathematical elegance and complexity. In this extraordinary field, our research takes a unique approach. The specific torsion subgroups chosen for this work are motivated by their relevance to open problems, their links to questions relating to the Mordell-Weil group, and uniform limitation. Similarly, these torsion groups provide essential data for understanding isogenies and moduli spaces in higher-dimensional abelian varieties.

From the Mordell-Weil theorem, we know that the group of $\aleph$-rational points on E forms a finitely generated abelian group.

$$E(\aleph) \cong \mathbb{Z}^r \oplus E(\aleph)_{\text{tors}} .$$

In particular, Kubert [14] and Mazur [15] determined $E(\aleph)_{tors}$ for $\aleph = \mathbb{Q}$ and EC defined on $\mathbb{Q}$.

$$G_n \; for \; 1 \leq n \leq 12, \; n \neq 11,$$
$$G_2 \oplus G_{2n} \; for \; 1 \leq n \leq 4.$$

Note that, $E(\aleph)_{tors}$ has been determined just for some other values of d. In particular, the set $E(\aleph)_{tors}$ has been classified by Kamienny, [11], Kenku and Momose, [13], for a quadratic number field.

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$$G_n \; for \; 1 \leq n \leq 18, \; n \neq 17,$$
$$G_2 \oplus G_{2n} \; for \; 1 \leq n \leq 6,$$
$$G_3 \oplus G_{3n} \; for \; 1 \leq n \leq 2,$$
$$G_4 \oplus G_4.$$

Remark 1.1. In the above result, we have a classification of possible torsion structures when the study varies over all number fields of degree d. However, for a fixed number field $\aleph$, there are generally no specific, well-defined torsion structures on $\aleph$. Najman [16, 17] investigated this problem for two imaginary quadratic number fields of class number one $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$.

- If $\aleph = \mathbb{Q}(i)$ and $E/\mathbb{Q}(i)$ an EC.

Then $E(\aleph)_{tors}$ is either one of the following groups

$$G_n \; for \; 1 \leq n \leq 12, \; n \neq 11,$$
$$G_2 \oplus G_{2n} \; for \; 1 \leq n \leq 4,$$
$$G_4 \oplus G_4.$$

- If $\aleph = \mathbb{Q}(\sqrt{-3})$ and $E/\mathbb{Q}(\sqrt{-3})$ an EC.

Then $E(\aleph)_{tors}$ is either one of the following groups

$$G_n \; for \; 1 \leq n \leq 12, \; n \neq 11,$$
$$G_2 \oplus G_{2n} \; for \; 1 \leq n \leq 4,$$
$$G_3 \oplus G_3,$$
$$G_3 \oplus G_6.$$

For the rest of the list of imaginary quadratic number fields of class number one, the torsion set has been studied by Sarma and Saikia in [19].
In the case of quartic number fields, Jeon, Kim and Park [10] determined which $G_n \oplus G_m$ groups appear infinitely often as $E(\aleph)_{tors}$ torsion groups when $\aleph$ varies over all quartic number fields and E on all ECs defined on $\aleph$

$$G_n \; for \; n = 1, \ldots, 18, 20, 21, 22, 24,$$
$$G_2 \oplus G_{2n} \; for \; n = 1, \ldots, 9,$$
$$G_3 \oplus G_{3n} \; for \; n_3 = 1, 2, 3,$$
$$G_4 \oplus G_{4n} \; for \; n = 1, 2,$$
$$G_5 \oplus G_5,$$
$$G_6 \oplus G_6.$$

In relation to imaginary quadratic number fields of class number one. Gauss demonstrated that $\mathbb{Q}(\sqrt{d_1})$, has a class number one for $d_1 \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$ [8]. Heegner [9], Baker [2], and Stark [20] all made substantial contributions to proving that this list is complete. This is exactly the complete list of imaginary quadratic number fields of class number one on which Najman and then Sarma studied ECs. For the imaginary biquadratic number fields of class number one, Brown and Parry have determined a complete list [4].

There are exactly 47 such fields noted by $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $\{d_1, d_2\}$ is one of the following sets.

$\{-1,2\}$	$\{2,-3\}$	$\{-3,5\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,3\}$	$\{2,-11\}$	$\{-3,-7\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,5\}$	$\{-2,-3\}$	$\{-3,-11\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,7\}$	$\{-2,5\}$	$\{-3,17\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,11\}$	$\{-2,-7\}$	$\{-3,-19\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,13\}$	$\{-2,-11\}$	$\{-3,41\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,19\}$	$\{-2,-19\}$	$\{-3,-43\}$	$\{-7,-163\}$	$\{-43,-67\}$
$\{-1,37\}$	$\{-2,29\}$	$\{-3,-67\}$	$\{-43,-163\}$	
$\{-1,43\}$	$\{-2,-43\}$	$\{-3,89\}$	$\{-67,-163\}$	
$\{-1,67\}$	$\{-2,-67\}$	$\{-3,-163\}$		
$\{-1,163\}$				

In this research, we investigate elliptic curves within an imaginary biquadratic number field $\aleph$ of class number one. Our study focuses on the existence of torsion subgroups isomorphic to G_{11}, G_{13}, G_{14}, G_{15}, G_{16}, G_{18}, $G_2 \oplus G_{10}$, and $G_2 \oplus G_{12}$ over $\aleph$.

2. $X_1(11)$, $X_1(14)$, $X_1(15)$, $X_1(2,10)$, and $X_1(2,12)$ over Imaginary biquadratic number fields of class number one

To know if there exists a curve with a torsion subgroup $G_N \oplus G_M$ on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, we need to determine if $X_1(N, M)$ has a $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$-rational point which is not a cusp. In this section, we will study the existence of ECs whose torsion contains a subgroup isomorphic to one of the following groups: G_{11}, G_{14}, G_{15}, $G_2 \oplus G_{10}$, and $G_2 \oplus G_{12}$ over an Imaginary biquadratic number field of class number one.

2.1. Odd torsion subgroup.

We would like to point out that the present study of ECs does not involve the application of Lemma 2.1 and Corollary 2.2. This is because, in general, these ECs are not defined on a subfield of the biquadratic number field. However, we use these mathematical principles specifically for certain modular curves. Specifically, we use them for $X_1(11)$, $X_1(14)$, $X_1(15)$, $X_1(2,10)$, and $X_1(2,12)$.

Lemma 2.1. *Let $E/\mathbb{Q}(\sqrt{d_1})$ be an EC. So, there exists a pair of homomorphisms*

$$E(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \xrightarrow{\Psi_1} E(\mathbb{Q}(\sqrt{d_1})) \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1}))$$

and,

$$E(\mathbb{Q}(\sqrt{d_1})) \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1})) \xrightarrow{\Psi_2} E(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$$

such that

(1) $\Psi_2 \circ \Psi_1 = [2]$,
(2) $\Psi_1 \circ \Psi_2 = [2] \oplus [2]$,
(3) $ker(\Psi_1) \subset E(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))[2]$,
(4) $ker(\Psi_2) \subset E(\mathbb{Q}(\sqrt{d_1}))[2] \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1}))[2]$,

(5) *$coker(\Psi_1)$ and $coker(\Psi_2)$ are groups where every non-zero element has order* 2.

Proof. We have $\left|Gal(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})/\mathbb{Q}(\sqrt{d_1}))\right| = 2$, let σ be the non-trivial element of $Gal(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})/\mathbb{Q}(\sqrt{d_1}))$ and $\alpha \in \mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$. Set, for any $\Theta = (t,s)$

$$\sigma\Theta = (\sigma(t),\sigma(s)) \text{ and } \Gamma(\Theta,\alpha) = (t,\alpha s).$$

Recall the canonical $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$-isomorphisms between E and its d_2-twist E^{d_2}

$$E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})) \overset{\Gamma(.,1/\sqrt{d_2})}{\longrightarrow} E^{d_2}(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})) \overset{\Gamma(.,-\sqrt{d_2})}{\longrightarrow} E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$$

Here we want to construct elements of $E(\mathbb{Q}(\sqrt{d_1}))$ and $E^{d_2}(\mathbb{Q}(\sqrt{d_1}))$ respectively, from an element Θ of $E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$.
So, let $\Theta \in E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$. As $\sigma(\Theta+\sigma\Theta) = \Theta+\sigma\Theta$, we have $\Theta+\sigma\Theta \in E(\mathbb{Q}(\sqrt{d_1}))$.
Consider now $\Theta-\sigma\Theta$. We have that, $\begin{cases} \sigma(\Theta-\sigma\Theta) = \mathcal{O}, \\ \sigma(\Theta-\sigma\Theta) = -(\Theta-\sigma\Theta) \neq \mathcal{O}. \end{cases}$
It follows that, if $\sigma(\Theta-\sigma\Theta) = \mathcal{O}$, we have $\Theta = \sigma\Theta$ and $\Theta \in E(\mathbb{Q}(\sqrt{d_1}))$.
On the other hand if $\sigma(\Theta-\sigma\Theta) = -(\Theta-\sigma\Theta) \neq \mathcal{O}$ we set

$$\Gamma(\Theta-\sigma\Theta, 1/\sqrt{d_2}) = (t_{\Theta-\sigma\Theta}, a) \in E^{d_2}(\mathbb{Q}(\sqrt{d_1})),$$

where $a \in \mathbb{Q}(\sqrt{d_1})$ and $a\sqrt{d_2} = s_{\Theta-\sigma\Theta}$.
So we define the homomorphism Ψ_1 as follows

$$\begin{aligned} \Psi_1 : E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})) &\longrightarrow E(\mathbb{Q}(\sqrt{d_1})) \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1})) \\ \Theta &\longmapsto (\Theta+\sigma\Theta, \Gamma(\Theta-\sigma\Theta, 1/\sqrt{d_2})) \end{aligned}$$

For Ψ_2, let $M = (a,b) \in E^{d_2}(\mathbb{Q}(\sqrt{d_1}))$. Clearly

$$\Gamma(M,\sqrt{d_2}) = (a, b\sqrt{d_2}) \in E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$$

and we define then

$$\begin{aligned} \Psi_2 : E(\mathbb{Q}(\sqrt{d_1})) \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1})) &\longrightarrow E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})) \\ (\Theta, M) &\longmapsto \Theta + \Gamma(M,\sqrt{d_2}) \end{aligned}$$

which is a homomorphism over $E(\mathbb{Q}(\sqrt{d_1})) \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1}))$.

Let $\Theta \in Ker(\Psi_1)$

$$\Psi_1(\Theta) = (\Theta+\beta\Theta, \Gamma(\Theta-\beta\Theta, \frac{1}{\sqrt{d_2}})) = (O,O)$$

So,

$$\begin{cases} \Theta+\beta\Theta = O \\ \Gamma(\Theta-\beta\Theta, \frac{1}{\sqrt{d_2}}) = O \end{cases} \Longrightarrow \begin{cases} \beta\Theta = -\Theta \\ 2\Theta = O \end{cases} \Longrightarrow \Theta \in E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))[2].$$

Thus, $ker(\Psi_1) \subset E(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))[2]$.

Let $\Theta \in E(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$

$$\begin{aligned}\Psi_2 \circ \Psi_1(\Theta) &= \Psi_2(\Theta + \beta\Theta, \Gamma(\Theta - \beta\Theta, \frac{1}{\sqrt{d_2}})) \\ &= \Theta + \beta\Theta + \underbrace{\Gamma(\Gamma(\Theta - \beta\Theta, \frac{1}{\sqrt{d_2}}), \sqrt{d_2})}_{\Theta - \beta\Theta} \\ &= \Theta + \beta\Theta + \Theta - \beta\Theta \\ &= 2\Theta.\end{aligned}$$

So, $\Psi_2 \circ \Psi_1 = [2]$.

Let $\Theta \in E(\mathbb{Q}(\sqrt{d_1}))$ and $M \in E^{d_2}(\mathbb{Q}(\sqrt{d_1}))$

$$\begin{aligned}\Psi_1 \circ \Psi_2(\Theta, M) =& \Psi_1(\Theta + \Gamma(M, \sqrt{d_2})) \\ =& (\Theta + \Gamma(M, \sqrt{d_2}) + \beta\Theta + \beta\Gamma(M, \sqrt{d_2}), \Gamma(\Theta + \Gamma(M, \sqrt{d_2}) \\ & - \beta\Theta - \beta\Gamma(M, \sqrt{d_2}), \frac{1}{\sqrt{d_2}})) \\ =& (\Theta + \beta\Theta + \underbrace{\Gamma(M, \sqrt{d_2}) + \beta\Gamma(M, \sqrt{d_2})}_{O}, \Gamma(\underbrace{\Theta - \beta\Theta}_{O} + \Gamma(M, \sqrt{d_2}) \\ & - \beta\Gamma(M, \sqrt{d_2}), \frac{1}{\sqrt{d_2}})) \\ =& (\Theta + \beta\Theta, \Gamma(\Gamma(M, \sqrt{d_2}) - \beta\Gamma(M, \sqrt{d_2}), \frac{1}{\sqrt{d_2}})) \\ =& (2\Theta, \Gamma(\Gamma(2M, \sqrt{d_2}), \frac{1}{\sqrt{d_2}})) \\ =& (2\Theta, 2M)\end{aligned}$$

Then, $\Psi_1 \circ \Psi_2 = [2] \oplus [2]$.

Following the same notation as in the previous we have that

$$ker(\Psi_2) \subset ker(\Psi_1 \circ \Psi_2)$$

then $ker(\Psi_2) \subset Ker([2] \circ [2])$. Therefore,

$$ker(\Psi_2) \subset E(\mathbb{Q}(\sqrt{d_1}))[2] \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1}))[2].$$

Since E^{d_2} is an abelian variety over $\mathbb{Q}(\sqrt{d_1})$ and is isomorphic to E over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, then E^{d_2} is a $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})/\mathbb{Q}(\sqrt{d_1})$-form of E. So, with these applications, it turns out that results can be deduced. □

Corollary 2.2. *Let $E/\mathbb{Q}(\sqrt{d_1})$ be an EC. If n is an odd integer, then there exists an isomorphism*

$$E(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))[n] \simeq E(\mathbb{Q}(\sqrt{d_1}))[n] \oplus E^{d_2}(\mathbb{Q}(\sqrt{d_1}))[n].$$

2.2. **The group G_{11}.**

Theorem 2.3. *There are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the following sets, such that G_{11} appear as a torsion subgroup.*

$\{-1,2\}$	$\{2,-3\}$	$\{-3,-7\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,7\}$	$\{2,-11\}$	$\{-3,-11\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-3,17\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,-19\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,19\}$	$\{-2,-7\}$	$\{-3,41\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,43\}$	$\{-2,-11\}$	$\{-3,-43\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,163\}$	$\{-2,-19\}$		$\{-7,-163\}$	$\{-43,-67\}$
	$\{-2,29\}$			$\{-43,-163\}$
	$\{-2,-43\}$			
	$\{-2,-67\}$			

Proof. From [1, 18] we have that an EC with torsion G_{11} are induced by solutions of the equation

$$X_1(11) : s^2 - s = t^3 - t^2,$$

and the equation $t(t-1)(t^5 - 18t^4 + 35t^3 - 16t^2 - 2t + 1) = 0$, give the t-coordinates of the cusp points.
We can reduce the equation of $X_1(11)$ to,

$$s^2 = t^3 - 432t + 8208.$$

Using the 2-descent, we can calculate the rank of $X_1^m(11)$ over all imaginary biquadratic number fields of class number one.

TABLE 1. Rank of $X_1(11)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

{d$_1, d_2$}	Rank	{d$_1, d_2$}	Rank	{d$_1, d_2$}	Rank	{d$_1, d_2$}	Rank
{-1, 2}	2	{2,-3}	2	{-3, 5}	0	{-7, 5}	2
{-1, 3}	0	{2,-11}	1	{-3,-7}	2	{-7,-11}	2
{-1, 5}	0	{-2,-3}	2	{-3,-11}	1	{-7, 13}	2
{-1, 7}	2	{-2, 5}	2	{-3, 17}	2	{-7,-19}	2
{-1, 11}	1	{-2,-7}	2	{-3,-19}	2	{-7,-43}	2
{-1, 13}	2	{-2,-11}	2	{-3, 41}	2	{-7, 61}	2
{-1, 19}	2	{-2,-19}	2	{-3,-43}	2	{-7,-163}	2
{-1, 37}	0	{-2, 29}	2	{-3,-67}	0	{-11, 17}	1
{-1, 43}	2	{-2,-43}	4	{-3, 89}	0	{-11,-19}	2
{-1, 67}	0	{-2,-67}	2	{-3,-163}	0	{-11,-67}	1
{-1, 163}	2	{-19,-67}	2	{-43,-67}	2	{-11,-163}	1
{-67,-163}	0	{-19,-163}	2	{-43,-163}	2		

So, as shown in the Table 1, the rank of $X_1(11)$ is ≥ 1, with the following sets $\{d_1, d_2\}$.

$\{-1,2\}$	$\{2,-3\}$	$\{-3,-7\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,7\}$	$\{2,-11\}$	$\{-3,-11\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-3,17\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,-19\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,19\}$	$\{-2,-7\}$	$\{-3,41\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,43\}$	$\{-2,-11\}$	$\{-3,-43\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,163\}$	$\{-2,-19\}$		$\{-7,-163\}$	$\{-43,-67\}$
	$\{-2,29\}$			$\{-43,-163\}$
	$\{-2,-43\}$			
	$\{-2,-67\}$			

Hence, there are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the previous sets, and G_{11} as a torsion subgroup.
Consider $\{d_1, d_2\}$ in the following cases $\{-1,3\}$, $\{-1,5\}$, $\{-1,37\}$, $\{-1,67\}$, $\{-3,5\}$, $\{-3,-67\}$, $\{-3,89\}$, $\{-3,-163\}$, and $\{-67,-163\}$. In this case, the rank of $X_1(11)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ is 0.

We then compute the torsion subgroup of $X_1(11)^m$ for $m = \pm1, \pm3, \pm5, -15$, $\pm37, \pm67, 89$, -163, 201, -267, 489, 10921. So, we have $X_1(11)(\mathbb{Q}) \simeq G_5$ and $X_1(11)^m \simeq O$ for the other values of m. Thus, the n-torsion of $X_1(11)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$ for an odd integer n, is determined by the above computation and the Lemma 2.1. On the other hand $f(t) = t^3 - 432t + 8208$ is irreducible on $\mathbb{Q}$, and thus on all biquadratic extensions of $\mathbb{Q}$. This confirms that $X_1(11)$ has no 2-torsion on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. It follows that,

$$X_1(11)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_5 \cong \{O, (0,0), (0,1), (1,0), (1,1)\}.$$

We see that all of these torsion points correspond to $t = 0$ or 1, and hence, are cusps of $X_1(11)$. Therefore, G_{11} cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,3\}$, $\{-1,5\}$, $\{-1,37\}$, $\{-1,67\}$, $\{-3,5\}$, $\{-3,-67\}$, $\{-3,89\}$, $\{-3,-163\}$, and $\{-67,-163\}$. □

Corollary 2.4. *G_{22} cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,3\}$, $\{-1,5\}$, $\{-1,37\}$, $\{-1,67\}$, $\{-3,5\}$, $\{-3,-67\}$, $\{-3,89\}$, $\{-3,-163\}$, and $\{-67,-163\}$.*

2.3. **The group G_{14}.**

Theorem 2.5. *There are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the following sets, such that G_{14} appear as a torsion subgroup.*

$\{-1,3\}$	$\{2,-3\}$	$\{-3,-7\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,5\}$	$\{2,-11\}$	$\{-3,-11\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,7\}$	$\{-2,-3\}$	$\{-3,17\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,11\}$	$\{-2,5\}$	$\{-3,41\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,13\}$	$\{-2,-7\}$	$\{-3,-43\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,19\}$	$\{-2,-11\}$	$\{-3,-67\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,37\}$	$\{-2,-19\}$	$\{-3,89\}$	$\{-7,-163\}$	$\{-43,-67\}$
$\{-1,43\}$	$\{-2,29\}$	$\{-3,-163\}$		$\{-43,-163\}$
$\{-1,67\}$	$\{-2,-43\}$			$\{-67,-163\}$
$\{-1,163\}$	$\{-2,-67\}$			

Proof. From [1, 18] we have that an EC with torsion G_{14} are induced by solutions of the equation

$$X_1(14) : s^2 + ts + s = t^3 - t,$$

Whose cusps satisfy the equation

$$t(t-1)(t+1)(t^3 - 9t^2 - t + 1)(t^3 - 2t^2 - t + 1) = 0.$$

We can reduce the equation of $X_1(14)$ to,

$$s^2 = t^3 - 675t + 13662.$$

Note that on $\mathbb{Q}(\sqrt{-7})$ there are already exactly 2 ECs with torsion isomorphic to G_{14}. See Theorem 16 in [12]. Let's move on to the other cases. So, using the 2-descent, we can calculate the rank of $X_1^m(14)$ over all imaginary biquadratic number fields of class number one.

TABLE 2. Rank of $X_1(14)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank
{-1, 2}	0	{2,-3}	1	{-3, 5}	0	{-7, 5}	1
{-1, 3}	1	{2,-11}	1	{-3,-7}	0	{-7,-11}	1
{-1, 5}	1	{-2,-3}	1	{-3,-11}	2	{-7, 13}	1
{-1, 7}	1	{-2, 5}	1	{-3, 17}	2	{-7,-19}	0
{-1, 11}	1	{-2,-7}	1	{-3,-19}	0	{-7,-43}	1
{-1, 13}	1	{-2,-11}	3	{-3, 41}	2	{-7, 61}	1
{-1, 19}	1	{-2,-19}	2	{-3,-43}	2	{-7,-163}	1
{-1, 37}	1	{-2, 29}	1	{-3,-67}	2	{-11, 17}	2
{-1, 43}	1	{-2,-43}	1	{-3, 89}	2	{-11,-19}	2
{-1, 67}	1	{-2,-67}	1	{-3,-163}	2	{-11,-67}	2
{-1, 163}	1	{-19,-67}	2	{-43,-67}	2	{-11,-163}	4
{-67,-163}	2	{-19,-163}	2	{-43,-163}	4		

So, as shown in the Table 2, the rank of $X_1(14)$ is ≥ 1, with the following sets $\{d_1, d_2\}$.

$\{-1,3\}$	$\{2,-3\}$	$\{-3,-11\}$	$\{-11,17\}$
$\{-1,5\}$	$\{2,-11\}$	$\{-3,17\}$	$\{-11,-19\}$
$\{-1,7\}$	$\{-2,-3\}$	$\{-3,41\}$	$\{-11,-67\}$
$\{-1,11\}$	$\{-2,5\}$	$\{-3,-43\}$	$\{-11,-163\}$
$\{-1,13\}$	$\{-2,-7\}$	$\{-3,-67\}$	$\{-19,-67\}$
$\{-1,19\}$	$\{-2,-11\}$	$\{-3,89\}$	$\{-19,-163\}$
$\{-1,37\}$	$\{-2,-19\}$	$\{-3,-163\}$	$\{-43,-67\}$
$\{-1,43\}$	$\{-2,29\}$		$\{-43,-163\}$
$\{-1,67\}$	$\{-2,-43\}$		$\{-67,-163\}$
$\{-1,163\}$	$\{-2,-67\}$		

Hence, there are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the previous sets, such that G_{14} appear as a torsion subgroup.

Consider $\{d_1, d_2\}$ in the following cases $\{-1,2\}$, $\{-3,5\}$ and $\{-3,-19\}$. In this case, the rank of $X_1(14)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ is 0. We then compute the torsion subgroup of $X_1^m(14)$ over $\mathbb{Q}$, for $m = -19, -15, -3, \pm 2, \pm 1,\ 5, 57$. So, we have $X_1(14)(\mathbb{Q}) \simeq G_6$ and $X_1(14)^m \simeq G_2$ for the other values of m. Thus, the n-torsion of $X_1(14)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$ for an odd integer n is determined by the above computation and the Lemma 2.1.

On the other hand $t^3 - 675t + 13662$ is irreducible over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,2\}$, $\{-3,5\}$ and $\{-3,-19\}$. This allows us to see that

$$X_1(14)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_6,\ \textit{for}\ \{-1,2\},\ \{-3,5\}\ \textit{and}\ \{-3,-19\}.$$

We have that all of these torsion points correspond to $t = -1$, 0 or 1, and hence, are cusps of $X_1(14)$. Therefore, G_{14} cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,2\}$, $\{-3,5\}$ and $\{-3,-19\}$. □

Corollary 2.6. *$G_2 \oplus G_{14}$ cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,2\}$, $\{-3,5\}$ and $\{-3,-19\}$. Otherwise it would induce a $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$-rational cyclic subgroup of order 14.*

2.4. The group G_{15}.

Theorem 2.7. *There are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the following sets, such that G_{15} appear as a torsion subgroup.*

$\{-1,3\}$	$\{2,-3\}$	$\{-3,5\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,5\}$	$\{2,-11\}$	$\{-3,-7\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,7\}$	$\{-2,5\}$	$\{-3,-11\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,11\}$	$\{-2,-7\}$	$\{-3,17\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,13\}$	$\{-2,-11\}$	$\{-3,-19\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,37\}$	$\{-2,29\}$	$\{-3,41\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,43\}$	$\{-2,-43\}$	$\{-3,-43\}$	$\{-7,-163\}$	$\{-43,-67\}$
$\{-1,67\}$	$\{-2,-67\}$	$\{-3,-67\}$		$\{-43,-163\}$
$\{-1,163\}$		$\{-3,89\}$		$\{-67,-163\}$
		$\{-3,-163\}$		

Proof. From [1, 18] we have that an EC with torsion G_{15} are induced by solutions of the equation

$$X_1(15) : s^2 + ts + s = t^3 + t^2,$$

Whose cusps satisfy the equation

$$t(t+1)(t^2+t+1)(t^4+3t^3+4t^2+2t+1)(t^4-7t^3-6t^2+2t+1) = 0.$$

We can reduce the equation of $X_1(15)$ to,

$$s^2 = t^3 - 27t + 8694.$$

Note that on $\mathbb{Q}(\sqrt{5})$ there are already exactly 2 ECs with torsion isomorphic to G_{15}. See Theorem 16 in [12]. So, using the 2-descent, we can calculate the rank of $X_1^m(15)$ over all imaginary biquadratic number fields of class number one.

Table 3. Rank of $X_1(15)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank
{-1, 2}	0	{2,-3}	1	{-3, 5}	0	{-7, 5}	1
{-1, 3}	1	{2,-11}	2	{-3,-7}	1	{-7,-11}	2
{-1, 5}	0	{-2,-3}	0	{-3,-11}	2	{-7, 13}	2
{-1, 7}	2	{-2, 5}	1	{-3, 17}	1	{-7,-19}	2
{-1, 11}	2	{-2,-7}	1	{-3,-19}	1	{-7,-43}	2
{-1, 13}	2	{-2,-11}	3	{-3, 41}	1	{-7, 61}	2
{-1, 19}	0	{-2,-19}	2	{-3,-43}	1	{-7,-163}	2
{-1, 37}	2	{-2, 29}	2	{-3,-67}	1	{-11, 17}	2
{-1, 43}	2	{-2,-43}	2	{-3, 89}	1	{-11,-19}	2
{-1, 67}	2	{-2,-67}	2	{-3,-163}	1	{-11,-67}	4
{-1, 163}	2	{-19,-67}	2	{-43,-67}	2	{-11,-163}	2
{-67,-163}	2	{-19,-163}	2	{-43,-163}	2		

So, as shown in the Table 3, the rank of $X_1(15)$ is ≥ 1, with the following sets $\{d_1, d_2\}$.

$\{-1,3\}$	$\{2,-3\}$	$\{-3,-7\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,7\}$	$\{2,-11\}$	$\{-3,-11\}$	$\{-7,-11\}$	$\{-11,-19\}$
$\{-1,11\}$	$\{-2,5\}$	$\{-3,17\}$	$\{-7,13\}$	$\{-11,-67\}$
$\{-1,13\}$	$\{-2,-7\}$	$\{-3,-19\}$	$\{-7,-19\}$	$\{-11,-163\}$
$\{-1,37\}$	$\{-2,-11\}$	$\{-3,41\}$	$\{-7,-43\}$	$\{-19,-67\}$
$\{-1,43\}$	$,\{-2,-19\}$	$\{-3,-43\}$	$\{-7,61\}$	$\{-19,-163\}$
$\{-1,67\}$	$\{-2,29\}$	$\{-3,-67\}$	$\{-7,-163\}$	$\{-43,-67\}$
$\{-1,163\}$	$\{-2,-43\}$	$\{-3,89\}$		$\{-43,-163\}$
	$\{-2,-67\}$	$\{-3,-163\}$		$\{-67,-163\}$

Hence, there are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ with the previous sets, such that G_{15} appear as a torsion subgroup.

Consider $\{d_1,d_2\}$ in the following cases $\{-1,2\}$, $\{-1,19\}$ and $\{-2,-3\}$. In this cases, the rank of $X_1(15)$ over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ is 0. We then compute the torsion subgroup of $X_1^m(15)$ over $\mathbb{Q}$, for $m=\pm 19$, $-3,\pm 2,\pm 1,6$. So, we have $X_1(15)(\mathbb{Q})\simeq G_4$ and $X_1(15)^m\simeq G_2$ for the other values of m. Thus, the n-torsion of $X_1(15)(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$ for an odd integer n is determined by the above computation and the Lemma 2.1.

On the other hand $t^3-27t+8694$ is irreducible over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ in the following cases $\{-1,2\}$, $\{-1,19\}$ and $\{-2,-3\}$. This allows us to see that

$$\begin{cases} X_1(15)(\mathbb{Q}(\sqrt{-2},\sqrt{-3}))\cong G_8, & \\ X_1(15)(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))\cong G_4, & \text{for the other values of } \{d_1,d_2\}. \end{cases}$$

All of these points are cusps of $X_1(15)$. Therefore, G_{15} cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ in the following cases $\{-1,2\}$, $\{-1,19\}$ and $\{-2,-3\}$. □

2.5. **The group $G_2\oplus G_{10}$.**

Theorem 2.8. *There are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ with the following sets, such that $G_2\oplus G_{10}$ appear as a torsion subgroup.*

$\{-1,2\}$	$\{2,-11\}$	$\{-3,5\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-7,-11\}$	$\{-7,13\}$	$\{-11,-19\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,-11\}$	$\{-7,-19\}$	$\{-11,-67\}$
$\{-1,19\}$	$\{-2,-7\}$	$\{-3,17\}$	$\{-7,61\}$	$\{-11,-163\}$
$\{-1,37\}$	$\{-2,-11\}$	$\{-3,-19\}$		$\{-19,-67\}$
	$\{-2,-19\}$	$\{-3,41\}$		$\{-19,-163\}$
	$\{-2,29\}$	$\{-3,89\}$		
	$\{-2,-43\}$			
	$\{-2,-67\}$			

Proof. From [1, 18] we have that an EC with torsion $G_2\oplus G_{10}$ are induced by solutions of the equation

$$X_1(2,10): s^2=t^3+t^2-t,$$

Whose cusps satisfy the equation

$$t(t^2-1)(t^2+t-1)(t^2-4t-1)=0.$$

We see that $X_1(2,10)$ is an EC. So, using 2-descent we compute the rank of $X_1^m(2,10)$ over all imaginary biquadratic number fields of class number one.

TABLE 4. Rank of $X_1(2,10)$ over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$.

$\{d_1,d_2\}$	Rank	$\{d_1,d_2\}$	Rank	$\{d_1,d_2\}$	Rank	$\{d_1,d_2\}$	Rank
{-1, 2}	1	{2,-3}	0	{-3, 5}	1	{-7, 5}	1
{-1, 3}	0	{2,-11}	2	{-3,-7}	0	{-7,-11}	2
{-1, 5}	0	{-2,-3}	2	{-3,-11}	2	{-7, 13}	2
{-1, 7}	0	{-2, 5}	1	{-3, 17}	2	{-7,-19}	2
{-1, 11}	2	{-2,-7}	2	{-3,-19}	2	{-7,-43}	0
{-1, 13}	2	{-2,-11}	2	{-3, 41}	2	{-7, 61}	2
{-1, 19}	2	{-2,-19}	3	{-3,-43}	0	{-7,-163}	0
{-1, 37}	2	{-2, 29}	2	{-3,-67}	0	{-11, 17}	2
{-1, 43}	0	{-2,-43}	2	{-3, 89}	2	{-11,-19}	2
{-1, 67}	0	{-2,-67}	2	{-3,-163}	0	{-11,-67}	2
{-1, 163}	0	{-19,-67}	2	{-43,-67}	0	{-11,-163}	2
{-67,-163}	0	{-19,-163}	2	{-43,-163}	0		

So, as shown in the Table 4, the rank of $X_1^m(2,10)$ is ≥ 1, with the following sets $\{d_1,d_2\}$.

$\{-1,2\}$	$\{2,-11\}$	$\{-3,5\}$	$\{-7,5\}$	$\{-11,17\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-7,-11\}$	$\{-7,13\}$	$\{-11,-19\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,-11\}$	$\{-7,-19\}$	$\{-11,-67\}$
$\{-1,19\}$	$\{-2,-7\}$	$\{-3,17\}$	$\{-7,61\}$	$\{-11,-163\}$
$\{-1,37\}$	$\{-2,-11\}$	$\{-3,-19\}$		$\{-19,-67\}$
	$\{-2,-19\}$	$\{-3,41\}$		$\{-19,-163\}$
	$\{-2,29\}$	$\{-3,89\}$		
	$\{-2,-43\}$			
	$\{-2,-67\}$			

Hence, there are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ with the previous sets, such that $G_2\oplus G_{10}$ appear as a torsion subgroup.

Consider $\{d_1,d_2\}$ in the following cases $\{-1,3\}$, $\{-1,5\}$, $\{-1,7\}$, $\{-1,43\}$, $\{-1,67\}$, $\{-1,163\}$, $\{2,-3\}$, $\{-3,-7\}$, $\{-3,-43\}$, $\{-3,-67\}$, $\{-3,-163\}$, $\{-7,-43\}$, $\{-7,-163\}$, $\{-43,-67\}$, $\{-43,-163\}$, and $\{-67,-163\}$. In this cases, the rank of $X_1(2,10)$ over $\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ is 0. We then compute the torsion subgroup of $X_1^m(2,10)$ over $\mathbb{Q}$, for $m=\pm163,\pm67,\pm43$, $\pm7,-6,\pm5$, $\pm3,\pm1$, $2,18$, $129,201$, $301,489$, $1141,2881$, 7009, 10921. So, we have $X_1(2,10)(\mathbb{Q})\simeq G_6$ and $X_1(2,10)^m\simeq G_2$ for the other values of m. Thus, the n-torsion of $X_1(2,10)(\mathbb{Q}(\sqrt{d_1},\sqrt{d_2}))$ for an odd integer n is determined by the above computation and the Lemma

2.1. For the 2-torsion, we find that the 2-division polynomial splits only over $\mathbb{Q}(\sqrt{-1}, \sqrt{5})$. This shows that

$$\begin{cases} X_1(2,10)(\mathbb{Q}(\sqrt{-1}, \sqrt{5})) \cong G_2 \oplus G_6, & \\ X_1(2,10)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_6, & \text{for the other values of } \{d_1, d_2\}. \end{cases}$$

We have that all of the points in $G_2 \oplus G_6$ correspond to $t = -1$, 0 or 1, and hence, are cusps of $X_1(2,10)$. Therefore, $G_2 \oplus G_{10}$ cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the previous sets $\{d_1, d_2\}$. □

2.6. The group $G_2 \oplus G_{12}$.

Theorem 2.9. *There are infinitely many isomorphism classes of ECs defined over* $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ *with the following sets, such that* $G_2 \oplus G_{12}$ *appear as a torsion subgroup.*

$\{-1,5\}$	$\{2,-11\}$	$\{-3,5\}$	$\{-7,13\}$	$\{-11,17\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-3,-7\}$	$\{-7,-19\}$	$\{-11,-19\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,17\}$	$\{-7,-43\}$	$\{-11,-67\}$
$\{-1,19\}$	$\{-2,29\}$	$\{-3,-19\}$	$\{-7,61\}$	$\{-11,-163\}$
$\{-1,37\}$	$\{-2,-43\}$	$\{-3,41\}$	$\{-7,-163\}$	$\{-19,-67\}$
$\{-1,43\}$	$\{-2,-11\}$	$\{-3,-43\}$		$\{-19,-163\}$
$\{-1,67\}$	$\{-2,-67\}$	$\{-3,-67\}$		$\{-43,-67\}$
$\{-1,163\}$		$\{-3,89\}$		$\{-43,-163\}$
		$\{-3,-163\}$		$\{-67,-163\}$

Proof. From [1, 18] we have that an EC with torsion $G_2 \oplus G_{12}$ are induced by solutions of the equation

$$X_1(2,12) : s^2 = t^3 - t^2 + t,$$

Whose cusps satisfy the equation

$$t(t-1)(2t-1)(2t^2-2t+1)(3t^2-3t+1)(6t^2-6t+1) = 0.$$

We have that $X_1(2,12)$ is an EC. So, in the same way in the proof of the previous Theorem and from the table 5, we find that the rank of $X_1^m(2,12)$ is ≥ 1, with the following sets $\{d_1, d_2\}$.

$\{-1,5\}$	$\{2,-11\}$	$\{-3,5\}$	$\{-7,13\}$	$\{-11,17\}$
$\{-1,11\}$	$\{-2,-3\}$	$\{-3,-7\}$	$\{-7,-19\}$	$\{-11,-19\}$
$\{-1,13\}$	$\{-2,5\}$	$\{-3,17\}$	$\{-7,-43\}$	$\{-11,-67\}$
$\{-1,19\}$	$\{-2,-19\}$	$\{-3,-19\}$	$\{-7,61\}$	$\{-11,-163\}$
$\{-1,37\}$	$\{-2,-11\}$	$\{-3,41\}$	$\{-7,-163\}$	$\{-19,-67\}$
$\{-1,43\}$	$\{-2,29\}$	$\{-3,-43\}$		$\{-19,-163\}$
$\{-1,67\}$	$\{-2,-43\}$	$\{-3,-67\}$		$\{-43,-67\}$
$\{-1,163\}$	$\{-2,-67\}$	$\{-3,89\}$		$\{-43,-163\}$
		$\{-3,-163\}$		$\{-67,-163\}$

Table 5. Rank of $X_1(2,12)$.

$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank	$\{d_1, d_2\}$	Rank
{-1, 2}	0	{2,-3}	0	{-3, 5}	1	{-7, 5}	0
{-1, 3}	0	{2,-11}	1	{-3,-7}	1	{-7,-11}	0
{-1, 5}	1	{-2,-3}	1	{-3,-11}	0	{-7, 13}	2
{-1, 7}	0	{-2, 5}	1	{-3, 17}	1	{-7,-19}	2
{-1, 11}	1	{-2,-7}	0	{-3,-19}	1	{-7,-43}	2
{-1, 13}	1	{-2,-11}	1	{-3, 41}	1	{-7, 61}	2
{-1, 19}	1	{-2,-19}	2	{-3,-43}	1	{-7,-163}	2
{-1, 37}	1	{-2, 29}	1	{-3,-67}	1	{-11, 17}	2
{-1, 43}	1	{-2,-43}	1	{-3, 89}	1	{-11,-19}	2
{-1, 67}	1	{-2,-67}	3	{-3,-163}	1	{-11,-67}	2
{-1, 163}	1	{-19,-67}	2	{-43,-67}	2	{-11,-163}	2
{-67,-163}	2	{-19,-163}	2	{-43,-163}	2		

Hence, there are infinitely many isomorphism classes of ECs defined over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the previous sets, such that $G_2 \oplus G_{12}$ appear as a torsion subgroup.

Consider $\{d_1, d_2\}$ in the following cases $\{-1,2\}$, $\{-1,3\}$, $\{-1,7\}$, $\{2,-3\}$, $\{-2,-7\}$, $\{-3,-11\}$, $\{-7,5\}$, and $\{-7,-11\}$. In this cases, the rank of $X_1(2,12)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ is 0. We then compute the torsion subgroup of $X_1^m(2,12)$ over $\mathbb{Q}$, for $m = -35, -19, -11, \pm 7, -6, \pm 3, \pm 2, \pm 1, 5, 14, 28, 33, 77$. So, we have $X_1(2,12)^m(\mathbb{Q}) \simeq G_4$ for $m = \pm 1$ and $X_1(2,12)^m \simeq G_2$ for the other values of m. Thus, the n-torsion of $X_1(2,12)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}))$ for an odd integer n is determined by the above computation and the Lemma 2.1. For the 2-torsion, we find that the 2-division polynomial splits only over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ in the following cases $\{-1,3\}$, $\{2,-3\}$, and $\{-3,-11\}$. This shows that

$$\begin{cases} X_1(2,12)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_8, & \text{for } \{-1,2\} \text{ and } \{-1,7\} \\ X_1(2,12)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_2 \oplus G_4, & \text{for } \{-1,3\},\ \{2,-3\}, \text{ and } \{-3,-11\} \\ X_1(2,12)(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})) \cong G_4, & \text{for the other values of } \{d_1, d_2\}. \end{cases}$$

We have that all of the points in $G_2 \oplus G_4$ are cusps of $X_1(2,12)$. Therefore, $G_2 \oplus G_{12}$ cannot be considered as a torsion subgroup on $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with the previous sets $\{d_1, d_2\}$. □

3. $X_1(13)$, $X_1(16)$, and $X_1(18)$ over imaginary biquadratic number fields of class number one

In this section, we study the hyperelliptic curves $X_1(13)$, $X_1(16)$, and $X_1(18)$ over imaginary biquadratic number fields of class number one. We focus on fields where the rank of their Jacobian variety is zero.

Theorem 3.1. *Let $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be an imaginary biquadratic number field with class number one. Suppose the rank of the Jacobian varieties of $X_1(13)$, $X_1(16)$, and $X_1(18)$ is zero over this field. Then the torsion subgroups G_{13}, G_{16}, and G_{18} cannot appear as torsion subgroups over any of these fields, respectively.*

Proof. **The case G_{13}:** We have that $X_1(13)$ is a hyperelliptic curve. So we will study the torsion points of $J_1(13)$ over $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.
Since the Jacobian is an abelian variety [22], it follows that for a prime p of good reduction over $\mathbb{Q}(\sqrt{d})$, we have

- **•:** If $d \equiv 1 \ mod\ 4$ and $(\frac{d}{p}) = 1$, then the prime-to-p part of $J_1(13)(\mathbb{Q}(\sqrt{d}))_{\text{tors}}$ injects into $J_1(13)(\mathbb{F}_p)$.
- **•:** If $d \equiv 1 \ mod\ 4$ and $(\frac{d}{p}) = -1$, then the prime-to-p part of $J_1(13)(\mathbb{Q}(\sqrt{d}))_{\text{tors}}$ injects into $J_1(13)(\mathbb{F}_p(\sqrt{d})) \simeq J_1(13)(\mathbb{F}_{p^2})$.
- **•:** If $d \equiv 2,\ 3 \ mod\ 4$ and $(\frac{4d}{p}) = 1$, then the prime-to-p part of $J_1(13)(\mathbb{Q}(\sqrt{d}))_{\text{tors}}$ injects into $J_1(13)(\mathbb{F}_p)$.
- **•:** If $d \equiv 2, 3 \ mod\ 4$ and $(\frac{4d}{p}) = -1$, then the prime-to-p part of $J_1(13)(\mathbb{Q}(\sqrt{d}))_{\text{tors}}$ injects into $J_1(13)(\mathbb{F}_p(\sqrt{d})) \simeq J_1(13)(\mathbb{F}_{p^2})$.

On the other hand, the discriminant of $X_1(13)$ is $-2^{12}.13^2$, so 2 and 13 are the only rational primes with bad reduction.
We will discuss the cardinal of the Jacobian variety of $X_1(13)$ on imaginary quadratic fields $\mathbb{Q}(\sqrt{d_1})$ in the following cases $d_1 = -1, -2, -3, -7, -11, -19 - 43, -67$. So, when $d_1 = -1$, we have 3 is inert and 5 splits into $\mathbb{Q}(i)$. Thus, the prime-to-3 part and the prime-to-5 part of $J_1(13)(\mathbb{Q}(i))_{\text{tors}}$ are injected respectively into $J_1(13)(\mathbb{F}_3(i)) \simeq J_1(13)(\mathbb{F}_9)$ and $J_1(13)(\mathbb{F}_5)$. We see that $\sharp J_1(\mathbb{Q}(i)) \leq 19$. Therefore, it follows that $J_1(\mathbb{Q}(i)) = J_1(\mathbb{Q}) \simeq G_{19}$.
As $J_1^m(\mathbb{Q}(i)) \simeq \{O\}$ for $m = 2, 3,\ 5, 7, 11,\ 13, 19,\ 37, 43,\ 67, 163$ and $J_1(\mathbb{Q})$ is generated by the cusps of $X_1(13)$, we have for an odd number n, the n-part of $J_1(\mathbb{Q}(\sqrt{-1}, \sqrt{d_2}))$ for $d_2 = 2, 3,\ 5, 7, 11,\ 13, 19,\ 37, 43,\ 67, 163$ is generated by the cusps of $X_1(13)$.

On the other hand, we have the 2-division polynomial of $X_1(13)$ is irreducible over $\mathbb{Q}(\sqrt{-1}, \sqrt{d_2})$ for $d_2 = 2, 3,\ 5, 7, 11,\ 13, 19,\ 37, 43,\ 67, 163$.
Similarly for the other values of d_1. So, as a conclusion, G_{13} cannot appear as torsion subgroups over any of these fields.

The case G_{16}: Similarly, we will discuss the cardinal of the Jacobian variety of $X_1(16)$ on imaginary quadratic fields $\mathbb{Q}(\sqrt{d_1})$ in the following cases $d_1 = -1, -2, -3, -7, -11, -19 - 43, -67$. So, we will do this proof for $d_1 = -3$, and similarly for the other values.
Since the discriminant of $X_1(16)$ is -2^{19}, so 2 is the only rational primes with bad reduction. We have 5 is inert and 7 splits into $\mathbb{Q}(\sqrt{-3})$. Thus, the prime-to-5 part and the prime-to-7 part of $J_1(16)(\mathbb{Q}(\sqrt{-3}))_{\text{tors}}$ are injected respectively into $J_1(16)(\mathbb{F}_5(\sqrt{-3})) \simeq J_1(16)(\mathbb{F}_{25})$ and $J_1(16)(\mathbb{F}_7)$. We see that $\sharp J_1(16)(\mathbb{Q}(\sqrt{-3})) \leq 40$. Therefore, it follows that $J_1(\mathbb{Q}(\sqrt{-3})) \simeq G_2 \oplus G_{10}$ or $J_1(\mathbb{Q}(\sqrt{-3})) \simeq G_2 \oplus G_2 \oplus G_{10}$.
As $J_1(16)(\mathbb{Q})$ is generated by the cusps of $X_1(16)$, and we have for an odd prime number p, $J_1^m(16)(\mathbb{Q}(\sqrt{-3}))[p] \simeq \{O\}$ for $m = 2,\ -7, -11,\ -19,\ -43, -67,\ 89$, thus the p-part of $J_1(16)(\mathbb{Q}(\sqrt{-3}, \sqrt{d_2}))$ for $d_2 = 2,\ -7, -11,\ -19,\ -43, -67,\ 89$

are generated by the cusps of $X_1(16)$.
On the other hand, we have the 2-division polynomial of $X_1(16)$ is irreducible over $\mathbb{Q}(\sqrt{-3}, \sqrt{d_2})$ for all d_2.
Note that when we take the field $\mathbb{Q}(\sqrt{-1}, \sqrt{-2})$, we have the 2-division polynomial of $X_1(16)$ which splits only on this one, but again all these points are cusps.
So, as a conclusion, G_{16} cannot appear as torsion subgroups over any of these fields.

The case G_{18}: We have that $X_1(18)$ is a hyperelliptic curve, and according to [16] G_{18} does not appear on $\mathbb{Q}(i)$. So, we will study its Jacobian $J_1(18)$. We obtain via 2-descent that, $rank(J_1(18)(\mathbb{Q}(\sqrt{-1}, \sqrt{d_2}))) = 0$ for $d_2 = 2, 3, 5, 7, 13, 19, 37, 163$.
So we will study the torsion points of $J_1(18)$ over $\mathbb{Q}(\sqrt{-1}, \sqrt{d_2})$ for $d_2 = 2, 3, 5, 7, 13, 19, 37, 163$.
As the discriminant of $X_1(18)$ is $-2^{15}.3^4$, so 2 and 3 are the only rational primes with bad reduction, and we have 7 is inert and 5 splits into $\mathbb{Q}(i)$. Thus, the prime-to-7 part and the prime-to-5 part of $J_1(18)(\mathbb{Q}(i))_{\text{tors}}$ are injected respectively into $J_1(18)(\mathbb{F}_7(i)) \simeq J_1(18)(\mathbb{F}_{49})$ and $J_1(18)(\mathbb{F}_5) \simeq J_1(18)(\mathbb{F}_5)$. We see that $\sharp J_1(\mathbb{Q}(i)) \leq 21$. Therefore, it follows that $J_1(\mathbb{Q}(i)) = J_1(\mathbb{Q}) \simeq G_{21}$.
As $J_1^m(\mathbb{Q}(i)) \simeq \{O\}$ for $m = 2, 5, 7, 13, 19, 37, 163$ and $J_1(\mathbb{Q})$ is generated by the cusps of $X_1(18)$, we have for an odd prime number p, the p-part of $J_1(\mathbb{Q}(\sqrt{d_1}))$ for $d_1 = 2, 5, 7, 13, 19, 37, 163$ is generated by the cusps of $X_1(18)$.
If $m = 3$, $J_1^3(\mathbb{Q}(i)) \simeq G_3$ and also generated by the cusps of $X_1(18)$.
On the other hand, we have that the 2-division polynomial of $X_1(18)$ is irreducible over $\mathbb{Q}(\sqrt{-1}, \sqrt{d_2})$ for all d_2.
So, as a conclusion, G_{18} cannot appear as torsion subgroups over any of these fields. □

Corollary 3.2. *The groups $G_2 \oplus G_{16}$ and $G_2 \oplus G_{18}$ cannot appear as torsion subgroups over any of the fields $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ mentioned in the above Proof. Otherwise, it would induce a $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$-rational cyclic subgroup of order 16 (respectively of order 18).*

From the results obtained in this section, we can conclude that G_{13}, G_{16}, and G_{18} cannot appear as torsion subgroups over any imaginary biquadratic number field of class number one mentioned in the above Proof.

4. Conclusion

In conclusion, this research investigates the structure and existence of specific torsion subgroups for elliptic curves over the imaginary biquadratic number field $\aleph = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ of class number one. By concentrating on the subgroups isomorphic to G_{11}, G_{13}, G_{14}, G_{15}, G_{16}, G_{18}, $G_2 \oplus G_{10}$ and $G_2 \oplus G_{12}$. Furthermore, the analysis shows that certain torsion subgroups, namely G_{13}, G_{16} and G_{18} cannot exist as torsion subgroups of elliptic curves over $\aleph$ if the rank of the Jacobian varieties of $X_1(13)$, $X_1(16)$ and $X_1(18)$ over this field is zero. The methods developed and used in this study are not restricted to $\aleph$ and show potential for application to more general number fields. Extending these techniques to analyse torsional structures over a wider class of number fields is a natural progression and an exciting direction for future work. This opens up the possibility of further exploring the relationship between the arithmetic properties of number fields and the torsion points of elliptic curves.

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