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ON INVARIANT DISTRIBUTIONALLY SCRAMBLED SETS IN NON-COMPACT SYSTEMS

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Abstract. We study here the invariance of topological distributionally scrambled sets for maps on uniform spaces (not necessarily compact or metrizable). For a uniformly continuous surjective self-map defined on a uniformly locally compact Hausdorff uniform space having topological weak specification, a fixed point, and countably many periodic points with distinct periods, we prove that the map admits an invariant topological distributionally scrambled set of type 1. Further, if the uniform space is second countable, the map admits a dense Mycielski invariant topological distributionally scrambled set of type 1.

1. Introduction

Since its origin, the concept of "chaos" has been the most intriguing area of research in dynamical systems. Li-Yorke gave the first mathematical definition of chaos for interval maps through the existence of an uncountable scrambled set in 1975 [10]. Over the past few decades, many researchers have studied various extensions of Li-Yorke chaos based on the properties of scrambled sets, such as dense chaos, generic chaos and ϵ-Li-Yorke chaos. Schweizer and Smítal introduced a stronger notion of chaos for interval maps based on the probabilistic measure of the distance between two trajectories, popularly termed as distributional chaos [15]. Balibrea et al. observed that for continuous maps on compact metric spaces, the notion of distributional chaos could be further classified into three nonequivalent variants, namely $DC1$, $DC2$ and $DC3$ [4].

Specification is another widely studied notion of chaos introduced by Bowen [5]. Many researchers have explored the relationship between distributional chaos and various types of specification properties. Sklar and Smítal proved that for continuous self-maps defined on compact metric spaces, specification property implies distributional chaos of type 3 [19]. Furthermore, Oprocha and Štefánková proved that a continuous self-map on a compact metric space with the weak specification property and a distal pair is distributionally chaotic of type 1 and possesses a dense scrambled set [12].

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In [7], Du studied the invariance of Li-Yorke scrambled sets for interval maps and proved that if a continuous map of the unit interval is turbulent, then there exists an invariant Li-Yorke scrambled set. The results proved by Du yielded interesting characterization for the class of interval maps with positive topological entropy in terms of the existence of an invariant Li-Yorke scrambled set. Later, Balibrea et al. studied the invariance of Li-Yorke scrambled sets for maps on compact metric spaces [3]. In [11], Oprocha studied the invariance of distributionally scrambled sets for interval maps and proved that there exists a uniform and invariant distributionally scrambled set for a continuous turbulent map of the unit interval. He posed the following open question: "Does every map with the specification property and a fixed point contain an invariant distributionally scrambled set?" Doleželová answered this question positively with additional assumptions and proved the existence of dense invariant distributionally scrambled sets for continuous self-maps on compact metric spaces [6].

Recently, various notions of chaos have been extended to general topological spaces, which are not necessarily compact or metrizable (see [1, 2, 13, 14, 21]). Shah et al. have defined and studied the notion of specification property for homeomorphisms on uniform spaces, which is termed as topological specification property [16]. The notion of topological distributional chaos for self-maps on uniform spaces is introduced in [18], wherein it is proved that a maps with the topological weak specification property admits a topological distributionally scrambled set of type 3. We extended this result in [20], proving that such maps are topologically distributionally chaotic of type 1.

In this paper, we strengthen the above results by proving the existence of an invariant topological distributionally scrambled set for maps on uniform spaces. Essentially, we prove that a uniformly continuous surjective self-map on a uniformly locally compact Hausdorff uniform space with topological weak specification property, a fixed point and countably many periodic points with mutually different periods has an invariant topological distributionally scrambled set of type 1. Furthermore, if the underlying space is second countable, then we show that the map admits a dense Mycielski invariant topological distributionally scrambled set of type 1. As a consequence, we obtain that a self-homeomorphism on a uniformly locally compact second countable Hausdorff uniform space with topological specification property and a fixed point has a dense Mycielski invariant topological distributionally scrambled set of type 1.

2. Preliminaries

In this section, we recall the definitions and preliminaries required for the development of the paper.

2.1. **Uniform Spaces.** Let X be a non-empty set. The set $\triangle_X = \{(x,x) \mid x \in X\}$ denotes the diagonal of the Cartesian product $X \times X$. A subset U of $X \times X$ is said to be symmetric if $U = U^{-1}$, where $U^{-1} = \{(y,x) \mid (x,y) \in U\}$. For subsets U and V of $X \times X$, the composition $U \circ V$ denotes the set of all pairs (x,y), for which there exists some point $z \in X$ satisfying $(x,z) \in U$ and $(z,y) \in V$. For $n \in \mathbb{N}$ and $U \in \mathcal{U}$, U^n denotes the subset $U \circ U \circ \cdots \circ U$ (n-times) of $X \times X$.

Definition 2.1. [9] A uniformity or a uniform structure on X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that

(1) for each $U \in \mathcal{U}$, $\triangle_X \subset U$,
(2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,
(3) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$,
(4) if $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
(5) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The elements of $\mathcal{U}$ are called entourages and the pair $(X,\mathcal{U})$ is called a uniform space.

For $\dot{A} \subset X$ and $U \in \mathcal{U}$, the set $U[\dot{A}]$ denotes the collection of all points y in X such that $(x, y) \in U$, for some point $x \in \dot{A}$. The set $U[x] = \{y \in X \mid (x, y) \in U\}$ is called the cross-section of U at $x \in X$. A uniform structure $\mathcal{U}$ on X induces a topology on X by defining a subset $\dot{G}$ of X to be open if for each point $x \in G$ there exists $U \in \mathcal{U}$ such that $U[x] \subset \dot{G}$. A uniform space $(X,\mathcal{U})$ is said to be Hausdorff if $\bigcap\{U \mid U \in \mathcal{U}\} = \triangle_X$. A space X with uniformity $\mathcal{U}$ is said to be uniformly locally compact if there exists $U \in \mathcal{U}$ such that $U[x]$ is compact, for each $x \in X$.

Throughout by a dynamical system, we mean a pair (X, f), where f is uniformly continuous self-map on a Hausdorff uniform space $(X,\mathcal{U})$. For a point $x \in X$, the set $\{f^n(x) \mid n \in \mathbb{N} \cup \{0\}\}$, denoted by $O_f(x)$, is called the orbit of x. If there exists $k \in \mathbb{N}$ such that $f^k(x) = x$ and $f^i(x) \neq x$ for all i $(1 \leq i < k)$, then x is said to be a periodic point of f with period k. A point $x \in X$ is said to be a fixed point if $f(x) = x$. A set $A \subset X$ is said to be invariant if $f(A) \subset A$. A countable union of Cantor sets is said to be a Mycielski set. For a given dynamical system (X, f), we denote the product map $f \times f$ on $X \times X$ by F and by $F^i(x, y)$ we mean $(f^i(x), f^i(y))$.

2.2. **Topological Specification Property.** The notion of specification property for self-maps on uniform spaces is termed as topological specification property.

Definition 2.2. [17] Let f be a uniformly continuous surjective self-map on a uniform space $(X,\mathcal{U})$. A map f is said to have topological specification property (briefly TSP) if for every $E \in \mathcal{U}$ there exists a positive integer M such that for any finite sequence $x_1, x_2, \ldots, x_k$ in X, any integers $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$, for $j = 2, \ldots, k$ and $p > M + (b_k - a_1)$, there exists $x \in X$ such that $f^p(x) = x$ and $F^i(x, x_j) \in E$, for all integers i with $a_j \leq i \leq b_j$, $j = 1, \ldots, k$.

If f fulfills both the above-mentioned conditions for the special case $k = 2$, and additionally the periodicity condition is omitted, then we say that f has topological weak specification property (TWSP). For any entourage $E \in \mathcal{U}$, if $F^j(x, y) \in E$ for all integers j with $a \leq j \leq b$, then we say that x E-traces y on the interval $[a, b]$.

2.3. **Topological Distributional Chaos.** For any positive integer n, points $a, b \in X$ and $E \in \mathcal{U}$, define the lower and upper distribution functions $F_{ab}(E)$

and $F^*_{ab}(E)$ as follows:

$$F_{ab}(E) = \liminf_{k\to\infty} \frac{1}{k}\#\{0 \le i < k \mid F^i(a,b) \in E\},$$
$$F^*_{ab}(E) = \limsup_{k\to\infty} \frac{1}{k}\#\{0 \le i < k \mid F^i(a,b) \in E\},$$

where $\#A$ denotes the cardinality of the set A.

Definition 2.3. [18] A pair of points a, $b \in X$ is called

(1) topologically distributionally chaotic of type 1 (TDC1) if $F_{ab}(E) = 0$, for some $E \in \mathcal{U}$, and $F^*_{ab}(U) = 1$, for all $U \in \mathcal{U}$,
(2) topologically distributionally chaotic of type 2 (TDC2) if $F_{ab}(E) < F^*_{ab}(E)$, for some $E \in \mathcal{U}$, and $F^*_{ab}(U) = 1$, for all $U \in \mathcal{U}$,
(3) topologically distributionally chaotic of type 3 (TDC3) if $F_{ab}(E) < F^*_{ab}(E)$, for some $E \in \mathcal{U}$.

A set S containing at least two points is called a topological distributionally scrambled set of type k if every pair of distinct points in S is topologically distributionally chaotic of type k, where $k \in \{1, 2, 3\}$ [18].

Definition 2.4. [18] A map f is said to be topologically distributionally chaotic of type k, where $k \in \{1, 2, 3\}$, if there exists an uncountable topological distributionally scrambled set of type k.

3. Main results

In this section, we prove our main results and related results.

Theorem 3.1. *Let $(X,\mathcal{U})$ be a uniformly locally compact Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with distinct periods, then there is a point $x \in X$ whose orbit is a topological distributionally scrambled set of type* 1.

Proof. Since closed entourages forms a base for the uniform structure [8], without loss of generality we can work with the closed entourages in $\mathcal{U}$.

Let $p \in X$ be a fixed point of f and let $\{q_i\}_{i=1}^{\infty}$ be the set of periodic points of f with distinct periods. Consider the sequence

$$p, q_1, p, q_1, q_2, p, q_1, q_2, q_3, p, q_1, q_2, q_3, q_4, p, \ldots,$$

which we denote by $\{u_j\}_{j=1}^{\infty}$. Since X is uniformly locally compact, there exists an entourage $U \in \mathcal{U}$ such that $U[t]$ is compact for each $t \in X$.

Let $V_0 = U$ and let V_n's be closed entourages in $\mathcal{U}$ such that $V_n^2 \subset V_{n-1}$, for any $n \in \mathbb{N}$. Let $M_n = M(V_n)$, $n \ge 1$, be as in the definition of TWSP. Let $0 = a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots$, be an increasing sequence of integers such that $a_{n+1} - b_n > M_n$, for any $n \ge 1$, and $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$.

Step 1: Let $g \in X$ be an arbitrary point. Since (X, f) has TWSP, for $V_1 \in \mathcal{U}$, points g, u_1 in X, and integers a_1, b_1, a_2, b_2 there exists a point $x_1 \in X$ such that

$$F^i(x_1, g) \in V_1, \text{ for } a_1 \le i \le b_1,$$
$$F^i(x_1, u_1) \in V_1, \text{ for } a_2 \le i \le b_2.$$

Let $W_1 = \bigcap_{i=0}^{b_2} F^{-i}(V_1) \in \mathcal{U}$. Then $W_1 \subset V_1$ implies that $W_1[x_1] \subset V_1[x_1] \subset U[x_1]$. Since $U[x_1]$ is compact, it follows that $W_1[x_1]$ is compact. Further, for any $x \in W_1[x_1]$, $(x, x_1) \in W_1$ implies that $F^i(x, x_1) \in V_1$, for all $a_2 \le i \le b_2$. Also, $F^i(x_1, u_1) \in V_1$, for all $a_2 \le i \le b_2$, implies that $F^i(x, u_1) \in V_1^2$, for all $a_2 \le i \le b_2$. Thus, each $x \in W_1[x_1]$ V_1^2-traces u_1, for all $a_2 \le i \le b_2$. Without loss of generality we can assume that $W_1 \subset V_2$ (if W_1 is not contained in V_2, we can always work with $W_1' = W_1 \cap V_2 \in \mathcal{U}$).
Step 2: For $V_2 \in \mathcal{U}$, points x_1, u_2 in X, and integers a_1, b_2, a_3, b_3, by TWSP there exists a point $x_2 \in X$ such that

$$F^i(x_2, x_1) \in V_2, \text{ for } a_1 \le i \le b_2,$$
$$F^i(x_2, u_2) \in V_2, \text{ for } a_3 \le i \le b_3.$$

Let $W_2' = \bigcap_{i=0}^{b_3} F^{-i}(V_2) \in \mathcal{U}$ and W_2'' be a closed entourage in $\mathcal{U}$ such that $W_2''[x_2] \subset W_1[x_1]$. Take $W_2 = W_2' \cap W_2''$, then $W_2[x_2] \subset W_1[x_1]$. Since $W_1[x_1]$ is compact, it follows that $W_2[x_2]$ is compact. Note that any $x \in W_2[x_2]$ V_2^2-traces u_2, for all $a_3 \le i \le b_3$. Again without loss of generality we can assume that $W_2 \subset V_3$.

Continuing like this, for each $n \in \mathbb{N}$, there is a compact set $W_n[x_n]$ such that $W_n[x_n] \subset W_{n-1}[x_{n-1}]$ and each $x \in W_n[x_n]$ V_n^2-traces u_n, for all $a_{n+1} \le i \le b_{n+1}$. This gives rise to a nested sequence $W_n[x_n]$ of compact sets and hence there is at least one point $x \in \bigcap_{n=1}^{\infty} W_n[x_n]$ such that for every $n \in \mathbb{N}$,

$$F^i(x, u_n) \in V_n^2, \text{ for } a_{n+1} \le i \le b_{n+1}$$

We now prove that the orbit of x is topological distributionally scrambled set of type 1. Let $y = f^l(x)$ and $z = f^m(x)$, for some non-negative integers l, m with $l > m$. Note that, $u_{s_n} = p$, where $s_n = \sum_{j=1}^{n} j$, for every $n \ge 1$. From our choice of s_n, it follows that $F^i(x, p) = (f^i(x), p) \in V_{s_n}^2$, for all $a_{s_n+1} \le i \le b_{s_n+1}$. This implies that $(f^i(y), p) \in V_{s_n}^2$, for all $a_{s_n+1} - l \le i \le b_{s_n+1} - l$ and $(f^i(z), p) \in V_{s_n}^2$, for all $a_{s_n+1} - m \le i \le b_{s_n+1} - m$. As $l > m$, we have $F^i(y, z) \in V_{s_n}^4$, for all $a_{s_n+1} - m \le i \le b_{s_n+1} - l$. Thus,

$$\#\{0 \le i \le b_{s_n+1} \mid F^i(y, z) \in V_{s_n}^4\} \ge b_{s_n+1} - a_{s_n+1} - (l - m).$$

Now for any entourage $E \in \mathcal{U}$, since $V_n^2 \subset V_{n-1}$, there will be an N such that $V_n^4 \subset E$, for all $n > N$. Thus, $F_{yz}^*(E) = 1$, for all $E \in \mathcal{U}$.

We now prove that $F_{yz}(E) = 0$, for some $E \in \mathcal{U}$. Choose a periodic point q_r with period $k > l - m$. Then, for $i \ne j$, $i, j \in \{0, 1, 2, \ldots, k-1\}$, $f^i(q_r) \ne f^j(q_r)$. Since X is Hausdorff, we can find an entourage $E_r \in \mathcal{U}$ such that $(f^i(q_r), f^j(q_r)) \notin E_r$, for all $i \ne j$, $i, j \in \{0, 1, 2, \ldots, k-1\}$. Let $E \in \mathcal{U}$ be such that $E \circ E \circ E \subset E_r$. Since $V_n^2 \subset V_{n-1}$, there will be N such that $V_n^2 \subset E$, for all $n > N$. Note that, $u_{t_n} = q_r$, where $t_n = r + \sum_{j=1}^{n} j$, for every $n \ge r$. Thus, $F^i(x, q_r) \in V_{t_n}^2$, for all $a_{t_n+1} \le i \le b_{t_n+1}$. This implies that $F^i(y, f^l(q_r)) \in V_{t_n}^2$ and $F^i(z, f^m(q_r)) \in V_{t_n}^2$, for all $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$. Observe that $F^i(y,z) \notin V^2_{t_n}$, for all $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$ and $t_n > N$. For if, $F^i(y,z) \in V^2_{t_n}$, for some $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$, then $F^i(y, f^l(q_r)) \in V^2_{t_n}$ and $F^i(z, f^m(q_r)) \in V^2_{t_n}$ imply that $F^i(f^l(q_r), f^m(q_r)) \in E \circ E \circ E \subset E_r$, a contradiction. Therefore,

$$\#\{0 \leq i \leq b_{t_n+1} \mid F^i(y,z) \in E\} \leq a_{t_n+1} + (l - m), \text{ for } t_n > N.$$

Hence $F_{yz}(E) = 0$. □

Lemma 3.2. [6] *There is a Cantor set $B \subset \{0,1\}^{\mathbb{N}}$ such that for any distinct $\alpha = \{\alpha(i)\}_{i=1}^{\infty}$ and $\beta = \{\beta(i)\}_{i=1}^{\infty}$ in B, the set $\{j \in \mathbb{N} \mid \alpha(j) \neq \beta(j)\}$ is infinite.*

Theorem 3.3. *Let $(X, \mathcal{U})$ be a uniformly locally compact Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with distinct periods, then X has a Mycielski invariant topological distributionally scrambled set of type 1. Moreover, each point $g \in X$ has a neighborhood which contains a Cantor topological distributionally scrambled set of type 1.*

Proof. Let $\alpha = \{\alpha(i)\}_{i=1}^{\infty} \in B$, where B is the set as in Lemma 3.2. Let $p \in X$ be a fixed point of f and let $\{q_i\}_{i=1}^{\infty}$ be the set of periodic points of f with distinct periods. Define the sequence $\{w_j^{(\alpha)}\}_{j=1}^{\infty}$ in the following manner:

$$w_j^{(\alpha)} = \begin{cases} p, & \text{if } \alpha(j) = 0; \\ q_1, & \text{if } \alpha(j) = 1. \end{cases}$$

Consider the sequence

$$p, q_1, w_1^{(\alpha)}, p, q_1, q_2, w_2^{(\alpha)}, p, q_1, q_2, q_3, w_3^{(\alpha)}, p, q_1, q_2, q_3, q_4, w_4^{(\alpha)}, p, \ldots,$$

which we denote by $\{u_j^{(\alpha)}\}_{j=1}^{\infty}$. Following the recursive steps as discussed in Theorem 3.1, we get a point $x^{(\alpha)}$ such that for every $n \geq 1$, $F^i(x^{(\alpha)}, u_n^{(\alpha)}) \in V_n^2$, for all $a_{n+1} \leq i \leq b_{n+1}$.

Let $C = \{x^{(\alpha)} \mid \alpha \in B\}$ and $\hat{C} = \bigcup_{i=0}^{\infty} f^i(C)$. We now prove that $\hat{C}$ is a TDC1 scrambled set. For $y, z \in \hat{C}$, let $y = f^l(x^{(\alpha)})$ and $z = f^m(x^{(\beta)})$, for some $l, m \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \in B$.

Case 1: If $\alpha = \beta$, $l \neq m$, then (y,z) is a TDC1 pair follows from the proof of Theorem 3.1.

Case 2: Suppose $\alpha \neq \beta$ and $l \geq m$. Then choose a sequence $\{s_n\}_{n=1}^{\infty}$ such that $u_{s_n} = p$, for every $n \geq 1$. Note that $(f^i(y), p) \in V^2_{s_n}$ and $(f^i(z), p) \in V^2_{s_n}$, for all $a_{s_n+1} - m \leq i \leq b_{s_n+1} - l$. Thus, we get $F^i(y,z) \in V^4_{s_n}$, for all $a_{s_n+1} - m \leq i \leq b_{s_n+1} - l$ and for every $n \geq 1$. Hence, $F^*_{yz} \equiv 1$.

Since $\alpha, \beta \in B$ and $\alpha \neq \beta$, there is a subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ of $\{w_j\}_{j=1}^{\infty}$ such that $w_{j_k}^{(\alpha)} \neq w_{j_k}^{(\beta)}$, for every $k \geq 1$ (That is, $w_{j_k}^{(\alpha)} = p$ and $w_{j_k}^{(\beta)} = q_1$ or $w_{j_k}^{(\alpha)} = q_1$ and $w_{j_k}^{(\beta)} = p$). We choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $u_{t_n} = w_{j_k}$, for every $n \geq 1$. Then $F^i(y, f^l(w_{j_k}^{(\alpha)})) \in V^2_{t_n}$ and $F^i(z, f^m(w_{j_k}^{(\beta)})) \in V^2_{t_n}$, for all $a_{t_n+1} - m \leq i \leq b_{t_n+1} - l$. If q_1 is a periodic point of period k, consider the entourage $E_1 \in \mathcal{U}$ such that $(p, f^i(q_1)) \notin E_1$, for all $i \in \{0, 1, 2, \ldots, k-1\}$ and let $E \in \mathcal{U}$ be such that $E \circ E \circ E \subset E_1$. Using arguments similar to that given in proof of Theorem 3.1, one can obtain $F_{yz}(E) = 0$.

Consider the bijective map $h : B \to C$, given by $h(\alpha) = x^{(\alpha)}$, for all $\alpha \in B$. To show that h is a homeomorphism, it is sufficient to show that h is continuous. Let $\{\alpha_m\}_{m=1}^{\infty}$ be a convergent sequence in B such that $\alpha_m \to \alpha$. Then for any $i \geq 1$, there is an m_0 such that, the first i members of sequences α_m and α are equal for all $m > m_0$. Therefore, the first i members of $\{u_j^{(\alpha_m)}\}_{j=1}^{\infty}$ and $\{u_j^{(\alpha)}\}_{j=1}^{\infty}$ are also equal, and hence the corresponding $x^{(\alpha_m)}$ and $x^{(\alpha)}$ belong to the same $W_i[x_i]$. Thus, as $m \to \infty$, $\alpha_m \to \alpha$ implies that $x^{(\alpha_m)} \to x^{(\alpha)}$. Hence, h is a homeomorphism and C is a Cantor set. Since $\hat{C}$ is a TDC1 scrambled set, the mapping $f^i|_C : C \to f^i(C)$ is also a bijective mapping. Continuity of $f^i|_C$ implies that $f^i|_C$ is a homeomorphism, for every $i \geq 1$. Thus, $\hat{C}$ is a Mycielski invariant topological distributionally scrambled set of type 1. Moreover, for every $\alpha \in B$, the corresponding $x^{(\alpha)} \in U[g]$, and hence $C \subset U[g]$. Since $g \in X$ is arbitrary, it follows that each point in X has a neighborhood which contains a Cantor topological distributionally scrambled set of type 1. □

Theorem 3.4. *Let $(X,\mathcal{U})$ be a uniformly locally compact second countable Hausdorff uniform space without isolated points. If $f : X \to X$ is a uniformly continuous surjective map having topological weak specification property, a fixed point and countably many periodic points with distinct periods, then X has a dense Mycielski invariant topological distributionally scrambled set of type* 1.

Proof. Let $\{\dot{F}_i\}_{i=1}^{\infty}$ denote the countable closed base for the uniform topology on X induced by $(X,\mathcal{U})$, where $\dot{F}_i = U_i[g_i]$ for some $U_i \in \mathcal{U}$, $g_i \in X$. Since X is uniformly locally compact there exists an entourage $U \in \mathcal{U}$ such that $U[t]$ is compact for each $t \in X$. Let $V_0 = U$ and let V_n be closed entourages in $\mathcal{U}$ such that $V_n^2 \subset V_{n-1}$, for any $n \in \mathbb{N}$. For each $g_i \in X$, using the arguments given in proof of Theorem 3.3, for the sequence

$$\begin{aligned} \{u_j^{(\alpha,i)}\}_{j=1}^{\infty} = \quad & \{p, q_1, w_1^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, w_2^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, q_3, \\ & w_3^{(\alpha)}, \mathbf{q_i}, p, q_1, q_2, q_3, q_4, w_4^{(\alpha)}, \mathbf{q_i}, p, \ldots\}, \end{aligned}$$

$\alpha \in B$, there exists a Cantor TDC1 scrambled set $C_i = \{x^{(\alpha,i)} \mid \alpha \in B\}$ such that $C_i \subset \dot{F}_i$. Let $M = \bigcup_{i=1}^{\infty} \hat{M}_i$, where $\hat{M}_i = \bigcup_{k=0}^{\infty} f^k(C_i)$. Then, M is a dense Mycielski invariant set in X.

We now prove that M is a TDC1 scrambled set. Clearly, $\hat{M}_i$ is a $TDC1$ scrambled set for each i. Consider $y, z \in M$ such that $y \in \hat{M}_i$ and $z \in \hat{M}_j$, $i \neq j$. Then $y = f^l(x^{(\alpha,i)})$ and $z = f^m(x^{(\beta,j)})$, for some $l, m \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \in B$. We assume $l \geq m$. Choose a sequence $\{s_n\}_{n=1}^{\infty}$ such that $u_{s_n}^{(\alpha,i)} = u_{s_n}^{(\beta,j)} = p$, for every $n \geq 1$. Then $(f^k(y), p) \in V_{s_n}^2$ and $(f^k(z), p) \in V_{s_n}^2$, for all $a_{s_n+1} - m \leq k \leq b_{s_n+1} - l$. Thus, we get $F^k(y, z) \in V_{s_n}^4$, for all $a_{s_n+1} - m \leq k \leq b_{s_n+1} - l$ and for every $n \geq 1$. Hence, $F_{yz}^* \equiv 1$.

Next, choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $u_{t_n}^{(\alpha,i)} = q_i$ and $u_{t_n}^{(\beta,j)} = q_j$, for every $n \geq 1$. Then $F^k(y, f^l(q_i)) \in V_{t_n}^2$ and $F^k(z, f^m(q_j)) \in V_{t_n}^2$, for all $a_{t_n+1} - m \leq k \leq b_{t_n+1} - l$. If q_i is a periodic point of period λ and q_j is a periodic point of period μ, consider the entourage $E \in \mathcal{U}$ such that $(f^{k_1}(q_i), f^{k_2}(q_j)) \notin E$, for all $k_1 \in \{0, 1, 2, \ldots, \lambda - 1\}$, $k_2 \in \{0, 1, 2, \ldots, \mu - 1\}$ and let $E_0 \in \mathcal{U}$ be such that $E_0 \circ E_0 \circ E_0 \subset E$. Since $V_n^2 \subset V_{n-1}$, there will be N such that $V_n^2 \subset E$, for all $n > N$. For $t_n > N$, if $F^k(y, z) \in V_{t_n}^2$, for some $a_{t_n+1} - m \leq k \leq b_{t_n+1} - l$, then $F^k(y, f^l(q_i)) \in V_{t_n}^2$ and $F^k(z, f^m(q_j)) \in V_{t_n}^2$ imply that $F^k(f^l(q_i), f^m(q_j)) \in E_0 \circ E_0 \circ E_0 \subset E$, a contradiction. Thus,

$$\#\{0 \leq k \leq b_{t_n+1} | F^k(y, z) \in E\} \leq a_{t_n+1} + (l - m), \text{ for } t_n > N.$$

Therefore, $F_{yz}(E) = 0$. Hence, M is a TDC1 scrambled set. $\square$

In [16], it is proved that self-homeomorphisms on uniform spaces with topological specification property have a dense set of periodic points and the construction suggests that we can have infinitely many periodic points with distinct periods. As a consequence, we have the following result:

Corollary 3.5. *Let $(X, \mathcal{U})$ be a uniformly locally compact second countable Hausdorff uniform space without isolated points. If $f : X \to X$ is a homeomorphism having topological specification property and a fixed point, then X has a dense Mycielski invariant topological distributionally scrambled set of type* 1.

4. Conclusion.

In this paper, we have extended the study of distributional chaos to the broader context of uniform spaces, which are not necessarily compact or metrizable. By strengthening existing results, we have proved the existence of invariant topological distributionally scrambled sets of type 1 for uniformly continuous surjective self-maps on uniformly locally compact Hausdorff uniform spaces with the topological weak specification property, a fixed point, and countably many periodic points with distinct periods. Additionally, if the underlying space is second countable, we have obtained that the map admits a dense Mycielski invariant topological distributionally scrambled sets of type 1.

The size of the scrambled set and its invariance are two critical aspects in the study of chaos. Under the given assumptions, we ensure both by establishing the existence of a dense Mycielski invariant scrambled set.

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