ENTROPY SOLUTIONS FOR A NONLINEAR UNILATERAL 
$p(x)$ PARABOLIC PROBLEMS WITH $f – \text{div}F$ DATA

MOUIN MEKKOUR*

Abstract. We study the nonlinear $p(x)$ parabolic problems with obstacle and $f – \text{div}F$ data.

1. Introduction

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $T$ is a positive real number and $Q = \Omega \times (0,T)$, while the variable exponent $p : \Omega \to (1,\infty)$ is a continuous function, the data $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and $F \in (L^{p(x)}(Q))^N$. The objective of this paper is to study the existence of an entropy solution for the obstacle parabolic problems of the type:

\[
\begin{aligned}
&u \geq \psi \quad \text{a.e. in } \Omega \times (0,T), \\
&\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p(x)} - 2\nabla u) + g(u)|\nabla u|^{p(x)} + f - \text{div}F \quad \text{in } \Omega \times (0,T) \\
u(x,0) = u_0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega \times (0,T).
\end{aligned}
\]

The function $g$ is a bounded and continuous positive function that belongs to $L^1(\mathbb{R})$. We recall that the notion of solutions for the problem $(P)$ without obstacle has been proved by Dall’ Algio - Orsina [19] and Porretta [25] with $F=0$ and $-g(u)|\nabla u|^{p(x)} = H(x,t,u,\nabla u)$ and the nonlinearity $g$ satisfying the following "natural" growth condition (of order $p$):

\[
g(x,t,s,\xi) \leq b(s)(|\xi|^p + c(x,t))
\]

and the classical sign condition

\[
g(x,t,s,\xi) \cdot s \geq 0.
\]

In the case $F = 0$ the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui, D. Meskine, A. Souissi in [22] with the penalization methods. Recently, M. Bendahmane, P.

*Corresponding author.

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Wittbold and A. Zimmermann in [14] proved the existence of renormalized solutions, and the existence of renormalized solution in Orlicz spaces has been proved in E. Azroul, H. Redwane and M. Rhoudaf [13].

The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. In Section 3 we make precise all the assumptions on $A, g, f$ and $u_0$, the definition of an entropy solution of $(P)$. In Section 4 we establish the existence of such a solution in (Theorem 3.1).

2. Mathematical preliminaries

Some definitions and basic properties of the generalised Lebesgue–Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in $\Omega$ if

$$|p(x) - p(y)| \leq \frac{C}{\log |x - y|}$$

∀ $x, y \in \Omega$ such that $|x - y| < \frac{1}{2}$, with possible different constant $C$.

Let us set $C_+(\Omega) = \{ p \in C(\Omega) : \min_{x \in \Omega} p(x) > 1 \}$. For any $p \in C_+(\Omega)$, we define

$$p^- = \inf_{x \in \Omega} p(x) \quad \text{and} \quad p^+ = \sup_{x \in \Omega} p(x).$$

For any $p \in C_+(\Omega)$, we introduce the variable exponent Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable with } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxembourg norm

$$||u||_{L^{p(x)}(\Omega)} = ||u||_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|}{\lambda}^{p(x)} dx \leq 1 \right\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, (see[21, 27]). If $p(x)$ is a constant function, then the variable exponent Lebesgue space coincides with the classical Lebesgue space.

We define also the variable Sobolev spaces by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

where the norm is

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, i.e.,

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$$

and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N - p(x)}$ for $p(x) < N$. 
Proposition 2.1. (Young’s Inequality) Let $p, \ p' \in P^+(\Omega)$, where $p'$ the conjugate for all $a, b > 0$, we have:

$$ab \leq \frac{a^p(x)}{p(x)} + \frac{b^{p'(x)}}{p'(x)}.$$ 

Proposition 2.2. (Generalised Hölder inequality) (see [20, 23]).

i) For any functions $u \in L^p(x)(\Omega)$ and $v \in L^{p'}(x)(\Omega)$, we have:

$$\left|\int_{\Omega} uv dx\right| \leq \left(\frac{1}{p(x)} + \frac{1}{p'(x)}\right)||u||_{p(x)}||v||_{p'(x)} \leq 2||u||_{p(x)}||v||_{p'(x)}.$$

ii) For all $p, q \in C_+(\Omega)$ such that $p(x) \leq q(x)$ a.e. in $\Omega$, we have $L^q(x) \hookrightarrow L^p(x)$ and the embedding is continuous.

The modular of the space $L^p(x)(\Omega)$, which is the mapping $\rho : L^p(x)(\Omega) \to \mathbb{R}$ is defined by

$$\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx \text{ for all } u \in L^p(x)(\Omega).$$

The next proposition shows that there is a gap between the modular and the norm in $L^p(x)(\Omega)$.

Proposition 2.3. See ([21, 27]). For $u \in L^p(x)(\Omega)$ and $\{u_k\}_{k \in \mathbb{N}} \subset L^p(x)(\Omega)$ then, the following assertions hold

$$u \neq 0 \Rightarrow ||u||_{p(x)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1$$

$$||u||_{p(x)} > 1 \Rightarrow ||u||_{p(x)}^p \leq \rho(u) \leq ||u||_{p(x)}^{p^-}$$

$$||u||_{p(x)} < 1 \Rightarrow ||u||_{p(x)}^{p^+} \leq \rho(u) \leq ||u||_{p(x)}^-$$

$$\lim_{k \to \infty} ||u_k||_{p(x)} = 0 \Leftrightarrow \lim_{k \to \infty} \rho(u_k) = 0$$

$$\lim_{k \to \infty} ||u_k||_{L^p(x)(\Omega)} = \infty \Leftrightarrow \lim_{k \to \infty} \rho(u_k) = \infty.$$ 

Proposition 2.4. (See [21].)

i) Assuming $1 < p^- \leq p^+ < \infty$ the spaces $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces.

ii) If $q \in C_+(\Omega)$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then $W^{1,p(x)}(\Omega) \hookrightarrow L^q(x)(\Omega)$ is continuous and compact. In particular, we have $W^{1,p(x)}_0(\Omega) \hookrightarrow L^q(x)(\Omega)$ is continuous and compact.

iii) (See [23].) Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set and $p \in C_+(\Omega)$ satisfy the log-Hölder continuity condition (1). Then, for $u \in W^{1,p(x)}_0(\Omega)$, the $p(x)$-Poincaré inequality

$$||u||_{p(x)} \leq C||\nabla u||_{p(x)}$$

holds, where the positive constant $C$ depends on $p(x)$ and $\Omega$.

Remark 2.1. By (iii) of Proposition 2.4 we know that $||\nabla u||_{p(x)}$ and $||u||_{1,p(x)}$ are equivalent norms on $W^{1,p(x)}_0(\Omega)$. 

We will also use the standard notations for Bochner spaces, i.e. if \( q \geq 1 \) and \( X \) is a Banach space then \( L^q(0,T;X) \) denotes the space of strongly measurable functions \( u : (0,T) \rightarrow X \) for which \( t \mapsto \|u(t)\|_X \in L^q(0,T) \). Moreover, \( C([0,T];X) \) denotes the space of continuous functions \( u : [0,T] \rightarrow X \) endowed with the norm \( \|u\|_{C([0,T];X)} = \max_{t \in [0,T]} \|u(t)\|_X \).

Set \( L^p(0,T;W^{1,p(x)}_0(\Omega)) = \{ u : (0,T) \rightarrow W^{1,p(x)}_0(\Omega); \int_0^T \|u(t)\|_{W^{1,p(x)}_0(\Omega)}^p dt \frac{1}{p} < \infty \} \).

We introduce the functional space see [14]

\[
V = \{ f \in L^p(0,T;W^{1,p(x)}_0(\Omega)); |\nabla f| \in L^p(Q) \},
\]

which endowed with the norm

\[
\|f\|_V = \|\nabla f\|_{L^p(Q)},
\]

or, the equivalent norm

\[
\|f\|_V = \|f\|_{L^p(0,T;W^{1,p(x)}_0(\Omega))} + \|\nabla f\|_{L^p(Q)}.
\]

The space \( V \) is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding \( L^p(Q) \hookrightarrow L^p(0,T;L^p(\Omega)) \) and the Poincaré inequality. We state some further properties of \( V \) in the following lemma.

**Lemma 2.1.** (See [14]). Let \( V \) be defined as in (7) and let its dual space be denoted by \( V^* \). Then, we have the following continuous dense embeddings:

\[
L^{p^+}(0,T;W^{1,p(x)}_0(\Omega)) \hookrightarrow V \hookrightarrow L^{p^+}(0,T;W^{1,p(x)}_0(\Omega)).
\]

In particular, since \( D(Q) \) is dense in \( L^{p^+}(0,T;W^{1,p(x)}_0(\Omega)) \), it is dense in \( V \) and for the corresponding dual spaces, we have

\[
L^{(p^+)'}(0,T;(W^{1,p(x)}_0(\Omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;(W^{1,p(x)}_0(\Omega))^*).
\]

Note that, we have the following continuous dense embeddings

\[
L^{p^+}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^{p^+}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^p(0,T;L^{p(x)}(\Omega)).
\]

3. **Assumptions and definition**

Throughout this paper, we assume that the following assumptions hold true.
3.1. Basic assumptions. Let \( p \in C_+(\Omega) \) and assume that \( p(x) \) satisfies the log-Hölder condition (1) with \( 1 < p^- \leq p(x) \leq p^+ < \infty \). The differential operator \( A : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is defined by

\[
Au = -\text{div}(|\nabla u|^{p(x)-2}\nabla u),
\]

is a Leray-Lions operator which is coercive and \( g : \mathbb{R} \rightarrow \mathbb{R}^+ \) is a bounded and continuous positive function that belongs to \( L^1(\mathbb{R}) \), \( \phi : \mathbb{R} \rightarrow \mathbb{R}^N \) is a continuous function,

\[
f \text{is an element of } L^1(Q), u_0 \in L^1(\Omega) \cap K, \ u_0 \geq 0 \text{ and } p \in C_+(\Omega). \tag{11}\]

Let \( \psi \) be a measurable function with values in \( \mathbb{R} \) such that \( \psi \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega) \), (see [22]), \( K \) is defined by: \( K = \{ u \in W^{1,p(x)}_0(\Omega) ; \ u(x) \geq \psi(x) \ \text{a.e. in } \Omega \} \) and consider the convex set

\[
K_\psi = \{ u \in V, \ u(t) \in K \}.
\]

**Theorem 3.1.** Under assumptions (8)-(11), there exists at least one entropy solution of problem \((P)\): Let \( f \in L^1(Q) \) and \( u_0 \in L^1(\Omega) \). A measurable function \( u \) defined on \( Q \) is a unilateral entropy solution of problem \((P)\) if

\[
u \geq \psi \ \text{a.e. in } Q,
\]

\[
T_k(u) \in V, \ \text{for all } k \geq 0 \text{ and } u \in C(0, T; L^1(\Omega)),
\]

\[
\int_\Omega S_k(u-v)(T)dx + \int_Q \frac{\partial v}{\partial t}T_k(u-v)dxdt
\]

\[
+ \int_Q |\nabla u|^{p(x)-2}\nabla u\nabla T_k(u-v)dxdt \leq \int_Q g(u)|\nabla u|^{p(x)}T_k(u-v)dxdt
\]

\[
+ \int_Q fT_k(u-v)dxdt + \int_Q F\nabla T_k(u-v) + \int_\Omega S_k(u_0-v(0))dx,
\]

for all \( v \in K_\psi \cap L^\infty(Q) \), \( \frac{\partial u}{\partial t} \in V^* + L^1(Q) \) and \( \forall k > 0 \),

where \( S_k(s) = \int_0^s T_k(r)dr. \)

**Proof.** The proof is divided into 4 steps.

In Step 1, we introduce an approximate problem. In Step 2, we establish a few a priori estimates which allow us to prove that the approximate solutions \( u_n \) converge to \( u \) a.e. in \( Q \). In Step 3, we define a time regularisation of the field \( T_k(u) \) and prove that \( u_n \) satisfies (24). In this step using some techniques, we also prove the modular convergence of \( T_k(u_n) \) to \( T_k(u) \) in \( L^{p^-}(0, T; W^{1,p(x)}_0(\Omega)) \). In Step 4, we pass to the limit which is the final step to prove Theorem 3.1.
Step 1: The approximate problem. Let us introduce the following regularization of the data:

\[ f_n \in L^{p(x)}(Q), \quad f_n \to f \text{ a.e. in } Q, \text{ and strongly in } L^1(Q) \text{ as } n \to \infty, \quad \text{(15)} \]
\[ u_{0n} \in D(\Omega): \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \text{ and } u_{0n} \to u_0 \text{ in } L^1(\Omega) \text{ as } n \to \infty, \quad \text{(16)} \]

Let us now consider the following regularized approximate problem

\[
\begin{aligned}
(P_n) \left\{ \begin{array}{l}
\frac{\partial u_n}{\partial t} - \text{div}(|\nabla u_n|^{p(x)} - 2\nabla u_n) - nT_n((u_n - \psi)^-) \\
= g(u_n)|\nabla u_n|^{p(x)} + f_n - \text{div}F \quad \text{in } D'(Q),
\end{array} \right.
\end{aligned}
\]

\[
\begin{aligned}
u_n = 0 \quad \text{on } \partial \Omega \times (0, T),
u_n(t = 0) = u_{0n} \quad \text{in } \Omega.
\end{aligned}
\]

Moreover, since \( f_n \in V^* \), proving the existence of weak solution \( u_n \in V \) of \((P_n)\) is an easy task (see [24]).

Step 2: A Priori estimates. The estimate derived in this step rely on standard techniques for problems of the type \((P_n)\).

**Proposition 3.1.** Assume that (8)-(11) hold true and let \( u_n \) be a solution of the approximate problem \((P_n)\). Then for all \( k > 0 \), we have

\[ \|T_k(u_n)\|_{L^{p(x)}(0, T; W^{1, p(x)}(\Omega))} \leq C \quad \text{for all } n \in \mathbb{N}, \]

where \( C \) is a constant independent of \( n \).

**Proof.** Let \( h > k > 0 \) and consider the test function \( \varphi = T_h(u_n - T_k(u_n)) \exp(G(u_n)) \in V \cap L^\infty(Q) \) in the approximate problem \((P_n)\), where \( G(s) = \int_0^s g(r)dr \), we have

\[
\begin{aligned}
\left\langle \frac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n)) \exp(G(u_n)) \right\rangle + \int_{\{k \leq |u_n| \leq k + h\}} (|\nabla u_n|^{p(x)} - 2\nabla u_n) \nabla u_n \exp(G(u_n)) dx dt & \\
+ \int_Q (|\nabla u_n|^{p(x)} - 2\nabla u_n) \nabla u_n T_h(u_n - T_k(u_n)) g(u_n) \exp(G(u_n)) dx dt & \\
- T_k(u_n) \exp(G(u_n)) dx dt & \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) dx dt & \\
= \int_Q g(u_n)|\nabla u_n|^{p(x)} T_h(u_n - T_k(u_n)) \exp(G(u_n)) dx dt & \\
+ \int_Q f_n T_h(u_n - T_k(u_n)) \exp(G(u_n)) dx dt & \\
+ \int_Q F \nabla T_h(u_n - T_k(u_n)) \exp(G(u_n)) dx dt.
\end{aligned}
\]
Then,
\[
\int_{\{k \leq |u_n| \leq k+\}} |\nabla u_n|^p(x) \exp(G(u_n)) \, dx dt \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx dt \\
= \int_Q f_n T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx dt \\
+ \int_Q F \nabla \left( T_h(u_n - T_k(u_n)) \right) \exp(G(u_n)) \, dx dt.
\]

On the one hand, we have
\[
\int_{\{k \leq |u_n| \leq k+\}} |\nabla u_n|^p(x) \exp(G(u_n)) \, dx dt \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx dt \\
\leq h \exp(||g||_{L^1(\mathbb{R})}) \left[ ||f_n||_{L^1(Q)} + ||u_{0n}||_{L^1(\Omega)} \right] \\
+ \int_Q F T_h(u_n - T_k(u_n)) \nabla u_n g(u_n) \exp(G(u_n)) \, dx dt \\
+ \int_{\{k \leq |u_n| \leq k+\}} F \left[ \exp(G(u_n)) \right]^{1-\frac{1}{p(x)}} \left[ \exp(G(u_n)) \right]^{\frac{1}{p(x)}} |\nabla u_n| \, dx dt,
\]

hence
\[
\int_{\{k \leq |u_n| \leq k+\}} |\nabla u_n|^p(x) \exp(G(u_n)) \, dx dt \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx dt \\
\leq h \exp(||g||_{L^1(\mathbb{R})}) \left[ ||f_n||_{L^1(Q)} + ||u_{0n}||_{L^1(\Omega)} \right] \\
+ \int_Q F T_h(u_n - T_k(u_n)) \nabla u_n g(u_n) \exp(G(u_n)) \, dx dt \\
+ \int_{\{k \leq |u_n| \leq k+\}} \left[ \frac{1}{2} p(x) \right]^{\frac{1}{p(x)}} |\nabla u_n| \left[ \exp(G(u_n)) \right]^{\frac{1}{p(x)}} \, dx dt.
\]
and by Young’s inequality, we have

\[
\int_{\{k \leq |u_n| \leq k + h\}} |\nabla u_n|^{p(x)} \exp(G(u_n)) dxdt \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) dxdt \\
\leq h \exp(\|g\|_{L^1(\mathbb{R})}) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] \\
+ \int_Q F\left(T_h\left(u_n - T_k(u_n)\right) \nabla u_n g(u_n) \exp(G(u_n))\right) dxdt \\
+ \frac{1}{2} \int_{\{k \leq |u_n| \leq k + h\}} |\nabla u_n|^{p(x)} \exp(G(u_n)) dxdt \\
+ \frac{1}{2} \int_{\{k \leq |u_n| \leq k + h\}} \frac{|F|^{p'(x)} \exp(G(u_n))}{p'(x) \left(\frac{1}{2} p(x)\right)} dxdt \\
\leq h \exp(\|g\|_{L^1(\mathbb{R})}) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] \\
+ \int_Q F\left(T_h\left(u_n - T_k(u_n)\right) \nabla u_n g(u_n) \exp(G(u_n))\right) dxdt \\
+ \frac{1}{2} \int_{\{k \leq |u_n| \leq k + h\}} |\nabla u_n|^{p(x)} \exp(G(u_n)) dxdt \\
+ C \int_{\{k \leq |u_n| \leq k + h\}} |F|^{p'(x)} \exp(G(u_n)) dxdt,
\]

then

\[
\frac{1}{2} \int_{\{k \leq |u_n| \leq k + h\}} |\nabla u_n|^{p(x)} \exp(G(u_n)) dxdt \\
- \int_Q nT_n((u_n - \psi)^-) T_h(u_n - T_k(u_n)) \exp(G(u_n)) dxdt \\
\leq h \exp(\|g\|_{L^1(\mathbb{R})}) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] + C_1 \\
+ \int_Q F\left(T_h\left(u_n - T_k(u_n)\right) \nabla u_n g(u_n) \exp(G(u_n))\right) dxdt.
\]
Let us observe that if we take \( \varphi = \rho(u_n) = \int_0^{u_n} g(s) \chi_{\{k \leq |s| \leq k+h\}} ds \exp(G(u_n)) \) a test function in the approximate \((P_n)\), we obtain

\[
\left[ \int_\Omega \varphi_1(u_n) dx \right]_0^T + \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt \\
- \int_Q nT_n((u_n - \psi)^-) \rho(u_n) dx dt \\
\leq \left( \int_0^\infty g(s) ds \right) \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \|f_n\|_{L^1(Q)} \\
+ \int_Q F\nabla u_n g(u_n) \chi_{\{k \leq |u_n| \leq k+h\}} \exp(G(u_n)) dx dt \\
+ \left( \int_0^\infty g(s) ds \right) \int_Q |F\nabla u_n| g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} dx dt,
\]

where \( \varphi_1(r) = \int_0^r \rho(s) ds \), which implies, using \( \varphi_1(r) \geq 0 \), we obtain

\[
\int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt \\
- \int_Q nT_n((u_n - \psi)^-) \rho(u_n) dx dt \\
\leq \left( \int_0^\infty g(s) ds \right) \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \|f_n\|_{L^1(Q)} \\
+ \int_Q F\nabla u_n g(u_n) \chi_{\{k \leq |u_n| \leq k+h\}} \exp(G(u_n)) dx dt \\
+ \left( \int_0^\infty g(s) ds \right) \int_Q |F\nabla u_n| g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} dx dt,
\]

then

\[
\int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt - \int_Q nT_n((u_n - \psi)^-) \rho(u_n) dx dt
\]
\[ \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} |g(u_n)\exp(G(u_n))| dxdt - \int_Q nT_n((u_n - \psi^{-})\rho(u_n)) dxdt \]

\[ \leq \|g\|_\infty \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] + \int_Q \frac{|F|^{p'(x)} g(u_n) \exp(G(u_n))}{\frac{p'(x)}{2} p(x)} dxdt \]

\[ + \frac{1}{2} \int_Q |\nabla u_n|^{p(x)} |g(u_n)\exp(G(u_n))| \chi_{\{k \leq |u_n| \leq k+h\}} dxdt \]

\[ + \|g\|_\infty \int_Q \frac{|F|^{p'(x)} g(u_n) \exp(G(u_n))}{\frac{p'(x)}{2} p(x)} dxdt \]

\[ + \frac{1}{2} \|g\|_\infty \int_Q |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} dxdt, \]

Since \( g \) is bounded function, then we have

\[ \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dxdt - \int_Q nT_n((u_n - \psi^{-})\rho(u_n)) dxdt \]

\[ \leq \|g\|_\infty \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] + C'\|g\|_\infty \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \int_Q |F|^{p'(x)} dxdt \]

\[ + \frac{1}{2} \int_Q |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} dxdt \]

\[ + \|g\|_\infty C' \|g\|_{L^1(\mathbb{R})} \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \int_Q |F|^{p'(x)} dxdt \]

\[ + \frac{1}{2} \|g\|_\infty \int_Q |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} dxdt, \]
then

\[
\int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt - \int_Q nT_n((u_n - \psi)^-) \rho(u_n) \, dx \, dt \\
\leq \|g\|_\infty \exp \left(\|g\|_{L^1(\mathbb{R})}\right) \left[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right] \\
+ C' \|g\|_\infty \exp \left(\|g\|_{L^1(\mathbb{R})}\right) \left[1 + \|g\|_\infty\right] \int_Q |F|^{p'(x)} \, dx \, dt \\
+ \frac{1}{2} \int_Q |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} \, dx \, dt \\
+ \frac{1}{2} \|g\|_\infty \int_Q |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \chi_{\{k \leq |u_n| \leq k+h\}} \, dx \, dt,
\]

and since

\[
\int_Q |F|^{p'(x)} \, dx \, dt = \rho(F) \leq \max \left\{||F|^{p'_{(1)}(Q)}|^p_{(1)}(Q), |F|^{p'_{(2)}(Q)}|^p_{(2)}(Q)\right\} = C''
\]

then, we have

\[
\int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt \\
- \int_Q nT_n((u_n - \psi)^-) \rho(u_n) \, dx \, dt \\
\leq \|g\|_\infty \left(\exp(\|g\|_{L^1(\mathbb{R})})\right) \left[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right] \\
+ C_2 + \frac{1}{2} \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt \\
+ \frac{1}{2} \|g\|_\infty \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt
\]

where \(C_2 = C' \|g\|_\infty \exp(\|g\|_{L^1(\mathbb{R})}) \left[1 + \|g\|_\infty\right] C''\)

which give

\[
C_3 \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt - \int_Q nT_n((u_n - \psi)^-) \rho(u_n) \, dx \, dt \leq C_4,
\]

where \(C_3 = 1 - \frac{1}{2}(1 + \|g\|_\infty)\), then

\[
C_3 \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p(x) g(u_n) \exp(G(u_n)) \, dx \, dt \leq C_4 + \int_Q nT_n((u_n - \psi)^-) \rho(u_n) \, dx \, dt \\
\leq C_4 + (h+k) \|g\|_\infty (\exp(\|g\|_{L^1(\mathbb{R})}) \int_Q nT_n((u_n - \psi)^-) \, dx \, dt \\
\leq C_4 + hC_5 \int_Q nT_n((u_n - \psi)^-) \, dx \, dt
\]

with \(C_5 = 2\|g\|_\infty (\exp(\|g\|_{L^1(\mathbb{R})})\)

so, we obtain
\[
\int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt
\leq hC_6 \int_Q nT_n((u_n - \psi)^-) dx dt
\]
with \(C_6 = \max\left(\frac{C_4}{C_3}, \frac{C_5}{C_3}\right)\).

Let us take \(\rho_1(u_n) = \int_0^{u_n} g(s) \chi_{\{|s| \leq k\}} ds \exp(G(u_n))\) a test function in the approximate \((\mathcal{P}_n)\), we obtain
\[
\left[\int_{\Omega} \varphi_2(u_n) dx\right]^T + \int_Q |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{|u_n| \leq k\}} dx dt
- \int_Q nT_n((u_n - \psi)^-) \rho_1(u_n) dx dt
\leq \left( \int_0^{\infty} g(s) ds \right) \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \|f_n\|_{L^1(Q)}
+ \int_Q F \nabla u_n g(u_n) \chi_{\{|u_n| \leq k\}} \exp(G(u_n)) dx dt
+ \left( \int_0^{\infty} g(s) ds \right) \int_Q |F \nabla u_n| g(u_n) \exp(G(u_n)) \chi_{\{|u_n| \leq k\}} dx dt,
\]
where \(\varphi_2(r) = \int_0^r \rho_1(s) ds\), which implies, using \(\varphi_2(r) \geq 0\), by using Young’s Inequality, we obtain
\[
\int_{\{|u_n| \leq k\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt
\leq \|g\|_\infty \exp \left( \|g\|_{L^1(\mathbb{R})} \right) \left[ \|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \right] + C_9
+ \frac{1}{2} \int_{\{|u_n| \leq k\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt
+ \frac{1}{2} \|g\|_\infty \int_{\{|u_n| \leq k\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt,
\]
then
\[
C_{10} \int_{\{|u_n| \leq k\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt
\leq hC_{11} \int_Q nT_n((u_n - \psi)^-) dx dt
\]
Similarly, taking \(\rho_2 = \int_0^{u_n} g(s) \chi_{\{|s| \geq k+h\}} ds \exp(G(u_n))\) as a test function in the approximate \((\mathcal{P}_n)\), we conclude that
\[
C_{12} \int_{\{|u_n| \geq k+h\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt
\leq hC_{13} \int_Q nT_n((u_n - \psi)^-) dx dt
\]
Conclusion, we have:

\[
\begin{aligned}
C_{14} \int_{Q} |\nabla u_n|^p g(u_n) \exp(G(u_n)) \, dx \, dt & \leq C_{12} \int_{\{|u_n| \geq k+h\}} |\nabla u_n|^p g(u_n) \exp(G(u_n)) \, dx \, dt \\
& + C_{10} \int_{\{|u_n| \leq k\}} |\nabla u_n|^p g(u_n) \exp(G(u_n)) \, dx \, dt \\
& + C_{5} \int_{\{k \leq |u_n| \leq k+h\}} |\nabla u_n|^p g(u_n) \exp(G(u_n)) \, dx \, dt \\
& \leq hC_{15} \int_{Q} nT_n((u_n - \psi^-) \, dx \, dt + hC_7,
\end{aligned}
\]

where \( C_{14} = \min(C_5, C_{10}, C_{12}) \) and \( C_{15} = \max(C_9, C_{11}, C_{13}) \)

using (17), we have

\[
\int_{Q} |\nabla u_n|^p g(u_n) \exp(G(u_n)) \, dx \, dt \leq - \int_{Q} nT_n((u_n - \psi^-)T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx \, dt
\]

we obtain

\[- \int_{Q} nT_n(u_n - \psi^-)T_h(u_n - T_k(u_n)) \exp(G(u_n)) \, dx \, dt \leq hC_6 \int_{Q} nT_n(u_n - \psi^-) \, dx \, dt + hC_7
\]

so, that

\[- \int_{Q} nT_n(u_n - \psi^-) \frac{T_h(u_n - T_k(u_n))}{h} \exp(G(u_n)) \, dx \, dt \leq C_6 \int_{Q} nT_n(u_n - \psi^-) \, dx \, dt + C_7.
\]

Let us now fix \( k > \|\psi\|_\infty \), by the fact that

\[nT_n(u_n - \psi^-)(u_n - k)\chi_{\{u_n \leq \psi^-; k \leq u_n \leq k+h\}} \geq 0\]

and letting \( h \to 0 \), one has

\[
\int_{Q} nT_n(u_n - \psi^-) \, dx \, dt \leq C_8. \tag{18}
\]

Let use \( v = T_k(u_n)^+ \exp(G(u_n))\chi(0, \tau) \) as a test function in \( (P_n) \)

\[
\left[ \int_{\Omega} \varphi_4(u_n) \, dx \right]_0^T + \int_{Q^r} (|\nabla u_n|^p - 2\nabla u_n) \nabla T_k(u_n)^+ \exp(G(u_n)) \, dx \, dt
\]

\[+ \int_{Q^r} |\nabla u_n|^p T_k(u_n)^+ g(u_n) \exp(G(u_n)) \, dx \, dt
\]

\[- \int_{Q^r} nT_n((u_n - \psi^-)T_k(u_n)^+ \exp(G(u_n)) \, dx \, dt
\]

\[= \int_{Q^r} g(u_n)|\nabla u_n|^p T_k(u_n)^+ \exp(G(u_n)) \, dx \, dt
\]

\[+ \int_{Q^r} f_nT_k(u_n)^+ \exp(G(u_n)) \, dx \, dt
\]

\[+ \int_{Q^r} FT_k(u_n)^+ \nabla u_n g(u_n) \exp(G(u_n)) \, dx \, dt
\]

\[+ \int_{Q^r} F\nabla T_k(u_n)^+ \exp(G(u_n)) \, dx \, dt
\]
where $\varphi_4(r) = \int_0^r T_k(s)^+ \exp(G(s)) ds$.

Due the definition of $\varphi_4$ and $|G(u_n)| \leq \exp(\|g\|_{L^1(\mathbb{R})}) \|u_0\|_{L^1(\Omega)}$, we have

$0 \leq \int_\Omega \varphi_4(u_{0n}) dx \leq k \exp(\|g\|_{L^1(\mathbb{R})}) \|u_0\|_{L^1(\Omega)}$. By using (18) then,

$$
\int_{Q^r} |\nabla T_k(u_n) + p(x) \exp(G(u_n))| dxdt \\
\leq \int_{Q^r} |\nabla T_k(u_n) + p(x) \exp(G(u_n))| dxdt \\
\leq \frac{1}{2} \int_{Q^r} |\nabla T_k(u_n) + p(x) \exp(G(u_n))| dxdt \\
+ \frac{1}{2} \|g\|_{\infty} \int_{Q^r} |\nabla T_k(u_n) + p(x) \exp(G(u_n))| dxdt.
$$

Let us take $\rho_5(u_n) = \int_0^{u_n} g(s) \chi_{\{s \geq 0\}} ds \exp(G(u_n))$ a test function in the approximate $(P_n)$, we obtain

$$
\left[ \int_{\Omega} \varphi_5(u_n) dx \right]_0^T + \int_{Q} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{u_n \geq 0\}} dxdt \\
- \int_{Q} nT_n((u_n - \psi)^-) \rho_5(u_n) dxdt \\
\leq \left( \int_0^\infty g(s) ds \right) \exp(\|g\|_{L^1(\mathbb{R})}) \|f_n\|_{L^1(\Omega)} \\
+ \int_{Q} F \nabla u_n g(u_n) \chi_{\{u_n \geq 0\}} \exp(G(u_n)) dxdt \\
+ \left( \int_0^\infty g(s) ds \right) \int_{Q} |F \nabla u_n| g(u_n) \exp(G(u_n)) \chi_{\{u_n \geq 0\}} dxdt,
$$

where $\varphi_5(r) = \int_0^r \rho_5(s) ds$, which implies, using $\varphi_5(r) \geq 0$, by using Young’s Inequality, we obtain

$$
\int_{\{u_n \geq 0\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\
\leq \|g\|_{\infty} \exp(\|g\|_{L^1(\mathbb{R})}) \|f_n\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)} + C_{16} \\
+ \frac{1}{2} \int_{\{u_n \geq 0\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\
+ \frac{1}{2} \|g\|_{\infty} \int_{\{u_n \geq 0\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dxdt,
$$

then

$$
\int_{\{u_n \geq 0\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \leq C_{17}
$$
Similarly, taking \( p_0 = \int_{u_n}^0 g(s)\chi_{\{s \leq 0\}}ds \exp(G(u_n)) \) as a test function in the approximate \((P_n)\), we conclude that
\[
\int_{\{u_n \leq 0\}} |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt \leq C_{18}
\]
Consequently,
\[
\int_Q |\nabla u_n|^{p(x)} g(u_n) \exp(G(u_n)) dx dt \leq C_{19}.
\]
Above, \( C_1, \ldots, C_{19} \) are constants independent of \( n \), we deduce that
\[
\int_Q |\nabla k(u_n)^+|^{p(x)} dx dt \leq k C_{20}.
\]
Similarly to (19), we take \( \varphi = T_k(u_n)^- \chi(0, \tau) \) in \((P_n)\) to deduce that
\[
\int_Q |\nabla k(u_n)^-|^{p(x)} dx dt \leq k C_{21}.
\]
Combining (19), (20) and Proposition 2.3, we conclude that
\[
\int_0^T \min \left\{ \| \nabla T_k(u_n) \|_{p(x)}^{p^+}, \| \nabla T_k(u_n) \|_{p(x)}^{p^-} \right\} dt \leq \rho(\nabla T_k(u_n)) \leq k C_{22}.
\]
Then, we conclude that \( T_k(u_n) \) is bounded in \( L^{p^+}(0, T; W^{1,p(x)}_0(\Omega)) \), independently of \( n \) for any \( k > 0 \). Now we turn to proving the almost everywhere convergence of \( u_n \).
Consider a non decreasing function \( g_k \in C^2(\mathbb{R}) \) such that
\[
g_k(s) = \begin{cases} 
    s & \text{if } |s| \leq \frac{k}{2} \\
    k & \text{if } |s| \geq k.
\end{cases}
\]
Multiplying the approximate equation by \( g'_k(u_n) \), we get
\[
\frac{\partial g_k(u_n)}{\partial t} - \text{div} \left( |\nabla u_n|^{p(x)-2}\nabla u_n g'_k(u_n) \right) + |\nabla u_n|^{p(x)} g''_k(u_n)

- nT_k(u_n - \psi)^- \frac{\partial g_k(u_n)}{\partial t}

= g(u_n)|\nabla u_n|^{p(x)} g'_k(u_n) + f_n g_k(u_n) - \text{div}(F g'_k(u_n)) + F \nabla u_n g''_k(u_n) \tag{21}
\]
in the sense of distributions. This implies, thanks to the fact that \( g'_k \) has compact support, that \( g_k(u_n) \) is bounded in \( L^{p^+}(0, T; W^{1,p(x)}_0(\Omega)) \), while its time derivative \( \frac{\partial g_k(u_n)}{\partial t} \) is bounded in \( L^1(Q) + V^* \). Due to the choice of \( g_k \), we conclude that for each \( k \), the sequence \( T_k(u_n) \) converges almost everywhere in \( Q \), which implies that the sequence \( u_n \) converge almost everywhere to some measurable function \( v \) in \( Q \). Thus by using the same argument as in [16], [17], [18], we can show the following lemma.
Lemma 3.2. Let \( u_n \) be a solution of the approximate problem \( (P_n) \). Then,
\[
    u_n \to u \quad \text{a.e. in } Q. \tag{22}
\]
We can deduce from (19) that
\[
    T_k(u_n) \to T_k(u) \quad \text{in} \quad L^p(0, T; W_0^{1,p(x)}(\Omega)). \tag{23}
\]

Lemma 3.3. [12] Let \( u_n \) be a solution of the approximate problem \( (P_n) \). Then,
\[
    \lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla(u_n)) \nabla u_n dx dt = 0. \tag{24}
\]

Step 3: Almost every convergence of the gradients:
This step is devoted to introducing for a fixed \( k \geq 0 \) fixed, a time regularization of the function \( T_k(u) \) in order to perform the monotonicity method.
This specific time regularization of \( T_k(u) \) (for fixed \( k \geq 0 \)) is defined as follows.
Let \((v_0^\mu)\) be a sequence of functions defined on \( \Omega \) such that
\[
    v_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega) \quad \text{for all } \mu > 0, \tag{25}
\]
\[
    \|v_0^\mu\|_{L^\infty(\Omega)} \leq k \quad \text{for all } \mu > 0, \tag{26}
\]
\[
    v_0^\mu \to T_k(u_0) \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{1}{\mu} \|v_0^\mu\|_{L^p(\Omega)} \to 0, \quad \text{as } \mu \to \infty. \tag{27}
\]
For fixed \( k, \mu > 0 \), let us consider the unique solution \((T_k(u))_\mu \in L^\infty(Q) \cap L^p(0, T; W_0^{1,p(x)}(\Omega))\) of the monotone problem:
\[
    \frac{\partial (T_k(u))_\mu}{\partial t} + \mu ( (T_k(u))_\mu - T_k(u) ) = 0 \quad \text{in } D'(Q), \tag{28}
\]
\[
    (T_k(u))_\mu(t = 0) = v_0^\mu \quad \text{in } \Omega. \tag{29}
\]
Note that due to (28), we have for \( \mu > 0 \) and \( k \geq 0 \),
\[
    \frac{\partial (T_k(u))_\mu}{\partial t} \in L^p(0, T; W_0^{1,p(x)}(\Omega)). \tag{30}
\]
We just recall here that (28)–(29) imply that
\[
    (T_k(u))_\mu \to T_k(u) \quad \text{a.e. in } Q, \tag{31}
\]
as well as weakly in \( L^\infty(Q) \) and strongly in \( L^p(0, T; W_0^{1,p(x)}(\Omega)) \) as \( \mu \to \infty. \)
Note that for any \( \mu \) and any \( k \geq 0 \), we have
\[
    \|(T_k(u))_\mu\|_{L^\infty(Q)} \leq \max \left( \|T_k(u)\|_{L^\infty(Q)}; \|v_0^\mu\|_{L^\infty(\Omega)} \right) \leq k. \tag{32}
\]
We introduce a sequence of increasing \( C^\infty(\mathbb{R}) \)–functions \( S_m \) such that
\[
    S_m(r) = r \quad \text{for } |r| \leq m, \quad \text{supp}(S_m') \subset [-(m + 1), m + 1], \quad \|S_m''\|_{L^\infty(\mathbb{R})} \leq 1,
\]
for any \( m \geq 1 \), and we denote by \( \omega(n, \mu, \eta, m) \) the quantities such that
\[
    \lim_{m \to \infty} \lim_{\eta \to 0} \lim_{\mu \to \infty} \lim_{n \to \infty} \omega(n, \mu, \eta, m) = 0.
\]
Lemma 3.4. ([12, 18]). We have
\[
\int_0^T \left( \frac{\partial u_n}{\partial t} T_n\left(u_n - (T_k(u))_\mu^+\right) \exp(G(u_n)) S_m'(u_n) \right) dt \geq \omega(n, \mu, \eta) \quad \forall m \geq 1.
\] (33)

Taking now \( v = T_\eta\left(u_n - (T_k(u))_\mu^+\right) S_m'(u_n) \exp(G(u_n)) \), in \((P_n)\) and by adapting the same way as in \([1, 11, 10, 9, 8, 6]\), we obtain
\[
\int_Q \left( \left[ |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u) \right] \times \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \right)^\theta \, dx \, dt = w(n)
\] (34)

which implies that
\[
T_k(u_n) \to T_k(u) \quad \text{in} \quad L^p(0, T; W^{1,p(x)}_0(\Omega)) \quad \forall k \geq 0.
\] (35)

By using \([16, 17]\), there exist a subsequence also denoted by \( u_n \) such that
\[
\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q.
\] (36)

Proposition 3.2. Let \( u_n \) be a solution of the approximate problem \((P_n)\).
Then \( u \geq \psi \) a.e. in \( Q \).

Proof. Thanks to (18), we can write \( \int_Q T_n(u_n - \psi)^- \, dx \, dt \leq \frac{C}{n} \). And by using Fatou’s Lemma as \( n \to \infty \), we have that \( \int_Q (u - \psi)^- \, dx \, dt \) converges to zeros, we get \( (u - \psi)^- = 0 \) a.e. in \( Q \). Consequently we conclude that \( u \geq \psi \) a.e. in \( Q \).

Step 4: Passing to the limit

a) We claim that \( u \in C(0, T; L^1(\Omega)) \). We will show that
\[
u_n \to u \quad \text{in} \quad C(0, T; L^1(\Omega)).
\]
Since \( T_k(u) \in K_\psi \), for every \( k \geq ||\psi||_{L^\infty} \) there exists a sequence \( v_j \in K_\psi \cap D(\bar{Q}) \) such that
\[
v_j \to T_k(u) \quad \text{in} \quad L^p(0, T; W^{1,p(x)}_0(\Omega))
\]
for the modular convergence.

Let \( \omega_{j,\mu}^{i,\ell} = (T_i(v_j))_\mu + e^{-\mu}T_i(\eta_i) \) with \( \eta_i \geq 0 \) converge to \( u_0 \) in \( L^1(\Omega) \), where \( (T_i(v_j))_\mu \) is the mollification of \( T_i(v_j) \) with respect to time. Choosing now \( v = T_k(u_n - \omega_{j,\mu}^{i,\ell}e^{0,\tau}) \) as test function in \((P_n)\), we get
\[
\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,\ell}) \right\rangle_{Q^\tau} + \int_{Q^\tau} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,\ell}) \, dx \, dt
\]
\[
- \int_{Q^\tau} nT_n(u_n - \psi)^- T_k(u_n - \omega_{j,\mu}^{i,\ell}) \, dx \, dt
\]
\[
= \int_{Q^\tau} g(u_n) |\nabla u_n|^{p(x)} T_k(u_n - \omega_{j,\mu}^{i,\ell}) \, dx \, dt + \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,\ell}) \, dx \, dt
\]
\[
+ \int_{Q^\tau} F\nabla T_k(u_n - \omega_{j,\mu}^{i,\ell}) \, dx \, dt.
\]
By using the fact that
\[-\int_{Q^r} n T_n(u_n - \psi) - T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt \geq 0\]
we deduce that:
\[
\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^r} + \int_{Q^r} |\nabla u_n|^{p(x) - 2} \nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt
\]
\[
= \int_{Q^r} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt + \int_{Q^r} f_n T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt
\]
\[+ \int_{Q^r} F \nabla T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt.\]

On the one hand, we have
\[
I = \int_{Q^r} |\nabla u_n|^{p(x) - 2} \nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt
\]
\[
= \int_{\{|T_k(u_n) - \omega_{j,\mu}^{i,l}| \leq k\}} |\nabla T_k(u)|^{p(x) - 2} \nabla T_k(u) \left[ \nabla T_k(u) - \nabla \omega_{j,\mu}^{i,l} \right] dx dt + \epsilon(n)
\]
and let \( \mu \) tend to \( \infty \). Then,
\[
I = \int_{\{|T_k(u) - T_l(v_j)| \leq k\}} |\nabla T_k(u)|^{p(x) - 2} \nabla T_k(u) \left[ \nabla T_k(u) - \nabla T_l(v_j) \right] dx dt + \epsilon(n, \mu),
\]
finally \( j \) tend to \( \infty \), we have
\[I = \epsilon(n, \mu, j, l).\]

On the other hand, we have
\[
J = \int_{Q^r} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt. \tag{39}
\]
In the following, we pass to the limit in (39): first we let \( n \) tend to \( \infty \), since
\[g(u_n)|\nabla u_n|^{p(x)} \to g(u)|\nabla u|^{p(x)} \quad \text{in} \ L^1(Q),\]
we obtain \( J = \int_{Q^r} g(u)|\nabla u|^{p(x)} T_k(u - \omega_{j,\mu}^{i,l}) dx dt + \epsilon(n) \) and let \( \mu \) tend to \( \infty \) and \( j \to \infty \), we have
\[J = \epsilon(n, \mu, j, l).\]
Similarly to (39) and by using (15), we have
\[
\int_{Q^r} f_n [T_k(u_n - \omega_{j,\mu}^{i,l})] dx dt = \epsilon(n, \mu, j, l)
\]
and we have
\[ \int_{Q^r} F \nabla \left[ T_k(u_n - \omega_{j,\mu}^{i,l}) \right] dx dt = \epsilon(n, \mu, j, l) \]
and by using Vitali’s theorem, we get
\[ \limsup_{k \to \infty} \limsup_{i \to 0} \limsup_{j \to \infty} \limsup_{\mu \to \infty} \lim_{n \to \infty} \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^r} \leq 0. \] (40)
We have (see ([1]))
\[ \left\langle \frac{\partial \omega_{j,\mu}^{i,l}}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^r} = \mu \int_{Q^r} (T_k(v_j) - \omega_{j,\mu}^{i,l}) T_k(u_n - \omega_{j,\mu}^{i,l}) \geq \epsilon(n, j, \mu, l). \] (41)
uniformly on \( \tau \). Therefore, by writing
\[ \int_{\Omega} S_k \left( u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau) \right) dx = \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^r} - \left\langle \frac{\partial \omega_{j,\mu}^{i,l}}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^r} + \int_{\Omega} S_k \left( u_n(0) - T_l(\eta_i) \right) dx \] (42)
and using (40) and (41) and (42), we see that
\[ \int_{\Omega} S_k \left( u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau) \right) dx \leq \epsilon(n, j, \mu, l), \] (43)
which implies, by writing
\[ \int_{\Omega} S_k \left( \frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \frac{1}{2} \left( \int_{\Omega} S_k \left( u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau) \right) dx \right) + \int_{\Omega} S_k \left( u_m(\tau) - \omega_{j,\mu}^{i,l}(\tau) \right) dx \] (44)
that
\[ \int_{\Omega} S_k \left( \frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \epsilon_1(n, m). \]
We deduce then that
\[ \int_{\Omega} |u_n(\tau) - u_m(\tau)| dx \leq \epsilon_2(n, m), \text{ not depending on } \tau \] (45)
and thus \((u_n)\) is a Cauchy sequence in \(C(0, T; L^1(\Omega))\) and since \(u_n \to u\), a.e. in \(Q\), we deduce that
\[ u_n \to u \text{ in } C(0, T; L^1(\Omega)). \] (46)

b) we prove that \(u\) satisfies (14)
Indeed, let $v \in K_{\psi} \cap L^\infty(Q), \frac{\partial v}{\partial t} \in L^{(p^-)'}(0, T; W^{-1,p}(\Omega))$. By the pointwise multiplication of the approximate problem $(P_n)$ by $T_k(u_n - v)$, we get

$$
\int_\Omega S_k\left(u_n(T) - v(T)\right)dx - \int_\Omega S_k\left(u_0 - v(0)\right)dx
+ \int_\Omega \frac{\partial v}{\partial t} T_k(u_n - v)dxdt + \int_\Omega (|\nabla u|^{p(x)-2}\nabla u) \nabla T_k(u_n - v)dxdt
- \int_\Omega \phi_n(u_n) \nabla T_k(u_n - v)dxdt
- \int_\Omega nT_n(u_n - \psi)^- T_k(u_n - v)dxdt
= \int_\Omega g(u_n) |\nabla u_n|^{p(x)} T_k(u_n - v)dxdt
+ \int_\Omega f_n T_k(u_n - v)dxdt + \int_\Omega F \nabla T_k(u_n - v)dxdt,
$$

where $S_k(s) = \int_0^s T_k(r)dr$.

Since $v \in K_{\psi} \cap L^\infty(Q)$, we have $-\int_\Omega nT_n(u_n - \psi)^- T_k(u_n - v)dxdt \geq 0$, we deduce that

$$
\int_\Omega S_k\left(u_n(T) - v(T)\right)dx - \int_\Omega S_k\left(u_0 - v(0)\right)dx + \int_\Omega \frac{\partial v}{\partial t} T_k(u_n - v)dxdt
+ \int_\Omega (|\nabla u|^{p(x)-2}\nabla u) \nabla T_k(u_n - v)dxdt - \int_\Omega \phi_n(u_n) \nabla T_k(u_n - v)dxdt
\leq \int_\Omega g(u_n) |\nabla u_n|^{p(x)} T_k(u_n - v)dxdt
+ \int_\Omega f_n T_k(u_n - v)dxdt + \int_\Omega F \nabla T_k(u_n - v)dxdt.
$$

\((47)\)

- Let us pass to the limit with $n \to \infty$ in each term in \((47)\). we saw that $u_n \to u$ in $C(0, T, L^1(\Omega))$. Therefore $u_n(t) \to u(t)$ in $L^1(\Omega)$ for all $t \leq T$.

As $S_k$ is lipschitz of coefficient $k$, when $n \to \infty$, we have

$$
\int_\Omega S_k(u_n - v)(T)dx \to \int_\Omega S_k(u - v)(T)dx
$$

and

$$
\int_\Omega S_k(u_n - v)(0)dx = \int_\Omega S_k(u_0 - v(0))dx \to \int_\Omega S_k(u_0 - v(0))dx.
$$

- Since $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0, T; W^{-1,p}(\Omega))$, that is

$$
\int_0^T \left< \frac{\partial v}{\partial t}, T_k(u_n - v) \right> dt \to \int_0^T \left< \frac{\partial v}{\partial t}, T_k(u - v) \right> dt.
$$
Let us pass to the limit with $n \to \infty$ for the term $\int_Q (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt$.

Since $v \in L^\infty(Q)$, we note $M = \|v\|_\infty$, we get

$$
\int_Q (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt
= \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla T_k(T_{k+M}(u_n) - v) \, dx \, dt
= \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla T_{k+M}(u_n) \, dx \, dt
- \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla v \chi_{\{|T_{k+M}(u_n) - v| \leq k\}} \, dx \, dt.
$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$, $\nabla u_n \to \nabla u$ a.e. in $Q$,

$$
\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u) \text{ almost everywhere}
$$

and by using Lebesgue theorem, we deduce that

$$
\int_Q (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla T_{k+M}(u_n) \chi_{\{|T_{k+M}(u_n) - v| \leq k\}} \, dx \, dt
\to \int_Q (|\nabla u|^{p(x)-2} \nabla u) \chi_{\{|T_{k+M}(u) - v| \leq k\}} \, dx \, dt
$$
and

$$
\int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n) \nabla v \chi_{\{|T_{k+M}(u_n) - v| \leq k\}} \, dx \, dt \to
\int_0^T \int_\Omega (|\nabla u|^{p(x)-2} \nabla u) \nabla v \chi_{\{|T_{k+M}(u) - v| \leq k\}} \, dx \, dt
$$
then

$$
\int_Q (|\nabla u_n|^{p(x)-2} \nabla u_n) T_k(u_n - v) \, dx \, dt \to \int_Q (|\nabla u|^{p(x)-2} \nabla u) T_k(u - v) \, dx \, dt.
$$

Let us pass to the limit for other term

$$
\int_Q \phi_n(u_n) \nabla T_k(u_n - v) \, dx \, dt = \int_Q \phi_n(u_n) \nabla T_k(T_{k+M}(u_n) - v) \, dx \, dt
$$

Since $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$ and $\phi_n$ is a Lipschitz continuous bounded function $\mathbb{R}$ in to $\mathbb{R}^N$, such that $\phi_n$ uniformly converge to $\phi$ on any compact subset of $\mathbb{R}$ as $n$ tend to $+\infty$. Then, we have

Due to (11), $T_k(u_n) \to T_k(u)$ in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) \forall k \geq 0$ and $u_n \to u$ a.e. in $Q$, we have

$$
f_n T_k(u_n - v) \to f T_k(u - v) \text{ strongly in } L^1(Q)
$$
and by Lebesgue theorem, we have

$$
\int_Q f_n T_k(u_n - v) \to \int_Q f T_k(u - v) \text{ strongly in } L^1(Q).
$$
Similarly, since $g$ is a bounded and continuous functions belong to $L^1(\mathbb{R})$ and $u_n \to u$ a.e. in $Q$, we obtain
\[
\int_Q g(u_n) |\nabla u_n|^{p(x)-2} T_k(u_n - v) \to \int_Q g(u) |\nabla u|^{p(x)-2} T_k(u - v) \text{ strongly in } L^1(Q).
\]

For the second term of the right hand side of (47), we have
\[
\int_Q F \nabla T_k(u_n - v) \to \int_Q F \nabla T_k(u - v) \text{ as } n \to +\infty,
\]
since $\nabla T_k(u_n - v) \to \nabla T_k(u - v)$ in $(L^{p(x)}(Q))^N$, while $F \in (L^{p'(x)}(Q))^N$ and Lebesgue theorem. Then, we conclude that $u$ satisfies (14).

As a conclusion of Step 1 to Step 4, the proof of Theorem 3.1 is complete. □

References


SIDI MOHAMED BEN ABDELLAH UNIVERSITY, POLY-DISCIPPLINARY FACULTY OF TAZA, LABORATORY LSI, DEPARTMENT MPI, P.O. BOX 1223 TAZA GARE, MOROCCO

Email address: mounir.mekkour@usmba.ac.ma