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THE EVEN-ORDER HARMONIC OSCILLATOR PERTURBED BY A DECREASING SCALAR POTENTIAL

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Abstract. This paper continues a previous work studying the perturbation $L = H + V$ on $\mathbb{R}$, with $H = (-1)^h \frac{d^{2h}}{dx^{2h}} + x^{2h}$, where $h \in \mathbb{N}^*$, and V is a decreasing scalar potential. It is assumed that $\left|V^{(n)}(x)\right| \leq c_n \left(1 + x^2\right)^{\frac{-s}{2}}$ for $x \in \mathbb{R}$, $s \in \mathbb{R}_+^* \backslash \{1\}$ and $n \in \mathbb{N}$. Let λ_k denote the k^{th} eigenvalue of H, and suppose the eigenvalues of L near λ_k can be expressed as $\lambda_k + \mu_k$. The authors prove an asymptotic formula for the fluctuation $\{\mu_k\}$, which is given by a transformation of V. This paper specifically examines the case where $s = 1$.

1. Introduction and preliminaries

In the space $L^2(\mathbb{R})$, we examine the operator H defined as:

$$H = (-1)^h \frac{d^{2h}}{dx^{2h}} + x^{2h}, \quad h \in \mathbb{N}^*.$$

We will see that most of the results obtained in [1] have a natural extension in this context. It is established that H is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ and has a compact resolvent [10]. The spectrum of H consists of an increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 0}$, each of which has finite multiplicity. There exists a positive integer k_0 such that for $k \geq k_0$, the eigenvalues λ_k are simple and have the following asymptotic expansion:

$$\lambda_k^{\frac{1}{h}} = \frac{2\pi}{T}\left(k + \frac{1}{2}\right) + O\left(\frac{1}{k}\right), \quad (k \to +\infty), \tag{1.1}$$

and

$$T = \frac{1}{h} B\left(\frac{1}{2h}, \frac{1}{2h}\right), \tag{1.2}$$

where B is the beta function. Let $V \in C^\infty(\mathbb{R}, \mathbb{R})$ which satisfies the following estimate:

$$\left|V^{(n)}(x)\right| \leq c_n \left(1 + x^2\right)^{\frac{-1}{2}}, \quad x \in \mathbb{R}, n \in \mathbb{N}, \tag{1.3}$$

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The operator $L = H + V$ is essentially self-adjoint with compact resolvent[13]. The Min-Max theorem [15] shows that the spectrum of L around λ_k, for $k \geq k_0$ can be written in the form $\lambda_k + \mu_k$. The goal is to study the asymptotic behavior of the fluctuation μ_k when $\lambda_k \to +\infty$. Our main results are:

Theorem 1.1. *The asymptotic behavior of μ_k is:*
for $h \geq 2$

$$\mu_k = \frac{h}{B(\frac{1}{2h}, \frac{1}{2h})} \int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}} dy + O\left(\lambda_k^{-\frac{1}{h}} \log(\lambda_k^{\frac{1}{h}})\right). \tag{1.4}$$

To measure the validity of theorem 1.1, we give the following theorem :

Theorem 1.2. *for all $h \geq 2$ we have:*

$$\mu_k = \frac{h\lambda_k^{-\frac{1}{2h}}}{B(\frac{1}{2h}, \frac{1}{2h})} \int_{-\lambda_k^{\frac{1}{2h}}}^{\lambda_k^{\frac{1}{2h}}} V(x)dx + O(\lambda_k^{-\frac{1}{2h}}). \tag{1.5}$$

In the case of $h = 1$ (the harmonic oscillator), several mathematicians have studied related problems. For instance, [6] investigated the perturbation $L = A + B$ where $B(x) \sim |x|^{-\alpha} \sum_m a_m \cos \omega_m x$. Additionally, in [14], the authors examined the operator $A = -\frac{d^2}{dx^2} + x^2 + q(x)$ under the condition that the functions q, q', and the integral $\int_0^x q(s)ds$ are bounded. For the case where V is periodic, one can refer to [4].

Our goal is to apply Weinstein's averaging method [8, 3, 17]. However, this method cannot be directly applied because H, viewed as a pseudo-differential operator (ΨDO), does not have a periodic flow. Nevertheless, the operator $H^{\frac{1}{h}}$ does possess this property [10]. Therefore, we begin by considering a perturbation of the operator $H^{\frac{1}{h}}$.

$$L_h = H^{\frac{1}{h}} + B, \quad h \in \mathbb{N}^*, \tag{1.6}$$

Where

$$B = \left(\frac{1}{h}\right) H^{\frac{1}{h}-1} V. \tag{1.7}$$

We apply the Average method in replacing B in perturbation (1.6) by the average:

$$\overline{B} = \frac{1}{T} \int_0^T e^{-itH^{\frac{1}{h}}} B e^{itH^{\frac{1}{h}}} dt, \tag{1.8}$$

where T is the period of the flow of $H^{\frac{1}{h}}$, T is given by (1.2) [9]. The main advantage of this method is that $\overline{B}$ is a compact operator and the operators L_h, $\overline{L}_h = H^{\frac{1}{h}} + \overline{B}$ are almost unitary equivalent, that means it exists a unitary operator U such as $UL_hU^{-1} - \overline{L}_h$ is compact. Note that L_h and $\overline{L}_h$ are also with a compact resolvent [13]. Using min-max Theorem, the parts of their spectrum

around $\lambda_k^{\frac{1}{h}}$ are respectively of the form $\lambda_k^{\frac{1}{h}} + v_k$ and $\lambda_k^{\frac{1}{h}} + \overline{v}_k$. Then we study $\overline{v}_k$ by using a functional calculus of the operator H.
We start by establishing the link between v_k and μ_k.

Proposition 1.3. *for $h \geq 1$ we have:*

$$\mu_k = h\lambda_k^{1-\frac{1}{h}} v_k + O\left(\lambda_k^{-1}\right), \quad (\lambda_k \to +\infty). \tag{1.9}$$

Using a functional calculus for H, we obtain

Proposition 1.4. *for $h \geq 1$ we have:*

$$h\lambda_k^{1-\frac{1}{h}} \bar{v}_k = \frac{1}{T} \int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}} dy + O(\lambda_k^{-\frac{1}{h}} \log(\lambda_k^{\frac{1}{h}})). \tag{1.10}$$

The following Proposition gives the relation between v_k and $\overline{v}_k$

Proposition 1.5. *for $h \geq 2$ we have:*

$$v_k = \overline{v}_k + O(\lambda_k^{-(2-\frac{1}{h}-\frac{\eta}{h})} \log^2(\lambda_k^{\frac{1}{h}})), \tag{1.11}$$

Where $\eta \in \left]0; \min(2, h-1)\right[$.

The work is structured into six sections followed by an Appendix. In the next section, we present essential definitions, and properties of Weyl pseudo-differential operators and the functional calculus for the operator H that will be used subsequently. In Section 3, we study the operator L_h and show the relation between v_k and μ_k by proving Proposition 1.3. The section 4 is devoted to proving Proposition 1.4. In Section 5, we study the relation between the spectrum of L_h and $\overline{L}_h$ and we prove Proposition 1.5. Finally, we justify the asymptotic expansion given by Theorem 1.1, and we prove Theorem 1.2 that ensure the validity of Theorem 1.1.

2. Weyl pseudo-differential operator and functional calculus

The weight function used in [1] is not suitable for our specific problem. Therefore, we need to use a different weight function that is better suited to our situation in order to obtain more relevant results. Let $\rho \in [0,1]$, $p, q \in \mathbb{R}$ and we consider the weight function [16]:

$$(x,\xi) \longrightarrow \left(1 + x^2 + \xi^2\right)^{\frac{p}{2}} \log^q\left(3 + x^2 + \xi^2\right), \qquad (x,\xi) \in \mathbb{R}^2.$$

We define $\Gamma_\rho^{p,q}(\mathbb{R} \times \mathbb{R})$ as the space of symbols associated with this tempered weight function. Specifically:

$$\begin{aligned} \Gamma_\rho^{p,q} &= \{a \in C^\infty(\mathbb{R}^2) : \forall \alpha, \beta \in \mathbb{N}^2, \exists\, c_{\alpha,\beta} > 0 \,/\, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \\ &\leq c_{\alpha,\beta}(1 + x^2 + \xi^2)^{\frac{p-\rho(\alpha+\beta)}{2}} \log^q(3 + x^2 + \xi^2)\}. \end{aligned}$$

We use the standard Weyl quantization for these symbols. Specifically, if $a \in \Gamma_\rho^{p,q}$, then for $u \in S(\mathbb{R})$, the corresponding operator is defined as:

$$op^w\,(a)\,u\,(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a\left(\frac{x+y}{2},\xi\right) u\,(y)\,dyd\xi.$$

We denote by $G_\rho^{p,q}$ the class of operators whose symbols are in $\Gamma_\rho^{p,q}$. For example, $H \in G_1^{2h,0}$ and $V \in G_0^{0,0}$.

Now, let us introduce the concept of the asymptotic expansion of symbols.

Definition 2.1. Let $a_j \in \Gamma_\rho^{p_j,q}$, $j \in \mathbb{N}$, we suppose that p_j is a decreasing sequence tending towards $-\infty$. We say that $a \in C^\infty(\mathbb{R}\times\mathbb{R})$ has an asymptotic expansion and we write:

$$a = \sum_{j=0}^{+\infty} a_j,$$

if

$$a - \sum_{j=0}^{r-1} a_j \in \Gamma_\rho^{p_r,q}, \qquad \forall r \geq 1.$$

Theorem 2.2. *(Calderon-Vaillancourt Theorem)*
If $a \in \Gamma_0^{0,0}$ then the operator $op^w\,(a)$ is bounded on $L^2\,(\mathbb{R})$.

Theorem 2.3. *(Compactness) If $a \in \Gamma_\rho^{p,q}, p < 0$ and $\rho \in [0,1]$, then the operator $op^\omega(a)$ is compact on $L^2(\mathbb{R})$.*

We need the symbolic calculation of these classes of operators, thus, we present the following Proposition, which will be proven in Appendix A.

Proposition 2.4. *i) If $A \in G_1^{p,q}$ and $B \in G_0^{p',q'}$ then the operator $AB \in G_0^{p+p',q+q'}$. Its Weyl symbol admits an asymptotic development:*

$$c = \sum_{j=0}^{+\infty} c_j, \quad c_j \in \Gamma_0^{p+p'-j,q+q'},$$

where

$$c_j = \frac{1}{2^j}\sum_{\alpha+\beta=j}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(\partial_x^\alpha\partial_\xi^\beta b).$$

ii) If $(B_i)_{i\in\{1,\cdots,n\}}$ is the family of operators such as $B_i \in G_0^{p_i,q_i}$. Then the operator

$$B_1B_2\cdots B_nH^{-\frac{p_1+\cdots p_n}{2h}}\log^{-(q_1+\cdots+q_n)}(3+H^{\frac{1}{h}}),$$

is bounded.

In order to prove this Proposition, it is necessary to perform a functional calculation for operator H. In our work, we need the properties of a function f that satisfies, for all $p,q \in \mathbb{R}$, $k \in \mathbb{N}$ and $\rho \in \left[1-\frac{1}{2h};1\right]$:

$$|f^{(k)}(x)| \leq C_k(1+|x|)^{p-\rho k}\log^q(3+x^{\frac{1}{h}}).$$

Proposition 2.5. *$f(H)$ is a (ΨDO) included in $G^{2ph,q}_{1-2h(1-\rho)}$ and its weyl symbol admits the following development*

$$\sigma_{f(H)} = \sum_{j\geq 0} \sigma_{f(H),2j},$$

$$\sigma_{f(H),2j} = \sum_{k=2}^{3j} \frac{d_{jk}}{k!} f^{(k)}(\sigma_H), \quad \forall j \geq 1,$$

where

$$d_{jk} \in \Gamma_1^{2hk-4j,0}, \quad \sigma_{f(H),2j} \in \Gamma^{2ph-2j(2-3h(1-\rho)),q}_{1-2h(1-\rho)}, \tag{2.1}$$

in particular

$$\sigma_{f(H),0} = f(\sigma_H) = \sigma_{\overline{V},0}, \quad \sigma_{f(H),1} = 0.$$

Proof. For studying $f(H)$ we follow the same strategy in [11], using the Mellin transformation ([2], Definitin 2.1), the latter consists of the following steps:
(1) We prove by induction that $(H-\lambda)^{-1}, \lambda \in \ C$, is a (ΨDO) and its Weyl symbol admits the development $b_\lambda = \sum_{j=0}^{+\infty} b_{j,\lambda}$ where

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1}, \\ b_{2j+1,\lambda} = 0, \\ b_{2j,\lambda} = \displaystyle\sum_{k=2}^{3j} (-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{jk} \in \Gamma_1^{2hk-4j,0}. \end{cases}$$

(2) We study the operator H^s using the Cauchy's integral formula:

$$H^s = \frac{1}{2\pi i} \int_\Delta \lambda^s (H-\lambda)^{-1} d\lambda.$$

Δ is the same domain defined in the article [11], H^s is a (ΨDO) and its Weyl symbol is:

$$\sigma_s = \sum_{j=0}^{+\infty} \sigma_{s,2j}$$

with

$$\sigma_{s,0} = \sigma_H^s, \quad \sigma_{s,2j} = \sum_{k=2}^{3j} d_{j,k}.\frac{s(s-1)\cdots(s-k+1)}{k!} \sigma_H^{s-k},$$

$$\sigma_{s,2j} \in \Gamma_1^{2sh-4j,0}.$$

(3) We study $f(H)$ using the representation formula

$$f(H) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} M\left[f\right](s) H^{-s} ds,$$

$\sigma \in [0, -r[$, $r < 0$ and $M\left[f\right]$ is the Mellin transformation of f. □

3. Reduction to a perturbation of $H^{\frac{1}{h}}$

If we translate H by a sufficiently large positive constant, we can assume that L is positive and $\|H^{-1}V\| < 1$. We can then reduce our problem to a perturbation of $H^{\frac{1}{h}}$ by expressing:

$$(H+V)^{\frac{1}{h}} = H^{\frac{1}{h}} + W.$$

consequently

$$\begin{aligned} W \quad &= B + H^{\frac{1}{h}} \left(H^{-1}V\right)^2 \\ &\times \sum_{k=0}^{+\infty} \alpha_{k+2} \left(H^{-1}V\right)^k, \end{aligned}$$

where

$$\begin{aligned} B &= \binom{1}{h} H^{\frac{1}{h}-1} V, \\ \alpha_k &= \frac{\left(\frac{1}{h}\right)\left(\frac{1}{h}-1\right)\cdots\left(\frac{1}{h}-k+1\right)}{k!}. \end{aligned} \tag{3.1}$$

We can write

$$\begin{aligned} L^{\frac{1}{h}} - L_h = H^{\frac{1}{h}} \left(H^{-1}V\right)^2 \\ \times \sum_{k=0}^{+\infty} \alpha_{k+2} \left(H^{-1}V\right)^k. \end{aligned} \tag{3.2}$$

Since $\|H^{-1}V\| < 1$, then the operator $\sum_{k=0}^{+\infty} \alpha_{k+2} \left(H^{-1}V\right)^k$, is bounded in $L^2(\mathbb{R})$. We have $H \in G_1^{2h;0}$. Thus $H^{-1} \in G_1^{-2h;0}$, $V \in G_0^{0;0}$ and (ii) Proposition 2.4 we obtain that the opertaor $(L^{\frac{1}{h}} - L_h)H^{2-\frac{1}{h}}$ is bounded. Therefore, there exists a constant $c > 0$ such that

$$-cH^{-2+\frac{1}{h}} \le L^{\frac{1}{h}} - L_h \le cH^{-2+\frac{1}{h}}, \tag{3.3}$$

According to the min-max theorem, we obtain

$$(\lambda_k + \mu_k)^{\frac{1}{h}} = \lambda_k^{\frac{1}{h}} + v_k + O\left(\lambda_k^{\frac{1}{h}-2}\right). \tag{3.4}$$

Using the fact that $\{\mu_k\}$ is bounded and the Taylor's formula for the function $t \longrightarrow (1+\frac{\mu_k}{t})^{\frac{1}{h}}$ we obtain the estimate:

$$\mu_k = h\lambda_k^{1-\frac{1}{h}} v_k + O\left(\lambda_k^{-1}\right). \tag{3.5}$$

This completes the proof of Proposition 1.3.

4. The asymptotic behavior of $\overline{v}_k$

We recall that $\overline{L}_l$ is obtained by replacing B in L_h by $\overline{B}$ and $\lambda_k^{\frac{1}{h}} + \overline{v}_k$ is the part of the spectrum of $\overline{L}_h$ around $\lambda_k^{\frac{1}{h}}$ as $\lambda_k \longrightarrow +\infty$.
Putting:

$$\overline{V} = \frac{1}{T}\int_0^T W(t)dt, \quad W(t) = e^{-itH^{\frac{1}{h}}} V e^{itH^{\frac{1}{h}}}, \tag{4.1}$$

and from (3.1)

$$\overline{B} = \left(\frac{1}{h}\right) H^{\frac{1}{h}-1}\overline{V}. \tag{4.2}$$

We recall that, (Proposition 2.5), the operator $H^{\frac{1}{h}} \in G_1^{2,0}$ its Weyl symbol σ is

$$\sigma = \sum_{j=0}^{+\infty} \sigma_{2j},$$

where

$$\sigma_0 = \sigma_H^{\frac{1}{h}}, \quad \sigma_{2j} \in \Gamma_1^{2-4j,0}.$$

Which gives for the operator $\overline{V}$

Proposition 4.1. *We have:*

$$\overline{V} \in G_0^{-1,1} \tag{4.3}$$

and its weyl symbol admits an asymptotic development

$$\sigma_{\overline{V}} = \sum_{j=0}^{+\infty} \sigma_{\overline{V},2j}, \quad \sigma_{\overline{V},2j} \in \Gamma_0^{-1-j,1}, \tag{4.4}$$

in particular

$$\sigma_{\overline{V},0} = \frac{1}{T}\int_0^T V(x(t))dt = \frac{h}{B(\frac{1}{2h},\frac{1}{2h})}\int_{-1}^{1} \frac{V(y\sigma_H^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}}dy. \tag{4.5}$$

Proof. We recall that $\varphi(t) = (x(t), \xi(t))$ is a solution of the dynamic system

$$\begin{cases} \frac{dx(t)}{dt} = \frac{\partial \sigma_H^{\frac{1}{h}}}{\partial \xi} = 2E^{\frac{1}{h}-1}\xi^{2h-1}(t), \\ \frac{d\xi(t)}{dt} = \frac{-\partial \sigma_H^{\frac{1}{h}}}{\partial x} = -2E^{\frac{1}{h}-1}x^{2h-1}(t), \\ x(0) = x, \quad \xi(0) = \xi, \\ x^{2h}(t) + \xi^{2h}(t) = x^{2h} + \xi^{2h} = E. \end{cases} \tag{4.6}$$

To prove (4.5), we remind that $\varphi(t) = (x(t), \xi(t))$ is a solution of the dynamic system (4.6). We can assume the initial conditions $x(0) > 0$, $\frac{dx(0)}{dt} > 0$, other cases can be treated similarly. For now, we study the properties of the function $x(t)$ on $[0, T]$, from (4.6) we get

$$dt = \pm\frac{dx}{2E^{\frac{1}{h}-1}(E - x^{2h})^{1-\frac{1}{2h}}}. \tag{4.7}$$

Combining the fact that $x(t)$ is a smooth periodic function of period T with (4.7), we can ensure that the function $x(t)$ reaches its maximum at t_0, $x(t_0) = E^{\frac{1}{2h}}$ and its minimum at t_1, $x(t_1) = -E^{\frac{1}{2h}}$.
For now we have

$$\sigma_{\overline{V},0} = \frac{1}{T}\left[\int_0^{t_0} V(x(t))dt + \int_{t_0}^{t_1} V(x(t))dt + \int_{t_1}^{T} V(x(t))dt\right],$$

we make the change of variable $x(t)=u$, obviously $x(t)$ is increasing on $[0,t_0]$, we get

$$\int_0^{t_0} V(x(t))dt = \frac{1}{2}E^{1-\frac{1}{h}}\int_x^{E^{\frac{1}{2h}}} \frac{V(u)}{(E-u^{2h})^{1-\frac{1}{2h}}}du, \tag{4.8}$$

after the same calculation on $[t_0,t_1]$ and $[t_1,T]$ we obtain

$$\sigma_{\overline{V},0} = \frac{h}{B(\frac{1}{2h},\frac{1}{2h})}E^{1-\frac{1}{h}}\int_{-E^{\frac{1}{2h}}}^{E^{\frac{1}{2h}}} \frac{V(u)}{(E-u^{2h})^{1-\frac{1}{2h}}}du, \tag{4.9}$$

we make now the change of variable $y=\frac{u}{E^{\frac{1}{2h}}}$, we have

$$\sigma_{\overline{V},0} = \frac{h}{B(\frac{1}{2h},\frac{1}{2h})}\int_{-1}^{1} \frac{V(y\sigma_H^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}}dy. \tag{4.10}$$

Let us now determie the class to which $\sigma_{\overline{V},0}$ belongs. Using (1.3) we get

$$\left|\sigma_{\overline{V},0}\right| \le c\int_0^1 \frac{1}{(1+y\sigma_H^{\frac{1}{2h}})(1-y^{2h})^{1-\frac{1}{2h}}}dy, \tag{4.11}$$

writing

$$\int_0^1 \frac{1}{(1+y\sigma_H^{\frac{1}{2h}})(1-y^{2h})^{1-\frac{1}{2h}}}dy = I_1+I_2, \tag{4.12}$$

where

$$I_1=\int_0^{\frac{1}{2}} \frac{1}{(1+y\sigma_H^{\frac{1}{2h}})(1-y^{2h})^{1-\frac{1}{2h}}}dy, \quad I_2=\int_{\frac{1}{2}}^1 \frac{1}{(1+y\sigma_H^{\frac{1}{2h}})(1-y^{2h})^{1-\frac{1}{2h}}}dy. \tag{4.13}$$

On the one hand, we have

$$|I_1| \le C\int_0^{\frac{1}{2}} \frac{1}{1+y\sigma_H^{\frac{1}{2h}}}dy \le C\sigma_H^{-\frac{1}{2h}}\log(3+\sigma_H^{\frac{1}{2h}}). \tag{4.14}$$

On the other hand

$$|I_2| \le C\left(1+\frac{1}{2}\sigma_H^{\frac{1}{2h}}\right)^{-1}\int_{\frac{1}{2}}^1 \frac{1}{(1-y^{2h})^{1-\frac{1}{2h}}}dy \le C\left(1+\sigma_H^{\frac{1}{2h}}\right)^{-1}. \tag{4.15}$$

We substitute (4.12), (4.14) and (4.15) into (4.11), this gives the following inequality

$$\left|\sigma_{\overline{V},0}\right| \le C(1+\sigma_H)^{-\frac{1}{2h}}\log(3+\sigma_H), \tag{4.16}$$

finally

$$\left|\sigma_{\overline{V},0}\right| \le C(1+x^2+\xi^2)^{-\frac{1}{2}}\log(3+x^2+\xi^2). \tag{4.17}$$

The same estimates hold for $\partial_x^\alpha\partial_\xi^\beta\sigma_{\overline{V},0}(x,\xi), \alpha,\beta\in\mathbb{N}$ so we have $\sigma_{\overline{V},0}\in\Gamma_0^{-1,1}$. Now we prove (4.4), first by a direct calculation we get

$$\partial_x^\alpha\partial_\xi^\beta\sigma_{2j}\in L^\infty(\mathbb{R}\times\mathbb{R}), \quad \alpha,\beta,j\in\mathbb{N},\ \alpha+\beta+j>2,$$

and

$$\partial_x^\alpha \partial_\xi^\beta \varphi(t) \in L^\infty(\mathbb{R}\times\mathbb{R}), \quad \alpha+\beta \geq 1,$$

we apply Egorov's theorem, as in [16] with the Heisenberg-von Neumann equation, the expression of that symbol is obtained:

$$\sigma_{W(t)} = \sum_{j=0}^{+\infty} \sigma_{W(t),2j},$$

with

$$\sigma_{W(t),2j} = \frac{1}{i}\int_0^t \sum_{\substack{\alpha'+\beta'+l'+2k'=2j+1\\ 0\leq l\leq 2j-1}} C_{\alpha',\beta'}(\partial_\xi^{\alpha'}\partial_x^{\beta'}\sigma_{2k})(\partial_\xi^{\beta'}\partial_x^{\alpha'}\sigma_{W,l}(\tau))|\varphi^{t-\tau}d\tau,$$

where

$$C_{\alpha',\beta'} = \left(1-(-1)^{\alpha+\beta}\right)\Gamma(\alpha,\beta),$$

and

$$\sigma_{W(t),0}(x,\xi) = V o x(t), \quad \sigma_{W(t),1}(x,\xi) = 0.$$

Again, using the estimation (1.3), by a direct calculation, we obtain for all $\alpha,\beta \in \mathbb{N}$.

$$\left|\partial_x^\alpha\partial_\xi^\beta \sigma_{W(t),2j}\right| \leq C(1+x^2+\xi^2)^{-\frac{j}{2}} \int_0^t (1+x^2(u))^{\frac{-s}{2}}du,$$

using (4.1), we obtain, for all $\alpha,\beta\in\mathbb{N}$

$$\left|\partial_x^\alpha\partial_\xi^\beta \sigma_{\overline{V},2j}(x,\xi)\right| \leq C(1+x^2+\xi^2)^{-\frac{j}{2}} \int_0^T (1+x^2(t))^{\frac{-s}{2}}dt.$$

To measure the estimate of the right-hand integral, we repeat the same calculation made in (4.5), obtaining for any $\alpha,\beta\in\mathbb{N}$

$$\left|\partial_x^\alpha\partial_\xi^\beta \sigma_{\overline{V},2j}(x,\xi)\right| \leq C(1+x^2+\xi^2)^{-\frac{1}{2}-\frac{j}{2}}\log\left(3+x^2+\xi^2\right).$$

This gives (4.4), as required. □

Proof of Proposition 1.4:

Proof. We recall that

$$\sigma_{\overline{V},0} = f(\sigma_H),$$

where

$$f(x) = \frac{h}{B(\frac{1}{2h},\frac{1}{2h})}\int_{-1}^1 \frac{V(yx_H^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}}dy.$$

A simple calculation shows that for any $k\in\mathbb{N}$

$$\left|f^{(k)}(x)\right| \leq C_k\left(1+|x|\right)^{-\frac{1}{2h}-(1-\frac{1}{2h})k}\log\left(3+x^{\frac{1}{h}}\right).$$

We have, using Proposition 2.5

$$\sigma_{f(H)} - f(\sigma_H) \in \Gamma_0^{-2,1}. \tag{4.18}$$

And by using Proposition 4.1, we get

$$\sigma_{\overline{V}} - \sigma_{\overline{V},0} \in \Gamma_0^{-2,1}. \tag{4.19}$$

By combining (4.18) and (4.19), we get

$$\sigma_{\overline{V}} - \sigma_{f(H)} \in \Gamma_0^{-2,1}. \tag{4.20}$$

In terms of operators, we have

$$\overline{V} - f(H) \in G_0^{-2,1}, \tag{4.21}$$

we can write

$$\frac{1}{h}H^{\frac{1}{h}-1}(\overline{V} - f(H)) = \left[\overline{L}_h - \left(H^{\frac{1}{h}} + \frac{1}{h}H^{\frac{1}{h}-1}f(H)\right)\right]. \tag{4.22}$$

From (4.21), Proposition 2.5, Proposition 2.4 and Theorem 2.2 we deduce that the operator

$$\left[\overline{L}_h - \left(H^{\frac{1}{h}} + \frac{1}{h}H^{\frac{1}{h}-1}f(H)\right)\right] H \log^{-1}(3 + H^{\frac{1}{h}}), \tag{4.23}$$

is bounded. According to min-max Theorem we have

$$\overline{v}_k = \frac{1}{h}\lambda_k^{\frac{1}{h}-1} f(\lambda_k) + O\left(\lambda_k^{-1}\log(\lambda_k^{\frac{1}{h}})\right), \tag{4.24}$$

then

$$h\lambda_k^{1-\frac{1}{h}}\overline{v}_k = \frac{1}{T}\int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}}dy + O\left(\lambda_k^{-\frac{1}{h}}\log(\lambda_k^{\frac{1}{h}})\right). \tag{4.25}$$

□

5. The relation betwen the spectrum of L_h and $\overline{L}_h$

Proposition 5.1. *There exists a skew-symmetric operator $Q \in G_0^{-(2h-1),1}$ such that the operator*

$$(e^{Q}L_h e^{-Q} - \overline{L}_h)H^{2-\frac{1}{h}-\frac{\eta}{h}} \log^{-2}(3 + H^{\frac{1}{h}}),$$

is bounded, with $\eta \in \,]0; \min(2, h-1)[$.

Proof. The operator Q is constructed as follows

$$\begin{aligned} Q = Q_1 + Q_2,\ Q_1 &= \frac{i}{hT}H^{\frac{1}{h}-1}\int_0^T (T-t)\,W(t)\,dt, \\ Q_2 = \frac{-1}{2T}\int_0^T (T-t)\int_0^t &\left[\frac{1}{h}H^{\frac{1}{h}-1}W(t), \frac{1}{h}H^{\frac{1}{h}-1}W(r)\right] dr\,dt. \end{aligned} \tag{5.1}$$

The following commutation formulas are straightforward to verify:[7]

$$\left[Q_1, H^{\frac{1}{h}}\right] = \frac{1}{h}H^{\frac{1}{h}-1}(\overline{V} - V), \tag{5.2}$$

and

$$\begin{aligned}\left[Q_2, H^{\frac{1}{h}}\right] &= \tfrac{-1}{2T}\int_0^T (T-t)\int_0^t \\ &\quad \left[\left[\tfrac{1}{h}H^{\frac{1}{h}-1}W(t), \tfrac{1}{h}H^{\frac{1}{h}-1}W(r)\right], H^{\frac{1}{h}}\right] dr dt \\ &= -\overline{\overline{V}} - \tfrac{1}{2}\left[Q_1, \tfrac{1}{h}H^{\frac{1}{h}-1}V\right],\end{aligned} \tag{5.3}$$

where

$$\overline{\overline{V}} = \frac{1}{2Ti}\int_0^T\int_0^t \left[\frac{1}{h}H^{\frac{1}{h}-1}W(t), \frac{1}{h}H^{\frac{1}{h}-1}W(r)\right] dr\, dt. \tag{5.4}$$

We note that the differential equation

$$\frac{dX(t)}{dt} = [Q, X], \quad X(0) = L_h,$$

admits a unique solution

$$X(t) = e^{tAdQ}.L_h = e^{tQ}L_h e^{-tQ}, \tag{5.5}$$

where $AdQ.L_h = [Q, L_h]$.
As a result of the equations (5.1), (5.2) and (5.3), we deduce that

$$\begin{aligned} e^Q L_h\, e^{-Q} - \overline{L}_h &= -\overline{\overline{V}} + \tfrac{1}{2h}\left[Q_2, H^{\frac{1}{h}-1}V\right] \\ &\quad + \tfrac{1}{2h}\left[Q, H^{\frac{1}{h}-1}\overline{V}\right] + \tfrac{1}{4h}\left[Q,\left[Q_1, H^{\frac{1}{h}-1}V\right]\right] \\ &\quad + \tfrac{1}{2}\left[Q,\left[Q_2, \tfrac{1}{h}H^{\frac{1}{h}-1}V\right]\right] - \tfrac{1}{2}\left[Q, \overline{\overline{V}}\right] \\ &\quad + \sum_{n\geq 0}\frac{(AdQ)^n}{(n+3)!}\left[Q,[Q,[Q,L_h]]\right].\end{aligned} \tag{5.6}$$

To continue the proof of Proposition 5.1, we need the following Lemma

Lemma 5.2. *$Q_1 \in G_0^{-(2h-1),1}$ and $\overline{\overline{V}}, Q_2 \in G_0^{-(4h-2-2\eta),2}$.*

Proof. Using the same calculations as in the proof of Proposition 4.1, we find that $Q_1 \in G_0^{-(2h-1),1}$. Now let's determine the class of $\overline{\overline{V}}$, we can write

$$\overline{\overline{V}} = \frac{1}{2Ti}\int_0^T [S(t), B(t)]\, dt, \tag{5.7}$$

where

$$S(t) = \frac{1}{h}H^{\frac{1}{h}-1}W(t), \quad B(t) = \int_0^t S(r)dr.$$

We start by determining the class of the operator $\int_0^T S(t)B(t)dt$. For now, we focus on the operator $S(t)B(t)$, whose Weyl symbol is given by:

$$c_t(x,\xi) = \frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}\sigma_{S(t)}(x+\omega, \rho+\xi)\sigma_{B(t)}(x+r, \tau+\xi)\, d\rho\, d\omega\, d\tau\, dr, \tag{5.8}$$

for every $(x,\xi)\in\mathbb{R}\times\mathbb{R}$. We split the oscillatory integral c into two parts $c^{(1)}$ and $c^{(2)}$, and use the cuttof function

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho)=\chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}\right], \tag{5.9}$$

and

$$\omega_{2,\varepsilon}=1-\omega_{1,\varepsilon},$$

where $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ in $[-1,1]$, $\chi\equiv 0$ in $\mathbb{R}\backslash]-2,2[$ and $\eta>0$.
Let's consider for $j\in\{1,2\}$

$$d_j(x,\xi,\omega,\tau,r,\rho)=\omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho)\sigma_{S(t)}(x+\omega,\rho+\xi)\sigma_{B(t)}(x+r,\tau+\xi), \tag{5.10}$$

$c^{(1)}$ (resp. $c^{(2)}$) the integral obtained in (6.1) by replacing the amplitude by d_1 (resp. d_2)

study of $c^{(2)}$:

On the support of d_2, we have $\omega^2+\rho^2+r^2+\tau^2\geq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}$. We perform integration by parts using the operator:

$$M=\frac{1}{2i}(\omega^2+\rho^2+r^2+\tau^2)^{-1}(-\rho\partial_r-r\partial_\rho+\tau\partial_\omega+\omega\partial_\tau).$$

We have for all $k\in\mathbb{N}$,

$$c^{(2)}=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}({}^tM)^k d_2\,d\rho\,d\omega\,d\tau\,dr,$$

then we obtain for all $k>0$

$$\left|c^{(2)}\right|\leq\left(1+x^2+\xi^2\right)^{-\frac{\eta}{4}k}. \tag{5.11}$$

study of $c^{(1)}$:

On the support of d_1 we have:

$$\begin{aligned} c^{(1)}(x,\xi) &= \tfrac{1}{\pi^2}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} e^{-2i(r\rho-\omega\tau)}\sigma_{S(t)}(x+\omega,\rho+\xi) \\ &\quad\times\sigma_{B(t)}(x+r,\tau+\xi)\,d\rho\,d\omega\,d\tau\,dr, \end{aligned} \tag{5.12}$$

then

$$\begin{aligned} \int_0^T\left|c_t^{(1)}\right| &\leq c\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho\,d\omega\,d\tau\,dr\times\int_0^T\left|\sigma_{S(t)}(x+\omega,\rho+\xi)\right|dt \\ &\quad\times\int_0^T\left|\sigma_{B(t)}(x+r,\tau+\xi)\right|dt, \end{aligned} \tag{5.13}$$

and

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{S(t)}(x,\xi)\right|\leq C_{\alpha,\beta}(1+x^2+\xi^2)^{-(h-1)}(1+x^2(t))^{\frac{-1}{2}}, \tag{5.14}$$

by integrating along the interval $[0,T]$ and following the same resoning in Proposition 2.4, we get

$$\left|\partial_x^\alpha\partial_\xi^\beta\int_0^T\sigma_{S(t)}(x,\xi)dt\right|\leq C_{\alpha,\beta}(1+x^2+\xi^2)^{-\frac{(2h-1)}{2}}\log(3+x^2+\xi^2). \tag{5.15}$$

From (5.13) and (5.14) we have

$$
\begin{aligned}
\int_0^T \left|c_t^{(1)}\right| \;\leq\; & c\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} (1+(x+\omega)^2+(\xi+\rho)^2)^{-\frac{2h-1}{2}} \\
& \times \log(3+(x+\omega)^2+(\xi+\rho)^2) \\
& \times(1+(x+r)^2+(\xi+\tau)^2)^{-\frac{2h-1}{2}} \\
& \times \log(3+(x+r)^2+(\xi+\tau)^2)\, d\rho\, d\omega\, d\tau\, dr.
\end{aligned} \tag{5.16}
$$

On the support of d_1, for small enough ε and since $\eta \in]0,2[$ there are positive constants c,$c^{'}$,C and $C^{'}$ such that:

$$
c(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+u)^2+(\xi+v)^2)^{\frac{1}{2}} \leq C(1+x^2+\xi^2)^{\frac{1}{2}},
$$

for all x, ξ, u and v in $\mathbb{R}$. It follows that

$$
\begin{aligned}
\int_0^T \left|c_t^{(1)}\right| dt \;\leq\; & c(1+x^2+\xi^2)^{-(2h-1)} \times \log^2(3+x^2+\xi^2) \\
& \times \int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho\, d\omega\, d\tau\, dr,
\end{aligned} \tag{5.17}
$$

finally we get:

$$
\int_0^T \left|c_t^{(1)}\right| dt \leq c(1+x^2+\xi^2)^{-(2h-1)+\eta} \times \log^2(3+x^2+\xi^2). \tag{5.18}
$$

Denoting σ as the Weyl symbol of the operator $\displaystyle\int_0^T S(t)B(t)dt$, we have

$$
\begin{aligned}
|\sigma| \;\leq\; & \int_0^T \left|c_t^{(1)}\right| dt + \int_0^T \left|c_t^{(2)}\right| dt \\
\leq\; & C\left[(1+x^2+\xi^2)^{-\frac{\eta}{4}k} + (1+x^2+\xi^2)^{-(2h-1)+\eta} \times \log^2(3+x^2+\xi^2)\right] \\
\leq\; & (1+x^2+\xi^2)^{-(2h-1)+\eta} \times \log^2(3+x^2+\xi^2),
\end{aligned} \tag{5.19}
$$

by employing a similar method, we find that $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ can be obtained. This also confirms that $Q_2 \in G_0^{-(4h-2-2\eta),2}$. □

Returning to the proof of the Proposition 5.1. Since $V \in G_0^{0,0}$, $\overline{V} \in G_0^{-1,1}$, $Q_1, Q \in G_0^{-(2h-1),1}$, $\overline{\overline{\overline{V}}}, Q_2 \in G_0^{-(4h-2-2\eta),2}$. Thus, we have:

$$
\left\{
\begin{array}{ll}
\left\| \overline{\overline{V}}.H^{2-\frac{1}{h}-\frac{\eta}{h}} \log^{-2}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \left[Q_2, H^{\frac{1}{h}-1}V \right] H^{3-\frac{2}{h}-\frac{\eta}{h}} \log^{-2}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \left[Q, H^{\frac{1}{h}-1}\overline{V} \right] H^{2-\frac{1}{h}} \log^{-2}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \left[Q, \left[Q_1, H^{\frac{1}{h}-1}V \right] \right] H^{3-\frac{2}{h}} \log^{-2}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \left[Q, \left[Q_2, \frac{1}{h}H^{\frac{1}{h}-1}V \right] \right] H^{4-\frac{5}{2h}-\frac{\eta}{2h}} \log^{-3}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \left[Q, \overline{\overline{V}} \right] H^{3-\frac{3}{2h}-\frac{\eta}{2h}} \log^{-3}(3+H^{\frac{1}{h}}) \right\| & \leq C, \\
\left\| \frac{(AdQ)^n}{(n+3)!} \left[Q, \left[Q, \left[Q, L_h \right] \right] \right] H^{3-\frac{1}{2h}} \log^{-3}(3+H^{\frac{1}{h}}) \right\| & \leq C \left\| Q \right\|^n .
\end{array}
\right.
\tag{5.20}
$$

For the last inequality, we used (5.2) and the following identity

$$(AdQ)^n .W = \sum_{p=0}^{n} (-1)^{n-p} C_n^p Q^p W Q^{n-p}.$$

From the above, we deduce that

$$(e^Q L_h e^{-Q} - \overline{L}_h) H^{2-\frac{1}{h}-\frac{\eta}{h}} \log^{-2}(3+H^{\frac{1}{h}}),$$

is bounded. □

Returning to the Proof of Proposition 1.5. We deduce from Proposition 5.1 that there exists a constant $c > 0$ such that

$$-cH^{-(2-\frac{1}{h}-\frac{\eta}{h})} \log^2(3+H^{\frac{1}{h}}) \leq e^Q L_h e^{-Q} - \overline{L}_h \leq cH^{-(2-\frac{1}{h}-\frac{\eta}{h})} \log^2(3+H^{\frac{1}{h}}).$$

According to the min-max Theorem

$$v_k = \overline{v}_k + O(\lambda_k^{-(2-\frac{1}{h}-\frac{\eta}{h})} \log^2(\lambda_k^{\frac{1}{h}})),$$

we choose $\eta \in]0; \min(2, h-1)[$. Now, to prove Theorem 1.1, we combine Propositions 1.3, 1.4 and 1.5. For $m \geq 2$, we have

$$\mu_k = \frac{1}{T} \int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}} dy + O\left(\lambda_k^{-\frac{1}{h}} \log(\lambda_k^{\frac{1}{h}})\right).$$

Proof of Theorem 1.2

Proof. We define

$$\mathcal{V}(x) = V(x) + V(-x), \quad \beta_k = \int_0^1 \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})}{(1-y^{2h})^{1-\frac{1}{2h}}} dy. \tag{5.21}$$

To begin, we study the asymptotic behavior of β_k. By a direct calculation, there is a function $\theta \in C([0,1], \mathbb{R})$ such that

$$\frac{1}{(1-y^{2h})^{1-\frac{1}{2h}}} = 1 + \frac{y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}. \tag{5.22}$$

Using (5.22) and split β_k as

$$\beta_k = \beta_{k,1} + \beta_{k,2}, \quad \beta_{k,1} = \int_0^1 \mathcal{V}(y\lambda_k^{\frac{1}{2h}})dy, \quad \beta_{k,2} = \int_0^1 \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy.$$

Firstly

$$\beta_{k,2} = \int_0^{\frac{1}{2}} \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy + \int_{\frac{1}{2}}^1 \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy. \tag{5.23}$$

Next consider (1.3) and the second integral in (5.23), we obtain

$$\left| \int_{\frac{1}{2}}^1 \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy \right| \leq \frac{c}{(1+\lambda_k)^{\frac{1}{2h}}} \int_{\frac{1}{2}}^1 y^{2h}\theta(y)dy,$$

consequently

$$\int_{\frac{1}{2}}^1 \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy = O(\lambda_k^{-\frac{1}{2h}}). \tag{5.24}$$

Since $y^{2h} \leq y^2$, then

$$\begin{aligned}
\left| \int_0^{\frac{1}{2}} \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy \right| &\leq c\int_0^{\frac{1}{2}} (1+(y\lambda_k^{\frac{1}{2h}})^2)^{-\frac{1}{2}} y^2 dy \\
&\leq c\lambda_k^{-\frac{3}{2h}} \int_0^{\frac{1}{2}\lambda_k^{\frac{1}{2h}}} \frac{u^2}{\sqrt{1+u^2}}du \\
&\leq c\lambda_k^{-\frac{3}{2h}} \int_0^{\frac{1}{2}\lambda_k^{\frac{1}{2h}}} udu \\
&\leq c\lambda_k^{-\frac{1}{2h}},
\end{aligned} \tag{5.25}$$

thus

$$\int_0^{\frac{1}{2}} \frac{\mathcal{V}(y\lambda_k^{\frac{1}{2h}})y^{2h}\theta(y)}{(1-y^{2h})^{1-\frac{1}{2h}}}dy = O(\lambda_k^{-\frac{1}{2h}}). \tag{5.26}$$

We substitute (5.24) and (5.26) into (5.23), this gives $\beta_{k,2} = O(\lambda_k^{-\frac{1}{2h}})$
Next, for $\beta_{k,1}$, we apply the change of variables $y\lambda_k^{\frac{1}{2h}} = x$, and using $\mathcal{V}(x) = V(x) + V(-x)$, we get

$$\beta_{k,1} = \lambda_k^{-\frac{1}{2h}} \int_0^{\lambda_k^{\frac{1}{2h}}} \mathcal{V}(x)dx = \lambda_k^{-\frac{1}{2h}} \int_{-\lambda_k^{\frac{1}{2h}}}^{\lambda_k^{\frac{1}{2h}}} V(x)dx,$$

finally, for all $h \geq 2$, we have

$$\mu_k = \frac{h\lambda_k^{-\frac{1}{2h}}}{B(\frac{1}{2h}, \frac{1}{2h})} \int_{-\lambda_k^{\frac{1}{2h}}}^{\lambda_k^{\frac{1}{2h}}} V(x)dx + O(\lambda_k^{-\frac{1}{2h}}).$$

$\square$

6. Conclusion

The even-order harmonic oscillator is an interesting problem in spectral theory, particularly due to its applications in fields such as quantum mechanics and mathematical physics. Although well-established methods exist to analyze such operators, we have chosen to employ the averaging method. This approach allows us to effectively treat non-periodic operators by imposing an artificial periodic structure, which simplifies the spectral analysis. By using this method, we have been able to gain insight into the asymptotic behavior of the operator's spectrum. However, the problem remains complex, and further research will be necessary to refine our understanding of anharmonic oscillators. Additionally, new methods must be explored to address these more general systems. Extending these results could lead to significant applications in more complex physical models, particularly in nonlinear systems where spectral theory plays a crucial role.

Appendix A. Proof of Proposition 2.4

Proof. We proceed as in [16]. The weyl symbol of the operator AB is defined by([16], page 79):

$$c(x,\xi)=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}a(x+\omega,\rho+\xi)b(x+r,\tau+\xi)\,d\rho\,d\omega\,d\tau\,dr, \tag{6.1}$$

for every $(x,\xi)\in\mathbb{R}\times\mathbb{R}$. We split the oscillator integral c into two parts $c^{(1)}$ and $c^{(2)}$, then we use the cuttof function:

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho)=\chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)}\right], \tag{6.2}$$

and

$$\omega_{2,\varepsilon}=1-\omega_{1,\varepsilon},$$

where $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ in $[-1,1]$, $\chi\equiv 0$ in $\mathbb{R}\backslash]-2,2[$ and $\eta>0$.
Let's consider for $j\in\{1,2\}$,

$$d_j(x,\xi,\omega,\tau,r,\rho)=\omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho)a(x+\omega,\rho+\xi)b(x+r,\tau+\xi), \tag{6.3}$$

$c^{(1)}$ (resp $c^{(2)}$) the integral obtained in (6.1) by replacing the amplitude by d_1 (resp d_2)

study of $c^{(2)}$:

On the support of d_2 we have $\omega^2+\rho^2+r^2+\tau^2\geq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}$

We make an integration by parts using the operator:

$$M=\frac{1}{2i}(\omega^2+\rho^2+r^2+\tau^2)^{-1}(-\rho\partial_r-r\partial_\rho+\tau\partial_\omega+\omega\partial_\tau).$$

We have for all $k\in\mathbb{N}$,

$$c^{(2)}=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}({}^tM)^k d_2\,d\rho\,d\omega\,d\tau\,dr,$$

then we obtain for all $k>0$

$$c^{(2)}\in\Gamma_0^{p+p'-k,q+q'}. \tag{6.4}$$

study of $c^{(1)}$: the function $(\omega,\tau,r,\rho)\longrightarrow d_1(x,\xi,\omega,\tau,r,\rho)$ has a compact support, we deduce from ([16], Proposition (II-26)) that for every $N\in\mathbb{N}$

$$c^{(1)}=\sum_{j=0}^{N}c_j(x,\xi)+R_{N+1}(x,\xi), \tag{6.5}$$

where

$$c_j=\frac{1}{2^j}\sum_{\alpha+\beta=j}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(\partial_x^\alpha\partial_\xi^\beta b), \tag{6.6}$$

and

$$|R_{N+1}(x,\xi)|\leq c_N\left\|(\partial_\omega\partial_\tau-\partial_r\partial_\rho)^{N+1}d_1\right\|_{H^3(\mathbb{R}^4)}, \tag{6.7}$$

where $H^3(\mathbb{R}^4)$ is the Sobolev space.
Since $a\in\Gamma_1^{p,q}$ and $b\in\Gamma_0^{p',q'}$, we have for every $j\leq N$:

$$c_j\in\Gamma_0^{p+p'-j,q+q'}. \tag{6.8}$$

Let's study the rest (6.5), from (6.7) we have:

$$\begin{aligned}|R_{N+1}(x,\xi)| &\leq c_N\sum_{\substack{|\gamma|\leq 3\\ \gamma\in\mathbb{N}^4}}\left\|(\partial_\omega\partial_\tau-\partial_r\partial_\rho)^{N+1}d_1\partial^\gamma_{\omega,,\tau,r,\rho}d_1\right\|_{\mathbb{L}^2(\mathbb{R}^4)}\\ &\leq c_N(1+x^2+\xi^2)^2\\ &\quad\times\sup_{\substack{[\omega^2+\rho^2+r^2+\tau^2\leq 2\varepsilon(1+x^2+\xi^2)]\\ |\gamma|\leq 3}}|(\partial_\omega\partial_\tau-\partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,,\tau,r,\rho}d_1|,\end{aligned} \tag{6.9}$$

for $|\gamma|\leq 3$, we have:

$$|(\partial_\omega\partial_\tau-\partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,,\tau,r,\rho}d_1|=(N+1)!\sum_{\alpha+\beta=N+1}\frac{(-1)^\beta}{\alpha!\beta!}\partial_\omega^\beta\partial_\tau^\beta\partial_r^\alpha\partial_\rho^\alpha\partial^\gamma_{\omega,\tau,r,\rho}d_1. \tag{6.10}$$

Since a (resp b) are independent of (τ,r) (resp (ω,ρ)), for $\gamma=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ we have :

$$|\partial_\omega^\beta\partial_\tau^\beta\partial_r^\alpha\partial_\rho^\alpha\partial^\gamma_{\omega,\tau,r,\rho}d_1|\leq C\sum_{\substack{i_1+i_2=\beta+\gamma_1\\ i_p\leq\beta+\gamma_1\\ j_1+j_2=\beta+\gamma_2\\ j_p\leq\beta+\gamma_2\\ k_1+k_2=\alpha+\gamma_3\\ k_p\leq\alpha+\gamma_3\\ r_1+r_2=\alpha+\gamma_4\\ r_p\leq\alpha+\gamma_4}}|\partial_\omega^{i_1}\partial_\rho^{r_1}a||\partial_\tau^{j_1}\partial_r^{k_1}b||\partial_\omega^{i_2}\partial_\tau^{j_2}\partial_r^{k_2}\partial_\rho^{r_2}\omega_{1,\varepsilon}|, \tag{6.11}$$

for ε small enough and on the support of $\omega_{1,\varepsilon}$ we have:

$$|\partial_\omega^{i_2}\partial_\tau^{j_2}\partial_\tau^{k_2}\partial_\rho^{r_2}\omega_{1,\varepsilon}|\leq c(1+x^2+\xi^2)^{\frac{-1}{2}(i_2+j_2+k_2+r_2)}. \tag{6.12}$$

From (6.11),(6.12) and the fact that $a\in\Gamma_1^{p,q}$, $b\in\Gamma_0^{p',q'}$ we have

$$\begin{aligned}|\partial_\omega^\beta\partial_\tau^\beta\partial_r^\alpha\partial_\rho^\alpha\partial^\gamma_{\omega,\tau,r,\rho}d_1|&\leq C(1+x^2+\xi^2)^{\frac{p+p'}{2}}\log^{q+q'}(3+x^2+\xi^2)\\ &\quad\times\sum(1+x^2+\xi^2)^{-\frac{1}{2}(i_1+r_1+i_2+j_2+r_2+k_2)},\end{aligned} \tag{6.13}$$

as $i_1 + r_1 + i_2 + j_2 + r_2 + k_2 = N + 1 + \gamma_1 + \gamma_2 + j_2 + k_2$ then

$$|\partial_\omega^\beta \partial_\tau^\beta \partial_r^\alpha \partial_\rho^\alpha \partial_{\omega,\tau,r,\rho}^\gamma d_1| \leq C(1 + x^2 + \xi^2)^{\frac{p+p'-(N+1)}{2}} \log^{q+q'}(3 + x^2 + \xi^2). \tag{6.14}$$

from (6.10) and (6.14) we get:

$$|(\partial_\omega \partial_x - \partial_r \partial_\rho)^{N+1} \partial_{\omega,,\tau,r,\rho}^\gamma d_1| \leq C(1 + x^2 + \xi^2)^{\frac{p+p'-(N+1)}{2}} \log^{q+q'}(3 + x^2 + \xi^2). \tag{6.15}$$

Finally from(6.9) and (6.15) we deduce the following estimate of R_{N+1}

$$|R_{N+1}(x, \xi)| \leq C(1 + x^2 + \xi^2)^{\frac{p+p'-(N+1)+4}{2}} \log^{q+q'}(3 + x^2 + \xi^2), \tag{6.16}$$

we obtain the same estimate for $\partial_x^\alpha \partial_\xi^\beta R_{N+1}$, which proves that

$$R_{N+1} \in \Gamma_0^{m_1+m_2-(N+1)+4}.$$

The rest of the symbol c is given by:

$$\delta_{N+1}(x, \xi) = R_{N+1}(x, \xi) + c^{(2)}(x, \xi), \tag{6.17}$$

to obtain the estimate on δ_{N+1} we use (6.4),(6.16) and (6.17) and in pushing the development a bit further, i.e. by writing:

$$\delta_{N+1}(x, \xi) = c_{N+1} + \cdots + c_{N+k} + \delta_{N+1+k},$$

and choosing $k \geq 4$, we then have

$$\delta_{N+1} \in \Gamma_0^{m_1+m_2-(N+1)+4}.$$

ii) It is enough to do the demonstration for $n = 2$, by writing:

$$\begin{aligned} B_1 B_2 H^{-\frac{p_1+p_2}{2h}} \log^{-(q_1+q_2)}(3 + H^{\frac{1}{h}}) &= B_1 H^{-\frac{p_1}{2h}} \log^{-q_1}\left(3 + H^{\frac{1}{h}}\right) \\ &\quad \times \log^{q_1}\left(3 + H^{\frac{1}{h}}\right) H^{\frac{p_1}{2h}} B_2 H^{-\frac{p_1+p_2}{2h}} \\ &\quad \times \log^{-(q_1+q_2)}(3 + H^{\frac{1}{h}}). \end{aligned}$$

On the one hand, using i), we obtain:

$$B_1 H^{-\frac{p_1}{2h}} \log^{-q_1}\left(3 + H^{\frac{1}{h}}\right) \in G_0^{0,0}.$$

On the other hand:

$$B_2 H^{-\frac{p_1+p_2}{2h}} \log^{-(q_1+q_2)}(3 + H^{\frac{1}{h}}) \in G_0^{-p_1,-q_1}.$$

Finally, we conclude that:

$$\log^{-q_1}\left(3 + H^{\frac{1}{h}}\right) \log^{q_1}\left(3 + H^{\frac{1}{h}}\right) H^{\frac{p_1}{2h}} B_2 H^{-\frac{p_1+p_2}{2h}} \log^{-(q_1+q_2)}(3 + H^{\frac{1}{h}}) \in G_0^{0,0}.$$

This proves ii), as required. $\square$

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