EXISTENCE SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS AND $L^1$ DATA

C. YAZOUGH*

ABSTRACT. In this article, we study the problem

\[
\begin{aligned}
\frac{\partial b(u)}{\partial t} - \text{div } a(x, t, u, \nabla u) + \text{div } \phi(u) &= f \quad \text{in } \Omega \times [0, T], \\
u &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
b(x, u)(t = 0) &= b(x, u_0) \quad \text{in } \Omega,
\end{aligned}
\]

in the framework of generalized Sobolev spaces, with $b(x, u)$ unbounded function on $u$. The main contribution of our work is to prove the existence of renormalized solutions when the second term $f$ belongs to $L^1(Q_T)$.

1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $p : \Omega \to [2, +\infty)$ be a continuous, real-valued function and let $p^- = \min_{x \in \Omega} p(x)$ and $p^+ = \max_{x \in \Omega} p(x)$ such that $2 < p^- < p^+ < \infty$, $Q = \Omega \times [0, T]$. And let $Au = -\text{div}(a(x, t, u, Du))$ be a Leray-Lions operator defined from the generalized Sobolev space $V$ into its dual $V^*$.

Now, we consider the parabolic problem associated for the differential equation

\[
\begin{aligned}
\frac{\partial b(u)}{\partial t} + Au + \text{div } \phi(u) &= f \quad \text{in } Q, \\
u &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
b(u)(t = 0) &= b(u_0) \quad \text{on } \Omega,
\end{aligned}
\]

where the function $b$ is assumed to be strictly increasing $C^1$-function, $b(u_0)$ lie in $L^1(\Omega)$, and $a$ is the Caratheodory function.

For the parabolic equation (1.1) the existence of weak solution has been proved by J-M Rakotoson [22] with the strict monotonicity and a measure data, the existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [15] in the case where $a(x, t, s, \xi)$ is independent of $s$, and by D. Blanchard, F. Murat and H. Redwane [13] with the large monotonicity on $a$. For the degenerated parabolic equations the existence of weak solutions have

\begin{flushright}
\text{Date: Received: Dec 2, 2018.} \\
\text{*Corresponding author.} \\
\text{2010 Mathematics Subject Classification. 35K10, 35K59, 35K61.} \\
\text{Key words and phrases. Sobolev spaces with variable exponent, Quasilinear parabolic equations, Renormalized solutions.}
\end{flushright}
been proved by proved by L. Aharouch and al [1] in the case where \( a \) is strictly monotone and \( f \in L^p(0, T, W^{-1,p}(\Omega)) \). See also the existence of renormalized solution by Y. Afdim et al [11] in the case \( a(x, t, s, \xi) = |\xi|^{p(x) - 2} \xi \), and [4] if we replace \( g(x, s, \xi) \) by \( \text{div}(\phi(u)) \).

Our paper can be see as a continuous of [20] in the case where \( b \) monotone and been proved by L. Aharouch and al [1] in the case where \( a(x, t, s, \xi) = |\xi|^{p(x) - 2} \xi \), and [4] if we replace \( g(x, s, \xi) \) by \( \text{div}(\phi(u)) \).

The plan of our paper is as follow: In section 2 we give some preliminaries. In section 3 we make precise all the assumptions on \( a \) and \( b \). In section 4 we establish some technical lemmas. In section 5 we prove our main result: the existence of renormalized solutions of problem (1.1).

2. Preliminaries

For each open bounded subset \( \Omega \) of \( \mathbb{R}^N \) \( (N \geq 2) \), we denote

\[
C_+(\overline{\Omega}) = \{ \text{continuous function } p(\cdot): \overline{\Omega} \rightarrow \mathbb{R}^+ \text{ such that } 1 < p_- \leq p_+ < \infty \},
\]

where

\[
p_- = \min_{x \in \Omega} p(x) \quad \text{and} \quad p_+ = \max_{x \in \Omega} p(x).
\]

We define the variable exponent Lebesgue space for \( p(\cdot) \in C_+(\overline{\Omega}) \) by:

\[
L^{p(\cdot)}(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable \, / \, } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \}.
\]

The space \( L^{p(\cdot)}(\Omega) \) under the norm:

\[
\| u \|_{p(x)} = \inf \left\{ \lambda > 0, \quad \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}
\]

is a uniformly convex Banach space, then reflexive. and we denote by \( L^{p'(\cdot)}(\Omega) \) the conjugate space of \( L^{p(\cdot)}(\Omega) \) where \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \).

**Proposition 2.1. (cf[17])** (i) For any \( u \in L^{p(\cdot)}(\Omega) \) and \( v \in L^{p'(\cdot)}(\Omega) \), we have the H"older type inequality:

\[
\left| \int_{\Omega} u v \, dx \right| \leq \left( \frac{1}{p_-} + \frac{1}{p'_-} \right) \| u \|_{p(x)} \| v \|_{p'(x)}.
\]

(ii) For all \( p_1, p_2 \in C_+(\overline{\Omega}) \) such that \( p_1(x) \leq p_2(x) \) for any \( x \in \overline{\Omega} \), then \( L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega) \) and the embedding is continuous.

**Proposition 2.2. (cf[17])** If we denote

\[
\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx \quad \forall u \in L^{p(\cdot)}(\Omega),
\]

then the following assertions holds:

(i): \( \| u \|_{p(x)} < 1 \) (resp, \( = 1, > 1 \)) \iff \( \rho(u) < 1 \) (resp, \( = 1, > 1 \)),

(ii): \( \| u \|_{p(x)} > 1 \Rightarrow \| u \|_{p(x)}^{p_-} \leq \rho(u) \leq \| u \|_{p(x)}^{p_+} \) and \( \| u \|_{p(x)} < 1 \Rightarrow \| u \|_{p(x)}^{p_-} \leq \rho(u) \leq \| u \|_{p(x)}^{p_+} \);

(iii): \( \| u \|_{p(x)} \rightarrow 0 \iff \rho(u) \rightarrow 0 \), \text{ and } \( \| u \|_{p(x)} \rightarrow \infty \iff \rho(u) \rightarrow \infty \).
Let
\[ W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega) \}, \]
which is a Banach space equipped with the following norm
\[ \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \]
The space \((W^{1,p(x)}(\Omega), \| \cdot \|_{1,p(x)})\) is a separable and reflexive Banach space.

Next, we define \(W_0^{1,p(x)}(\Omega)\) as the closure of \(C^\infty(\Omega)\) in \(W^{1,p(x)}(\Omega)\) and
\[
p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}
\]

**Proposition 2.3.** (see [18])

(i): Assuming \(1 < p_- \leq p_+ < \infty\), the spaces \(W^{1,p(x)}(\Omega)\) and \(W_0^{1,p(x)}(\Omega)\) are separable and reflexive Banach spaces.

(ii): If \(q(x) \in C_+(\Omega)\) and \(q(x) < p^*(x)\) for a.e. \(x \in \Omega\), then the embedding \(W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)\) is continuous and compact.

(iii): Poincaré type inequality: there exists a constant \(C > 0\), such that
\[
\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).
\]

Extending the definition of variable exponent \(p(\cdot) : \bar{\Omega} \rightarrow [1, \infty)\) to \(\bar{\Omega}_T = \Omega \times [0,T]\) by setting \(p(x,t) := p(x)\) for all \((x,t) \in \bar{\Omega}\), we may also consider the generalized Lebesgue space
\[
L^{p(x)}(\bar{\Omega}_T) = \{ u : \bar{\Omega}_T \rightarrow IR \text{ measurable} / \int_{\bar{\Omega}_T} |u(x,t)|^{p(x)} \, dx \, dt < \infty \},
\]
endowed with the norm
\[
\|u\|_{L^{p(x)}(\bar{\Omega}_T)} = \inf \left\{ \lambda > 0, \int_{\bar{\Omega}_T} \left| \frac{u(x,t)}{\lambda} \right|^{p(x)} \, dx \, dt \leq 1 \right\}
\]
which, of course, shares the same type of properties as \(L^{p(x)}(\Omega)\).

Now, we introduce the functional space
\[
V = \{ u \in L^{p_-}(0,T,W_0^{1,p(x)}(\Omega)) : |\nabla u| \in L^{p(x)}(\bar{\Omega}_T) \}
\]
which, endowed with the norm
\[
\|u\|_V := \|\nabla u\|_{L^{p(x)}(\bar{\Omega}_T)}
\]
or, the equivalent norm
\[
\|u\|_V := \|u\|_{L^{p_-}(0,T,W_0^{1,p(x)}(\Omega))} + \|\nabla u\|_{L^{p(x)}(\bar{\Omega}_T)}
\]
is separable and reflexive Banach space.

We have used the standard notations for Bochner spaces, i.e. if \(X\) is a Banach space and \(q \geq 1\), then \(L^q(0,T;X)\) denotes the space of strongly measurable function
\[
u : (0,T) \hookrightarrow X, \quad t \mapsto \|u(t)\|_X \in L^q(0,T).
\]
Moreover, $C([0,T];X)$ denotes the space of continuous functions $u : [0,T] \to X$ endowed with the norm
\[
\|u\|_{C([0,T];X)} := \max_{t \in [0,T]} \|u(t)\|_X.
\]

**Lemma 2.4.** The dual of the Banach space $V$ is denoted by $V^*$, then we have the following continuous embedding:
\[
L^{p_+}(0,T,W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p_-}(0,T,W^0_0^{1,p(x)}(\Omega))
\]
In particular, since $\mathcal{D}(Q_T)$ is dense in $L^{p_+}(0,T,W_0^{1,p(x)}(\Omega))$, it is dense in $V$ and for the corresponding dual spaces we have
\[
L^{(p_-)'}(0,T,W^{-1,p'(x)}(\Omega)) \hookrightarrow V^* \hookrightarrow L^{(p_+)'}(0,T,W^{-1,p'(x)}(\Omega))
\]

**Proposition 2.5.** One can represent the elements of $V^*$ as follow: if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^{p'(x)}(Q))^N$ such that $T = \text{div} F$ and
\[
<T, \zeta>_{V^*,V} = \int_0^T \int_{\Omega} F \cdot \nabla \zeta \, dx \, dt \quad \text{for any} \; \zeta \in V
\]
Moreover, we have
\[
\|T\|_{V^*} = \max\{\|f_i\|_{L^{p'(x)}(Q)}, i = 1, ..., N\}
\]

3. **Basic Assumptions**

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N \ (N \geq 2)$, $T > 0$ and $p_- > 2$.

We consider a Leray-Lions operator $A$ acted from $V$ into its dual $V^*$, defined by the formula
\[
Au = -\text{div} \ a(x,t,u,\nabla u)
\]
where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function (measurable with respect to $(x,t)$ in $Q_T$ and continuous with respect to $(s,\xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $(x,t)$ in $Q_T$) which satisfies the following conditions
\[
|a(x,t,s,\xi)| \leq \beta(K(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad (3.2)
\]
\[
(a(x,t,s,\xi) - a(x,t,s,\eta)) \cdot (\xi - \eta) > 0 \quad \text{for} \quad \xi \neq \eta, \quad (3.3)
\]
\[
a(x,t,s,\xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \quad (3.4)
\]
for a.e. $(x,t) \in Q_T$, all $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$, where $K(x,t)$ is a positive function lying in $L^{p'(x)}(Q_T)$ and $\alpha, \beta > 0$.

We assume that
\[
b(\cdot,\cdot) : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{is a Carathéodory function such that} \quad b(x,0) = 0 \quad \text{and} \quad b(x,\cdot) \quad \text{is a strictly increasing function in} \quad C^1(\mathbb{R}) \quad \text{for a.e.} \quad x \in \Omega.
\]

Moreover, For any $K > 0$, there exists $\lambda_K > 0$, a function $A_K \in L^\infty(\Omega)$ and a function $B_K \in L^{p(x)}(\Omega)$ such that
\[
\lambda_K \leq \frac{\partial b(x,s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left|\nabla_x \left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_K(x), \quad (3.6)
\]
for almost every $x \in \Omega$, for every $s$ such that $|s| \leq K$.

We consider the quasilinear $p(x)$-parabolic problem

$$
\begin{align*}
\frac{\partial b(x,u)}{\partial t} - \text{div} \ a(x,t,u,\nabla u) + \text{div} \ \phi(u) &= f & \text{in } \Omega \times ]0,T[,
\end{align*}
$$

$$
\left\{ \begin{array}{ll}
u(x,t) = 0 & \text{on } \partial \Omega \times ]0,T[ \quad \text{in } \Omega,
\end{array} \right.
$$

with $\phi(\cdot) \in C(\mathbb{R}, \mathbb{R}^N)$, $f \in L^1(\Omega \times ]0,T[)$ and $u_0 \in L^1(\Omega)$ such that $b(x,u_0) \in L^1(\Omega)$.

**Remark 3.1.** The problem (3.7) does not admit a weak solution under assumptions (3.2) – (3.6) (even when $b(x,u) = u$) since the growths of $a(x,t,u,\nabla u)$ and $\phi(u)$ are not controlled with respect to $u$ (so that these fields are not in general defined as distributions, even when $u$ belongs to $V$).

### 4. Some technical results

Firstly, we establish some embedding and compactness results in generalized Sobolev spaces.

Let $p_- > 2$, we set $X = W^{1,p(x)}_0(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,p'(x)}(\Omega)$.

Denoting the space $W^{1,p(x)}_0(0,T;X,H) = \{v \in V \text{ and } v_t \in V^*\}$ endowed with the norm

$$
\|u\|_{W^{1,p(x)}_0(0,T;X,H)} = \|u\|_V + \|u_t\|_{V^*}
$$

is a Banach space. Here $u_t$ stands for the generalized derivative of $u$; i.e.,

$$
\int_0^T u_t(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{for all } \varphi \in C_0^\infty(0,T).
$$

**Lemma 4.1.** Let $B_0$, $B$ and $B_1$ be Banach spaces with $B_0 \subset B \subset B_1$. Let us set

$$
Y = \{u : u \in L^{p_0}(0,T;B_0) \quad \text{and} \quad u_t \in L^{p_1}(0,T;B_1)\}
$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.

Assuming that the embedding $B_0 \hookrightarrow B \hookrightarrow B_1$ be compact, then

$$
Y \hookrightarrow L^{p_0}(0,T;B)
$$

and this imbedding is compact.

**Remark 4.2.** Let $p_- > 2$, we set

$$
B_0 = W^{1,p(x)}_0(\Omega), \quad B = L^2(\Omega) \quad \text{and} \quad B_1 = W^{-1,p'(x)}(\Omega),
$$

with $p_0 = p_-$ and $p_1 = p'_-$. In view of the Lemma 4.1, we obtain

$$
W^{1,p(x)}_1(0,T,X,H) \subseteq Y \hookrightarrow L^2(Q_T).
$$

Moreover, in view of [9], we have

$$
W^{1,p(x)}_1(0,T,X,H) \subseteq C([0,T];L^2(\Omega)).
$$

Now, To deal with time derivative in the Sobolev space with variable exponents, we introduce a time mollification of a function $u \in V$ as follows
Proposition 4.3. [9, 27] The time mollification of a function $u \in V$ for any $\mu \geq 0$ is introduced by
\[
    u_\mu(x, t) = \mu \int_{-\infty}^{t} \pi(x, s) \exp(\mu(s - t)) \, ds \quad \text{where} \quad \pi(x, s) = u(x, s) \chi_{(0, T)}(s),
\]
the following assertions holds
\begin{enumerate}[(i)]
    \item If $u \in L^{p(x)}(Q_T)$, then $u_\mu$ is measurable in $Q_T$, $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and
    \[
        \int_{Q_T} |u_\mu|^p \, dx \, dt \leq \int_{Q_T} |u|^p \, dx \, dt.
    \]
    \item If $u \in V$, then $u_\mu \to u$ in $V$ as $\mu \to +\infty$.
    \item If $u_n \to u$ in $V$, then $(u_n)_\mu \to u_\mu$ in $V$.
    \item We have $\|(T_k(u))_\mu\| \leq k$ for any $u \in V$.
\end{enumerate}

Lemma 4.4. (see. [3]) Let $g \in L^{r}(x)(Q_T)$ and $g_n \in L^{r}(x)(Q_T)$, with $\|g_n\|_{L^{r}(x)(Q_T)} \leq C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in $Q_T$, then $g_n \to g$ in $L^{r}(x)(Q_T)$ where $n \to \infty$.

Lemma 4.5 (see. [1]). Assume that
\[
    \frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad \text{in} \quad D'(Q_T)
\]
where $\alpha_n$ and $\beta_n$ are bounded respectively in $V^*$ and in $L^1(Q_T)$. If $v_n$ is bounded in $V$, then $v_n \to v$ in $L^{p(x)}(Q_T)$. Further $v_n \to v$ strongly in $L^1(Q_T)$ where $n \to \infty$.

Lemma 4.6. Assuming that (3.2) – (3.4) holds, and let $(u_n)_n$ be a sequence in $V$ such that $u_n \to u$ in $V$ and
\[
    \int_{Q_T} \left( a(x, t, \nabla u_n) - a(x, t, \nabla u) \right) \cdot (\nabla u_n - \nabla u) \, dx \to 0, \quad (4.3)
\]
then $u_n \to u$ in $V$ for a subsequence.

The proof of this Lemma is the same as in the case of constant exponent $p$ (see. [6]).

5. Main results

Definition 5.1. A measurable function $u$ is an renormalized solution of the Dirichlet problem (3.7) if $T_k(u) \in V \quad \forall k \geq 0$,
\[
    b(x, u) \in L^{\infty}(0, T; L^1(\Omega)) \quad \text{and} \quad \lim_{m \to \infty} \int_{\{m < |u| \leq m+1\}} a(x, t, u, \nabla u) \cdot \nabla u \, dx \, dt = 0, \quad (5.1)
\]
\[
    \frac{\partial B_S(x, u)}{\partial t} = \text{div} \left( S'(u)a(x, t, u, \nabla u) \right) + S''(u)a(x, t, u, \nabla u) \cdot \nabla u \
    - \text{div} \left( \phi'(u) - S'(u)\phi(u) + S''(u)\phi(u) \cdot \nabla u = fS'(u) \quad \text{in} \quad D'(Q_T), \quad (5.2)
\]

\[
    - \text{div} \left( \phi'(u) + S''(u)\phi(u) \cdot \nabla u = fS'(u) \quad \text{in} \quad D'(Q_T), \quad (5.2)
\]
for any functions $S(\cdot) \in W^{2,\infty}(\mathbb{R})$ such that $S'(\cdot)$ has a compact support in $\mathbb{R}$, with
\[
B_S(x, z) = \int_0^z \frac{\partial b(x, r)}{\partial r} S'(r) dr \quad \text{and} \quad B_S(x, u)(t = 0) = B_S(x, u_0) \quad \text{in } \Omega.
\]
(5.3)

**Theorem 5.2.** Let $f \in L^1(Q_T)$ and $b(x, u_0) \in L^1(\Omega)$. Assume that (3.2)–(3.6) hold. Then, there exists at least one renormalized solution $u$ of problem (3.7).

**Proof of the Theorem 5.2.**

**Step 1: Approximate problems.** Let $(f_n)_n$ be a sequence in $V^* \cap L^1(Q_T)$ such that $f_n \to f$ in $L^1(Q_T)$ with $|f_n| \leq |f|$, and let $(u_{0,n})_n$ be a sequence in $C^1_0(\Omega)$ such that $b(x, u_{0,n}) \to b(x, u_0)$ in $L^1(\Omega)$ and $|b(x, u_{0,n})| \leq |b(x, u_0)|$. We define the following approximation
\[
b_n(x, r) = b(x, T_n(r)) + \frac{1}{n}, \quad a_n(x, t, s, \xi) = a(x, t, T_n(s), \xi) \quad \text{and} \quad \phi_n(s) = \phi(T_n(s)).
\]
(5.4)

We consider the approximate problem:
\[
\begin{cases}
\frac{\partial u_n}{\partial t} - \text{div } a_n(x, t, u_n, \nabla u_n) - \text{div } \phi_n(u_n) = f_n & \text{in } Q_T, \\
u_n = 0 & \text{on } \partial \Omega \times (0, T), \\
b_n(x, u_n)(t = 0) = b_n(x, u_{0,n}) & \text{in } \Omega.
\end{cases}
\]
(5.5)

In view of [21], there exists at least one weak solution $u_n \in V$ of the problem (5.5).

**Step 2: Weak convergence of truncations.** Let $\tau \in [0, T]$, taking $T_k(u_n)\chi(0,\tau)$ as a test function in (5.5), we obtain
\[
\int_\Omega \frac{\partial b_n(x, u_n)}{\partial t}, T_k(u_n)\chi(0,\tau) dx + \int_0^\tau \int_\Omega a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \\
+ \int_0^\tau \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n) dx dt = \int_0^\tau \int_\Omega f_n T_k(u_n) dx dt,
\]
\[
\int_\Omega B^n_k(x, u_n(\tau)) dx + \int_0^\tau \int_\Omega a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \\
+ \int_0^\tau \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n) dx dt = \int_0^\tau \int_\Omega f_n T_k(u_n) dx dt + \int_\Omega B^n_k(x, u_{0,n}) dx,
\]
(5.6)

with $B^n_k(x, r) = \int_0^r T_k(s)b'_n(x, s) ds$.

Taking $\Phi_n(s) = \int_0^s \phi_n(\sigma) d\sigma$, then $\Phi_n(0) = 0_{\mathbb{R}^N}$ and $\Phi_n(\cdot) \in C^1(\mathbb{R}, \mathbb{R}^N)$, in
view of the Divergence Theorem, we obtain
\[
\int_0^T \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n) \, dx \, dt = \int_0^T \int_\Omega \text{div} \Phi_n(T_k(u_n)) \, dx \, dt
= \int_0^T \int_\partial\Omega \Phi_n(T_k(u_n)) \cdot \vec{n} \, d\sigma \, dt = 0,
\]
(5.7)
since \( u_n = 0 \) on \( \partial\Omega \) and \( \vec{n} = (n_1, n_2, \ldots, n_N) \) the exterior normal vector on the boundary \( \partial\Omega \).

On the other hand, since \( b_n(s) \) have the same sign as \( s \), then
\[
0 \leq \int_\Omega B_k^{n}(x, u_{0,n}) \, dx \leq k \int_\Omega |b_n(x, u_{0,n})| \, dx \leq k ||b(x, u_0)||_{L^1(\Omega)}.
\]
(5.8)

By using (5.6) − (5.8), we deduce that
\[
0 \leq \int_\Omega B_k^{n}(x, u_{0,n}) \, dx + \int_0^T \int_\Omega a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt
\leq k(\|f_n\|_{L^1(Q_T)} + \|b(u_0)\|_{L^1(\Omega)})
\leq C_1 k
\]
for any \( \tau \in [0, T] \).

(5.9)

It’s clear that \( \int_\Omega B_k^{n}(x, u_{0,n}) \, dx \geq 0 \). Thanks to (3.4) we obtain
\[
\|\nabla T_k(u_n)\|_{L^p(\Omega \times (Q_T))} \leq \int_{Q_T} |\nabla T_k(u_n)|^p \, dx \, dt + 1 \leq C_2 k \quad \forall k \geq 1.
\]
(5.10)

We have \( T_k(u_n) \) is bounded in \( V \). In view of the Remark 4.2, there exists a subsequence still denoted \( T_k(u_n) \) and \( v_k \in V \) such that
\[
T_k(u_n) \rightarrow v_k \quad \text{in} \quad V \quad \text{and} \quad T_k(u_n) \rightarrow v_k \quad \text{a.e. in} \quad Q_T.
\]
(5.11)

Let \( k > 0 \) large enough, by virtue of Holder’s and Poincaré inequality, we have:
\[
k \cdot \text{meas} \{ |u_n| > k \} = \int_\Omega |T_k(u_n)| \, dx \, dt \leq \int_\Omega |T_k(u_n)| \, dx \, dt
\leq \left( \frac{1}{p} + \frac{1}{p'} \right) \|T_k(u_n)\|_{L^p(\Omega \times (Q_T)}} \|1\|_{L^{p'}(\Omega \times (Q_T))}
\leq C_3 \|\nabla T_k(u_n)\|_{L^{p'}(\Omega \times (Q_T))}
\leq C_4 k^{\frac{1}{p'}},
\]
which implies that
\[
\text{meas} \{ |u_n| > k \} \leq C_5 \frac{1}{k^{1 - \frac{1}{p'}}} \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty,
\]
(5.12)
since for all \( \delta > 0 \) that
\[
\text{meas} \{ |u_n - u_m| > \delta \} \leq \text{meas} \{ |u_n| > k \} + \text{meas} \{ |u_m| > k \}
+ \text{meas} \{ |T_k(u_n) - T_k(u_m)| > \delta \}.
\]
(5.13)

Using (5.12) we get that for all \( \varepsilon > 0 \), that exists \( k_0 > 0 \) such that
\[
\text{meas} \{ |u_n| > k \} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas} \{ |u_m| > k \} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0(\varepsilon).
\]
(5.14)
On the other hand, in view of (5.11), we can assume that \((T_k(u_n))_n\) is a Cauchy sequence in measure in \(Q_T\), then for all \(k > 0\) and \(\delta, \varepsilon > 0\) there exists \(n_0 = n_0(\delta, \varepsilon)\) such that
\[
\text{meas}\{ |T_k(u_n) - T_k(u_m)| > \delta \} \leq \varepsilon \quad \forall n, m \geq n_0. \tag{5.15}
\]
By combining (5.13) – (5.15), we deduce that for all \(\varepsilon, \delta > 0\), there exists \(n_0 = n(\delta, \varepsilon)\) such that
\[
\text{meas}\{ |u_n - u_m| > \delta \} \leq \varepsilon \quad \forall n, m \geq n_0. \tag{5.16}
\]
If follows that \((u_n)_n\) is a Cauchy sequence in measure, then there exists a subsequence still denoted \((u_n)_n\) such that
\[
u_n \to u \quad \text{a.e. in } Q_T, \tag{5.17}
\]
\[
b_n(x, u_n) \to b(x, u) \quad \text{a.e. in } Q_T. \tag{5.18}
\]
Using (5.11), we deduce that
\[
T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } V, \tag{5.19}
\]
and in the view of the Lebesgue dominated convergence theorem
\[
T_k(u_n) \longrightarrow T_k(u) \quad \text{in } L^{p(x)}(Q_T). \tag{5.20}
\]
We now establish that \(b(x, u)\) belongs to \(L^\infty(0, T; L^1(\Omega))\). Using (5.9), (5.17) and Fatou’s Lemma, we conclude that
\[
0 \leq \frac{1}{k} \int_\Omega B_k(x, u(\tau)) \, dx \leq (\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}) = C_6,
\]
for any \(\tau \in (0, T)\). Due to the definition of \(B_k(x, s)\) and the fact that \(\frac{1}{k} B_k(x, u)\) converges pointwise to \(b(x, u)\), as \(k\) tends to 0, shows that \(b(x, u)\) belong to \(L^\infty(0, T; L^1(\Omega))\).

**Step 3 : A priori estimates.** Taking \(T_1(u_n - T_h(u_n))\) as a test function in (5.5), we obtain
\[
\int_0^T \left( \frac{\partial b_n(x, u_n)}{\partial t}, T_1(u_n - T_h(u_n)) \right) \, dt + \int_{\{h < |u_n| \leq h+1\}} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt
\]
\[
+ \int_{\{h < |u_n| \leq h+1\}} \phi_n(u_n) \cdot \nabla u_n \, dx \, dt = \int_{Q_T} f_n T_1(u_n - T_h(u_n)) \, dx \, dt.
\]
We have
\[
\int_{\{h < |u_n| \leq h+1\}} \phi_n(u_n) \cdot \nabla u_n \, dx \, dt = \int_0^T \int_\Omega \text{div } \Phi_n(T_{h+1}(u_n)) \, dx \, dt - \int_0^T \int_\Omega \text{div } \Phi_n(T_h(u_n)) \, dx \, dt = 0,
\]
it follows that
\[
\int_{\Omega} B_n, h(x, u_n(T)) \, dx + \int_{\{h < |u_n| \leq h+1\}} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt
\]
\[
\leq \int_{\{|u_n| > h\}} |f_n| \, dx \, dt + \int_{\{|u_0| > h\}} B_n, h(x, u_0, n) \, dx
\]
with $B_{n,h}(x,r) = \int_0^r b'_n(x,s)T_1(s-T_h(s)) \, ds$, it’s clear that $B_{n,h}(x,u_n(T)) \geq 0$, then

$$\lim_{n \to \infty} \int_{\{|x| \geq n\}} a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla u_n \, dx \leq \int_{\{|u| > h\}} |f| \, dx + \int_{\{|u| > h\}} b(x,u_0) \, dx.$$ 

Since $f \in L^1(Q_T)$ and $b(x,u_0) \in L^1(\Omega)$, we obtain

$$\lim_{h,n \to \infty} \int_{\{h \leq |u_n| \leq h+1\}} a(x,t,u_n,\nabla u_n) \cdot \nabla u_n \, dx \, dt = 0. \quad (5.21)$$

and thanks to (3.4), we deduce that

$$\lim_{h,n \to \infty} \int_{\{h \leq |u_n| \leq h+1\}} |\nabla u_n|^{p(x)} \, dx \, dt = 0. \quad (5.22)$$

**Step 4 : Convergence of the gradient.** This step is devoted to establish the following limits

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \to a(x,t,T_k(u),\nabla T_k(u)) \quad \text{weakly in } (L^{p(x)}(Q_T))^N. \quad (5.23)$$

Let $h \geq k > 0$ and $n$ large enough ($n > h + 1$). In the sequel, we denote by $\varepsilon_i(n) \, i = 1, 2, \ldots$ a various functions of real numbers which converges to 0 as $n$ tends to infinity (respectively for $\varepsilon_i(n,\mu)$ and $\varepsilon_i(n,\mu,h)$).

Let $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$, where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$.

Taking $S_h(\cdot) \in W^{2,\infty}(\mathbb{R})$ an increasing function, such that $S_h(r) = r$ for $|r| \leq h$ and $\text{supp}(S'_h) \subset [-h-1, h+1]$, then $\text{supp}(S''_h) \subset [-h-1, -h] \cup [h, h+1]$.

Using $\omega_{n,\mu} S'''_h(u_n)$ as a test function in (5.5), we obtain

$$\int_0^T \left( \frac{\partial b_n(x,u_n)}{\partial t}, \omega_{n,\mu} S'''_h(u_n) \right) \, dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla \omega_{n,\mu} S''_h(u_n) \, dx \, dt$$

$$+ \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla u_n S''_h(u_n) \omega_{n,\mu} \, dx \, dt + \int_{Q_T} \phi_n(u_n) \cdot \nabla \omega_{n,\mu} S'_h(u_n) \, dx \, dt$$

$$+ \int_{Q_T} \phi_n(u_n) \cdot \nabla u_n S'_h(u_n) \omega_{n,\mu} \, dx \, dt = \int_{Q_T} f_n \omega_{n,\mu} S'_h(u_n) \, dx \, dt. \quad (5.24)$$

Now, we study each terms of the above inequality.

For the first term on the left hand side of (5.24), we have

$$\int_0^T \left( \frac{\partial b_n(x,u_n)}{\partial t}, \omega_{n,\mu} S'''_h(u_n) \right) \, dt = \int_0^T \int_\Omega \frac{\partial b_n(x,u_n)}{\partial t} (T_k(u_n) - (T_k(u))_\mu) S''_h(u_n) \, dx \, dt \geq \varepsilon_1(n,\mu),$$

see Lemma 3.2 of [13].

Concerning the third term on the left-hand side of (5.24). Let $n$ large enough,
since $\text{supp } (S''_h) \subset [-h - 1, -h] \cup [h, h + 1]$, and thanks to (5.21), we obtain

$$
\varepsilon_2(n, h) \leq \left| \int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n S''_h(u_n) \omega_{n, \mu} \, dx \, dt \right|
$$

$$
= \int_{\{h < |u_n| \leq h + 1\}} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n \cdot S''_h(u_n) ||\omega_{n, \mu}|| \, dx \, dt
$$

$$
\leq 2k ||S''_h||_{L^\infty(\mathbb{R})} \int_{\{h < |u_n| \leq h + 1\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dx \, dt \rightarrow 0 \quad \text{as } n, h \rightarrow \infty.
$$

(5.25)

For the fourth and fifth terms on the left-hand side of (5.24), it’s clear that

$$
\phi_n(T_{h+1}(u_n)) = \phi(T_{h+1}(u_n)) \rightarrow \phi(T_k(u)) \text{ in } (L^{p(x)}(Q_T))^N, \text{ and since } \nabla T_k(u_n) - \nabla (T_k(u)) \rightarrow 0 \text{ in } (L^{p(x)}(Q_T))^N,
$$

we conclude that

$$
\varepsilon_3(n, \mu) = \left| \int_{Q_T} \phi_n(u_n) \cdot \nabla \omega_{n, \mu} S'_h(u_n) \, dx \, dt \right|
$$

$$
= \left| \int_{\{u_n \leq \mu + 1\}} \phi_n(T_{h+1}(u_n)) \cdot (\nabla T_k(u_n) - \nabla (T_k(u)) \omega_{n, \mu} S'_h(u_n) \, dx \, dt \right|
$$

$$
\leq ||S'_h||_{L^\infty(\mathbb{R})} \int_{\{u_n \leq \mu + 1\}} |\phi(T_{h+1}(u_n))| ||\nabla T_k(u_n) - \nabla (T_k(u)) \omega_{n, \mu} S'_h(u_n) | \, dx \, dt \rightarrow 0 \quad \text{as } n, \mu \rightarrow \infty.
$$

(5.26)

In view of Young’s inequality and (5.22), we obtain

$$
\varepsilon_4(n, \mu, h) = \left| \int_{Q_T} \phi_n(u_n) \cdot \nabla u_n S''_h(u_n) \omega_{n, \mu} \, dx \, dt \right|
$$

$$
\leq ||S''_h||_{L^\infty(\mathbb{R})} \int_{\{h < |u_n| \leq h + 1\}} \phi_n(T_{h+1}(u_n)) |\nabla u_n| ||\omega_{n, \mu}|| \, dx \, dt
$$

$$
\leq ||S''_h||_{L^\infty(\mathbb{R})} \int_{\{h < |u_n| \leq h + 1\}} \frac{|\phi_n(T_{h+1}(u_n))| \psi'}{\psi'} |\omega_{n, \mu}|| \, dx \, dt
$$

$$
+ 2k ||S''_h||_{L^\infty(\mathbb{R})} \int_{\{h < |u_n| \leq h + 1\}} \frac{|\nabla u_n| \psi'}{\psi} \, dx \, dt \rightarrow 0 \quad \text{as } n, \mu \text{ then } h \rightarrow \infty.
$$

(5.27)

For the term on the right-hand side of (5.24), we have $f_n \rightarrow f$ in $L^1(Q_T)$, and since $T_k(u_n) - (T_k(u)) \rightarrow 0$ weak-$*$ in $L^\infty(Q_T)$, then

$$
|\varepsilon_5(n, \mu)| \leq ||S'_h||_{L^\infty(\mathbb{R})} \int_{Q_T} |f_n| \left| \phi_k(T_k(u_n) - (T_k(u)) \mu) \right| \, dx \, dt \rightarrow 0 \quad \text{as } n, \mu \rightarrow \infty.
$$

(5.28)

By combining (5.24) – (5.28), we deduce that

$$
\limsup_{n, \mu \rightarrow \infty} \int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla (T_k(u_n) - (T_k(u)) \mu) \, dx \, dt \leq 0, \quad (5.29)
$$

and since $S'_h(u_n) = 1$ on $\{|u_n| \leq k\}$, then

$$
\limsup_{n \rightarrow \infty} \int_{Q_T} a(x, t, T_n(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt
$$

$$
\leq \limsup_{n, \mu \rightarrow \infty} \int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla (T_k(u)) \mu \, dx \, dt.
$$

(5.30)
We have
\[
\int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla (T_k(u)) = \int_{\{|u_n| \leq k\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u)) + \int_{\{k < |u_n| \leq h+1\}} a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \cdot \nabla (T_k(u)) \mu S^*_h(u_n) dx dt
\]

(5.31)

we have \((a(x, t, T_k(u_n), \nabla T_k(u_n)))_n\) is bounded in \((L^{r'(x)}(Q_T))^N\), then there exists \(\xi_k \in \overline{(L^{r'(x)}(Q_T))^N}\) such that \(a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightarrow \xi_k\) in \((L^{r'(x)}(Q_T))^N\), it follows that
\[
\lim_{n, \mu \to \infty} \int_{\{|u_n| \leq k\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u)) = \int_{Q_T} \xi_k \cdot \nabla T_k(u) dx dt.
\]

(5.32)

Similarly, we have
\[
\lim_{n, \mu \to \infty} \int_{\{k < |u_n| \leq h+1\}} a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \cdot \nabla (T_k(u)) \mu S^*_h(u_n) dx dt = 0.
\]

(5.33)

By using (5.30) – (5.33), we conclude that
\[
\limsup_{n \to \infty} \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \leq \int_{Q_T} \xi_k \cdot \nabla T_k(u) dx dt. \tag{5.34}
\]

On the other hand, thanks to (3.3) we have
\[
\int_{Q_T} \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)) \right) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) dx dt \geq 0
\]
then
\[
\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \geq \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u) dx dt \geq \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) dx
\]
it follows that
\[
\liminf_{n \to \infty} \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \geq \int_{Q_T} \xi_k \cdot \nabla T_k(u) dx dt. \tag{5.35}
\]

Having in mind (5.34), we obtain
\[
\lim_{n \to \infty} \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt = \int_{Q_T} \xi_k \cdot \nabla T_k(u) dx dt. \tag{5.36}
\]
Then in view of the Lemma 4.6, we deduce that

\[ \lim_{n \to \infty} \int_{Q_T} \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u), \nabla T_k(u)) \right) \cdot \left( \nabla T_k(u_n) - \nabla T_k(u) \right) dx \, dt = 0. \] (5.37)

In view of the Lemma 4.6, we deduce that

\[ T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad V \quad \text{then} \quad \nabla u_n \rightharpoonup \nabla u \quad \text{a.e in} \quad Q_T. \] (5.38)

Then \( a(x, t, T_k(u_n), \nabla T_k(u_n)) \to a(x, t, T_k(u), \nabla T_k(u)) \) a.e. in \( Q_T \), and since \( a(x, t, T_k(u_n), \nabla T_k(u_n)) \) is bounded in \( (L^{p'}(Q_T))^N \), we get

\[ a(x, t, T_k(u_n), \nabla T_k(u_n)) \to a(x, t, T_k(u), \nabla T_k(u)) \quad \text{in} \quad (L^{p'}(Q_T))^N. \] (5.39)

Moreover, in view of (5.37), we deduce that

\[ \lim_{n \to \infty} \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx \, dt = \int_{Q_T} a(x, t, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) dx \, dt. \] (5.40)

**Step 5: Passage to the limit.** Let \( \varphi \in V \cap L^\infty(Q_T) \) and \( S(\cdot) \in C^\infty(\mathbb{R}) \), with \( \text{supp} \, S(\cdot) \subset [-M, M] \) for some \( M > 0 \). Taking \( S(\cdot) \varphi \) a test function in (5.5), we obtain

\[
\begin{align*}
&\int_0^T \left( \partial B^n_S(x, u_n) \cdot \varphi \right) dt + \int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot (S''(u_n) \varphi \nabla u_n + S'(u_n) \nabla \varphi) dx \, dt \\
&\quad + \int_{Q_T} \phi_n(u_n) \cdot (S''(u_n) \varphi \nabla u_n + S'(u_n) \nabla \varphi) dx \, dt = \int_{Q_T} f_n S'(u_n) \varphi dx \, dt.
\end{align*}
\] (5.41)

with \( B^n_S(x, z) = \int_0^z S'(\tau) \frac{\partial b_n(x, \tau)}{\partial \tau} d\tau \).

Firstly, since \( S'(\cdot) \) is bounded and \( B^n_S(x, u_n) \) converge to \( B^n_S(x, u_n) \) a.e. in \( Q_T \) and weak—* in \( L^\infty(Q_T) \), then

\[ \frac{\partial B^n_S(x, u_n)}{\partial t} \rightharpoonup \frac{\partial B_S(x, u_n)}{\partial t} \quad \text{in} \quad D'(Q_T). \] (5.42)

Concerning the second term on the left-hand side of (5.41), we have

\[
\begin{align*}
&\int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot (S''(u_n) \varphi \nabla u_n + S'(u_n) \nabla \varphi) dx \, dt \\
&= \int_{Q_T} a(x, t, T_M(u_n), \nabla T_M(u_n)) \cdot (S''(u_n) \varphi \nabla T_M(u_n) + S'(u_n) \nabla \varphi) dx \, dt,
\end{align*}
\]

and thanks to (5.39), we have

\[ a(x, t, T_M(u_n), \nabla T_M(u_n)) \to a(x, t, T_M(u), \nabla T_M(u)) \quad \text{in} \quad (L^{p'}(Q_T))^N, \]

and since

\[ S''(u_n) \varphi \nabla T_M(u_n) + S'(u_n) \nabla \varphi \to S''(u) \varphi \nabla T_M(u) + S'(u) \nabla \varphi \quad \text{in} \quad (L^{p'}(Q_T))^N, \]
hence
\[
\lim_{n \to \infty} \int_{Q_T} a(x, t, T_M(u_n), \nabla T_M(u_n)) \cdot (S''(u_n)\varphi \nabla T_M(u_n) + S'(u_n)\nabla \varphi) \, dx \, dt = \int_{Q_T} a(x, t, T_M(u), \nabla T_M(u)) \cdot (S''(u)\varphi \nabla T_M(u) + S'(u)\nabla \varphi) \, dx \, dt
\]
\[
= \int_{Q_T} a(x, t, u, \nabla u) \cdot (S''(u)\varphi \nabla u + S'(u)\nabla \varphi) \, dx \, dt.
\] (5.43)

Similarly, since \( \phi_n(u_n) = \phi(T_M(u_n)) \) for \( n \geq M \), then
\[
\lim_{n \to \infty} \int_{Q_T} \phi_n(u_n) \cdot (S''(u_n)\varphi \nabla u_n + S'(u_n)\nabla \varphi) \, dx \, dt = \int_{Q_T} \phi(T_M(u_n)) \cdot (S''(u_n)\varphi \nabla T_M(u_n) + S'(u_n)\nabla \varphi) \, dx \, dt
\]
\[
= \int_{Q_T} \phi(T_M(u)) \cdot (S''(u)\varphi \nabla T_M(u) + S'(u)\nabla \varphi) \, dx \, dt
\]
\[
= \int_{Q_T} \phi(u) \cdot (S''(u)\varphi \nabla u + S'(u)\nabla \varphi) \, dx \, dt.
\] (5.44)

Moreover, since \( S(u_n) \varphi \rightarrow S(u) \varphi \) weak-* in \( L^\infty(\Omega) \), then
\[
\int_{\Omega} f_n S'(u_n) \varphi \, dx \longrightarrow \int_{\Omega} f S'(u) \varphi \, dx.
\] (5.45)

By combining (5.41) – (5.45), we deduce that
\[
\int_{Q_T} \frac{\partial B_S^n(x, u_n)}{\partial t} \varphi \, dx \, dt + \int_{Q_T} a(x, t, u, \nabla u) \cdot (S''(u)\varphi \nabla u + S'(u)\nabla \varphi) \, dx \, dt
\]
\[
+ \int_{Q_T} \phi(u) \cdot (S''(u)\varphi \nabla u + S'(u)\nabla \varphi) \, dx \, dt = \int_{Q_T} f S'(u) \varphi \, dx \, dt,
\] (5.46)
we conclude that \( u \) is a renormalized solution to problem (5.30).

It remains to show that \( B_S(x, u) \) satisfies the initial condition (5.3). To this end, firstly remark that, \( S(\cdot) \) being bounded, \( B_S^n(x, u_n) \) is bounded in \( L^\infty(Q_T) \). Secondly, (5.3) and the above considerations on the behavior of the terms of this equation show that \( \frac{\partial B_S^n(u_n)}{\partial t} \) is bounded in \( L^1(Q_T) + V^* \). As consequence \( B_S^n(u_n) \) lies in a compact set of \( C^0([0, T], L^1(\Omega)) \). It follows that on the one hand, \( B_S^n(u_n)(t = 0) = B_S^n(u_n)(u_0^n) \) converges to \( B_S(u)(t = 0) \) strongly in \( L^1(\Omega) \). On the other hand, the smoothness of \( S \) implies that
\[
B_S(u)(t = 0) = B_S(u_0) \quad \text{in} \quad \Omega.
\]

As conclusion of step 1 to step 4, the proof of theorem 5.2 is complete.

References


**Department of Mathematics, Physics and Computing, LSI, FPT, University S.M. Ben Abdellah, Taza, Box 1223, Morocco.**

*Email address: chihabyazough@gmail.com*