Abstract. We give an existence result of a renormalized solution for a class of nonlinear parabolic equations
\[ \frac{\partial b(u)}{\partial t} - \text{div} \left( a(x,t,\nabla u) \right) + h(u)|\nabla u|^p = \mu, \]
where the right side is a general measure and \( b(u) \) is a strictly increasing \( C^1 \)-function with \( b(0) = 0 \), \(-\text{div}(a(x,t,\nabla u))\) is a Leray–Lions type operator with growth \( |\nabla u|^{p-1} \) in \( \nabla u \) and \( h : \mathbb{R} \to \mathbb{R}^+ \) is a continuous positive function that belongs to \( L^1(\mathbb{R}) \).

1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( (N \geq 1) \), \( T \) is a positive real number, and let \( Q := \Omega \times (0,T) \), \( p > 1 \). We will consider the following nonlinear parabolic problem
\[ \frac{\partial b(u)}{\partial t} - \text{div} \left( a(x,t,\nabla u) \right) + h(u)|\nabla u|^p = \mu \quad \text{in} \ Q, \tag{1.1} \]
\[ b(u)(t = 0) = b(u_0) \quad \text{in} \ \Omega, \tag{1.2} \]
\[ u = 0 \quad \text{on} \ \partial \Omega \times (0,T). \tag{1.3} \]

In Problem (1.1)-(1.3) the framework is the following: the data \( \mu \) is a bounded Radon measure on \( Q \), \( b(s) \) is a strictly increasing \( C^1 \)-function for every \( s \in \mathbb{R} \) with \( u_0 \) belongs to \( L^1(\Omega) \). The operator \(-\text{div}(a(x,t,\nabla u))\) is a Leray–Lions operator which is coercive and which grows like \( |\nabla u|^{p-1} \) with respect to \( \nabla u \), (see assumptions (3.3), (3.4) and (3.5) of Section 3).

In the case where \( b(u) = u \), and the right hand side is a bounded measure, the existence of a distributional solution was proved in [10], but due the lack of regularity of solution, the distributional formulation is not strong enough to provide uniqueness (see [40] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by DiPerna and Lions [23] for the study of Boltzmann equation was adapted to parabolic equations and (elliptic equations) with \( L^1 \) data (see [5, 11, 13, 32, 30, 31]).
equivalent notion of entropy solution has been developed independently by [6] for the study of nonlinear elliptic problems and by [36] in the parabolic case. Both renormalized or entropy solutions provide a convenient framework to deal with elliptic or parabolic equations with $L^1$ data. A large number of papers was then devoted to the study the existence of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and different contexts: in addition to the references already mentioned see, among others, [2, 3, 15, 16, 17, 19, 18, 20, 21, 33, 39].

Concerning the datum $\mu$, the existence and uniqueness of renormalized solution of (1.1)-(1.3) has proved in [38] in the case where $b(u) = u$, $u_0 \in L^1(\Omega)$, $h = 0$, and for every measure $\mu$ which do not charge the sets of zero p-capacity, the so-called diffuse measures or soft measures, and we will use the symbol $\mu \in M_0(Q)$ to denote them (see Section 2 for the definition). The importance of the measures not charging sets of null p-capacity was first observed in the stationary case in [8], and developed in the evolution case in [38].

For $\mu \in M_0(Q)$, $u_0 \in L^1(\Omega)$ and $h = 0$ the existence and uniqueness of renormalized solution was proved in [18].

In the case where $\mu \in M_0(Q)$, $h = 0$, and with the parabolic term on $b(x, u)$, the existence of renormalized solution of Problem (1.1)-(1.3) was proved in [37]. Concerning the lower order term, existence of renormalized solution of Problem (1.1)-(1.3) has proved in [19] in the case where $b(u) = b(x, u)$, $\mu \in L^1(Q)$, $u_0 \in L^1(Q)$ and $h$ satisfies the sign condition. In the case $b(u) = b(x, u)$, $\mu \in L^1(Q)$, $u_0 \in L^1(Q)$ and without sign condition, the existence of renormalized solution of Problem (1.1)-(1.3) has proved in [1].

Notice that the definition of renormalized solution of problem (1.1)-(1.3) can be extended to case of general measure by adapting the idea of [25] for elliptic case and [34] for parabolic case.

For $\mu \in \mathcal{M}(Q)$ (the space of all bounded Radon measures on $Q$), $b(u) = u$ and $u_0 \in L^1(\Omega)$, and $h = 0$, the existence of renormalized solution was proved in [25] for elliptic case and [34] for parabolic case.

Our goal is to extend the approach in [37] to general, and possibly singular measure data and with lower order term which does not satisfies the sign condition.

The paper is organized as follows as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties.

Section 3 is devoted to specify the assumptions on $b$, $a$, $h$, $u_0$ and $\mu$ and to give the definition of renormalized solution of (1.1)-(1.3) and see how the definition of renormalized solution does not depend on the decomposition (not uniquely determined) of the regular part of $\mu$ we mentioned above and to the statement of standard approximation argument we will use later. In Section 4 we establish (Theorem 4.1) the existence of such a solution.
We recall the notion of p-capacity associated to our problem. Let $Q = \Omega \times (0, T)$ for any fixed $T > 0$, and let us recall that $V = W^{1,p}_0(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\| \|_{W^{1,p}_0(\Omega)} + \| \|_{L^2(\Omega)}$ and

$$W = \left\{ u \in L^p(0, T; V), u_t \in L^{p'}(0, T; V') \right\},$$

endowed with its natural norm $\| \|_{L^p(0,T;V)} + \| \|_{L^{p'}(0,T;V')}$, remark that $W$ is continuously embedded in $C([0, T], L^2(\Omega))$, and if $1 < p < \infty$, then $C_c^\infty(\Omega \times [0, T])$ is dense in $W$.

Let $U \subseteq Q$ be an open set, we define the parabolic p-capacity of $U$ as

$$\text{cap}_p(U) = \inf \left\{ \| u \|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \right\},$$

where as usual we set $\inf \{ \emptyset \} = +\infty$, then for any Borel set $B \subseteq Q$ we define

$$\text{cap}_p(B) = \inf \left\{ \text{cap}_p(U) : U \text{ open set of } Q, B \subseteq U \right\}.$$

We define the space $S$ by

$$S = \left\{ u \in L^p(0, T; W^{1,p}_0(\Omega)), u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q) \right\},$$

endowed with its natural norm $\| \|_{L^p(0,T;W^{1,p}_0(\Omega))} + \| \|_{L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)}$.

We will denote by $\mathcal{M}(Q)$ the set of all Radon measures with bounded total variation on $Q$, while $\mathcal{M}_0(Q)$ is the set of all measures with bounded total variation over $Q$ that do not charge the sets of zero p-capacity, that is if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $\text{cap}_p(E) = 0$. We recall the following theorem.

**Theorem 2.1.** Let $\mu$ be a bounded measure in $Q$. If $\mu \in \mathcal{M}_0(Q)$ then there exists $(f, g_1, g_2)$ such that $f \in L^1(Q), g_1 \in L^p(0, T; W^{-1,p'}(\Omega)), g_2 \in L^p(0, T; V)$ and

$$\int_Q \phi \, d\mu = \int_Q f \phi \, dx \, dt + \int_0^T \langle g_1, \phi \rangle \, dt - \int_0^T \langle \phi_t, g_2 \rangle \, dt \quad \phi \in C_c^\infty(\Omega \times [0, T]).$$

Such a triplet $(f, g_1, g_2)$ will called the decomposition of $\mu$.

**Proof.** See [38].

So, if $\mu \in \mathcal{M}(Q)$, thanks to a well known decomposition result (see for instance [27]), we can split it into a sum (uniquely determined) of its absolutely continuous part $\mu_0$ with respect to p-capacity and its singular part $\mu_s$, that $\mu_s$ is concentrated on a set $E$ of zero p-capacity; we will say that $\mu_s \perp \text{cap}_p$. Hence, if $\mu \in \mathcal{M}(Q)$, by Theorem 2.1, we have

$$\mu = f - \text{div}(G) + g_t + \mu_s^+ - \mu_s^-,$$

in the sense of distributions, for $f \in L^1(Q), G \in (L^p(Q))^N, g \in L^p(0, T; V)$, where $\mu_s^+$ and $\mu_s^-$ are respectively the positive and the negative part of $\mu_s$; note that the decomposition of the absolutely continuous part of $\mu$ in Theorem 2.1 is not uniquely determined. Let us state the following result that will be very useful in the sequel; its proof relies on an easy application of Egorov and Dunford-Pettis theorems.
Proposition 2.2. Let $\rho_\varepsilon$ be a sequence of $L^1(Q)$ functions that converges to $\rho$ weakly in $L^1(Q)$, and let $\sigma_\varepsilon$ be sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to $\sigma$ almost everywhere on $Q$. Then
\[
\lim_{\varepsilon \to 0} \int_Q \rho_\varepsilon \sigma_\varepsilon \, dxdt = \int_Q \rho \sigma \, dxdt.
\]

Here are some notations we will use throughout this paper. For any non-negative real number $k$ we denote by $T_k(r) = \min(k, \max(r, -k))$ the truncation function at level $k$.

By $\langle \cdot, \cdot \rangle$ we mean the duality between suitable spaces in which functions are involved, in particular we will consider both the duality between $W^{1,p}_0(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega) + L^1(\Omega)$.

3. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: $\Omega$ is a bounded open set on $\mathbb{R}^N (N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

$$b : \mathbb{R} \to \mathbb{R}$$

is a strictly increasing $C^1$-function with $b(0) = 0$, and there exist $\gamma$ and $\Lambda > 0$ such that

$$\gamma \leq b'(s) \leq \Lambda, \quad (3.2)$$

for every $s \in \mathbb{R}$.

$$a : Q \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function} \quad (3.3)$$

$$a(x,t,\xi) \cdot \xi \geq \alpha |\xi|^p, \quad (3.4)$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ given real number.

$$|a(x,t,\xi)| \leq \beta (L(x,t) + |\xi|^{p-1}), \quad (3.5)$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ given real number, $L$ is a non negative function in $L^p(Q)$.

$$[a(x,t,\xi) - a(x,t,\xi')] [\xi - \xi'] > 0, \quad (3.6)$$

for any $(\xi,\xi') \in \mathbb{R}^{2N}$ and for almost every $(x,t) \in Q$.

$$h : \mathbb{R} \to \mathbb{R}^+ \quad (3.7)$$

is a continuous positive function that belongs to $L^1(\mathbb{R})$.

$$\mu \in \mathcal{M}(Q), \quad (3.8)$$

$u_0$ is an element of $L^1(\Omega)$ such that $b(u_0) \in L^1(\Omega). \quad (3.9)$

To simplify notation, let us also define $v = b(u) - g$, the definition of a renormalized solution for Problem (1.1)-(1.3) is given below.
Definition 3.1. A measurable function $u$ defined on $Q$ is a renormalized solution of Problem (1.1)-(1.3) if

$$T_k(v) \in L^p(0, T; W^{1,p}_0(\Omega)) \quad \forall k \geq 0, \quad v \in L^\infty(0, T; L^1(\Omega)) \quad \text{and} \quad h(u)|\nabla u|^p \in L^1(Q),$$

(3.10)

and, for every function $S$ in $W^{2,\infty}(\mathbb{R})$, which is piecewise $C^1$ and such that $S'$ has a compact support and $S(0) = 0$, we have

$$S(v)_t - \text{div} \left( S'(v)a(x, t, \nabla u) \right) + S''(v)a(x, t, \nabla u) \nabla v + S'(v)h(u)|\nabla u|^p = f S'(v) - \text{div} \left( G S'(v) \right) + G S''(v) \nabla v \quad \text{in} \quad \mathcal{D}'(Q),$$

(3.11)

$$S(v)(t = 0) = S(b(u_0)) \quad \text{in} \quad L^1(\Omega).$$

(3.12)

For every $\psi \in C(\overline{Q})$, we have

$$\lim_{n \to \infty} \frac{1}{n} \int_{\{(x, t) \in Q : n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \psi \, dx \, dt = \int_Q \psi d\mu_s^+, \quad (3.13)$$

$$\lim_{n \to \infty} \frac{1}{n} \int_{\{(x, t) \in Q : -2n < v \leq -n\}} a(x, t, \nabla u) \nabla v \psi \, dx \, dt = \int_Q \psi d\mu_s^- \quad (3.14)$$

Remark 3.2. Note that thanks to our assumptions and the choice of $S$ all terms in (3.11) are well defined (for more details see [18], Remark 3.2). Let also observe that $S(v)_t \in L^p(0, T; W^{-1,p}(\Omega)) + L^1(Q)$ and $S'(v) \in L^p(0, T; W^{1,p}_0(\Omega))$, which implies that $S(v) \in C([0, T], L^1(\Omega))$ (see [32]) and (3.11) makes a weak sense, indeed, since $\mu_0 \in \mathcal{M}_0(Q)$ and it is defined on the $\sigma$-algebra of the borelians of the open set $Q$, then $\mu_0$ does not charge set at $t = 0$, which implies, in the weak sense, that $g(x, 0) = 0$ for any $g$ such that $(f, \text{div}(G), g)$ is a decomposition of $\mu_0$, this explains (3.12), (see [38]).

A remark on the assumptions (3.2) is also necessary. As one could check later, due essentially to the presence of the term $g_t$ in the decomposition of the measure $\mu$, we are forced to assume $b'(s) \geq \gamma > 0$. We conjecture that this assumption is only technical and could be removed in order to deal with more general elliptic-parabolic problems (see for instance [4], [21]).

Now we give the following property of renormalized solutions; throughout the paper $C$ will indicate any positive constant whose value may change from line to line.

Proposition 3.3. Let $v = b(u) - g$ be a renormalized solution of problem (1.1)-(1.3). Then, for every $k > 0$, we have

$$\int_Q |\nabla T_k(v)|^p \, dx \, dt \leq C(k + 1),$$

(3.15)

where $C$ is a positive constant not depending on $k$.

Proof. Using assumptions (3.2), following the same arguments as in [34], yields (3.15). \qed

Here, we give two results which show that the renormalized solution does not depend on the decomposition of the regular part of $\mu$. 

Lemma 3.4. Let \( \mu_0 \in \mathcal{M}_0(Q) \), and let \((f, g_1, g_2)\) and \((\overline{f}, \overline{g}_1, \overline{g}_2)\) to be two different decomposition of \( \mu \) according to Theorem 2.1. Then we have \((g_2 - \overline{g}_2), = \overline{f} - f + \overline{g}_1 - g_1\) in distribution sense, \(g_2 - \overline{g}_2 \in C([0, T], L^1(\Omega))\) and \((g_2 - \overline{g}_2)(0) = 0\).

Proof. See [38], Lemma 2.29.

The following result shows that the definition of renormalized solution does not depend on the decomposition of the absolutely continuous part of \( \mu \) under the condition of bounded perturbations of time derivative part of \( \mu_0 \), and due the estimate (3.15).

Proposition 3.5. Let \( u \) be a renormalized solution of problem (1.1)-(1.3). Then, \( u \) satisfies definition 3.1 for every decomposition \((f, g_1, g_2)\) such that \( g_2 - \overline{g}_2 \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(\Omega)\).

Proof. See [34], Proposition 3 and Remark 6.

4. Existence result

This section is devoted to establish the following existence theorem.

Theorem 4.1. Under assumptions (3.1)-(3.9), there exists at least a renormalized solution \( u \) of Problem (1.1)-(1.3).

Proof. We will obtain the existence result by an approximation process, we approximate the measure \( \mu \in \mathcal{M}(Q) \) by a sequence defined by

\[
\mu^\varepsilon = f^\varepsilon - \text{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} + \lambda^\varepsilon_+ - \lambda^\varepsilon_-
\]

where \( f^\varepsilon \in C^\infty_c(Q) \) is a sequence of functions which converges to \( f \) weakly in \( L^1(Q) \), \( G^\varepsilon \in (C^\infty_c(Q))^N \) is a sequence of functions which converges to \( G \) strongly in \((L^p(Q))^N\), \( g^\varepsilon \in C^\infty_c(Q) \) is a sequence of functions which converges to \( g \) strongly in \( L^p(0, T; W^{1,p}_0(\Omega)) \), and \( \lambda^\varepsilon_+ \in C^\infty_c(Q) \) (respectively \( \lambda^\varepsilon_- \)) is a sequence of non negatives functions that converges to \( \mu^\varepsilon_+ \) (respectively \( \mu^\varepsilon_- \)) in the narrow topology of measures. Moreover let \( u^\varepsilon_0 \in C^\infty_c(\Omega) \) such that

\[
u^\varepsilon_0 \in C^\infty_c(\Omega) : u^\varepsilon_0 \to u_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0.\]

We also assume

\[
\|\mu^\varepsilon\|_{L^1(Q)} \leq C|\mu|_{\mathcal{M}(Q)} \text{ and } \|b(u^\varepsilon_0)\|_{L^1(Q)} \leq C\|b(u_0)\|_{L^1(Q)}.
\]

Let us now consider the following regularized problem:

\[
u^\varepsilon \in L^p(0, T; W^{1,p}_0(\Omega)),
\]

\[
\int_0^T \langle \frac{\partial \nu^\varepsilon}{\partial t}, \varphi \rangle \, dt + \int_Q a(x, t, \nabla \nu^\varepsilon) \varphi \, dx \, dt + \int_Q h(u^\varepsilon)\nabla u^\varepsilon \nabla \varphi \, dx \, dt = \int_Q f^\varepsilon \varphi \, dx \, dt + \int_Q G^\varepsilon \nabla \varphi \, dx \, dt
\]

\[
\forall \varphi \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q),
\]

\[
b(u^\varepsilon)(t = 0) = b(u^\varepsilon_0) \text{ in } \Omega.
\]
Proposition 4.2. Let $u^\varepsilon = b(u^\varepsilon) - g^\varepsilon$. As a consequence, proving existence of a weak solution $u^\varepsilon \in L^p(0, T; W_0^{1,p}(\Omega))$ of (4.3)-(4.5) is an easy task (see [9], [28]).

Now we prove the following proposition which gives some compactness results.

Proposition 4.2. Let $u^\varepsilon$ and $v^\varepsilon$ be defined as before. Then

$$\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$

(4.6)

$$\int_Q \|
abla T_k(u^\varepsilon)\|^p \, dx \leq C k,$$

(4.7)

$u^\varepsilon$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$,

(4.8)

$$\|v^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$

(4.9)

$$\int_Q \|
abla T_k(v^\varepsilon)\|^p \, dx \leq C(k+1),$$

(4.10)

$h(u^\varepsilon)|\nabla u^\varepsilon|^p$ is bounded in $L^1(Q)$,

(4.11)

and, up to a subsequence, for any $k > 0$ we have

$$u^\varepsilon \to u \text{ a.e. on } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q)$$

(4.12)

$$v^\varepsilon \to v \text{ a.e. on } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q)$$

(4.13)

$$T_k(u^\varepsilon) \to T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. on } Q,$$

(4.14)

$$T_k(v^\varepsilon) \to T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. on } Q,$$

(4.15)

Proof. We prove (4.6) and (4.7), using $T_k(u^\varepsilon)^+$ as a test function in (4.4) and we integrate in $|0,t]$ we get

$$\int_\Omega B_k(u^\varepsilon)(t) \, dx + \int_0^t \int_\Omega a(x,t,\nabla u^\varepsilon) \nabla T_k(u^\varepsilon)^+ \, dx \, ds + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p T_k(u^\varepsilon)^+ \, dx \, ds$$

(4.16)

$$= \int_0^t \int_\Omega \mu^\varepsilon T_k(u^\varepsilon)^+ \, dx \, ds + \int_\Omega B_k(u_0^\varepsilon) \, dx,$$

for almost every $t \in (0, T)$, and where $B_k(s) = \int_0^s T_k(r)^+ b'(r) \, dr$. Using assumption (3.4) and since $B_k(u^\varepsilon) \geq 0$, from (4.16) we obtain that $T_k(u^\varepsilon)^+$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, and similarly by taking $T_k(u^\varepsilon)^-$ as a test function in (4.4) yields (4.7). By (3.2) we have

$$B_1(s) \geq \gamma \int_0^s T_1(r)^+ \, dr \quad \forall s \in \mathbb{R},$$

and since $\int_0^s T_1(r)^+ \, dr \geq s - 1 \quad \forall s \in \mathbb{R}^+$, we obtain

$$\int_\Omega u^\varepsilon(t) \, dx \leq \frac{1}{\gamma} (\|b(u_0^\varepsilon)\|_{L^1(\Omega)} + |\mu|_{M(Q)}) + \text{meas}(\Omega).$$

Similarly by taking $T_k(u^\varepsilon)^-$ as a test function in (4.4) we conclude that $u^\varepsilon$ is bounded in $L^\infty(0,T;L^1(\Omega))$, which yields (4.6). Now, using (4.6), (4.7), and
arguing as in [10] we obtain (4.8). Taking $T_k(v^\varepsilon)^+$ as test function in (4.4) and we integrate in $[0,t]$, by assumptions (3.2), (3.4), (3.5), (3.7) and by means of Young’s inequality one obtains

$$
\int_\Omega \overline{T}_k(u^\varepsilon)(t)dx + \frac{\alpha}{2} \int_{\{|u^\varepsilon| \leq k\}} b'(u^\varepsilon)|\nabla u^\varepsilon|^pdxdt
\leq C(\|G^\varepsilon\|_{L^p(Q)}^p + \|L\|_{L^p(Q)}^p + \|\nabla f^\varepsilon\|_{L^p(Q)}^p)
+ k(\|f^\varepsilon\|_{L^1(Q)} + \|b(u_0^\varepsilon)\|_{L^1(\Omega)} + \|\lambda_+^\varepsilon\|_{L^1(Q)} + \|\lambda_+^\varepsilon\|_{L^1(\Omega)}),
$$

where $\overline{T}_k(s) = \int_0^s T_k(r)^+dr \forall s \in \mathbb{R}^+$. Similarly by taking $T_k(v^\varepsilon)^-$ as a test function in (4.4) we deduce that (4.9) and (4.10) hold true.

Now we prove that (4.11) holds, we use $\rho_k(u^\varepsilon) = \int_0^{u^\varepsilon} h(s)\chi_{\{s > k\}} ds$ as test function in (4.4) we obtain

$$
\int_\Omega B_k(u^\varepsilon)(T)dx + \int_{\partial \Omega} a(x, t, \nabla u^\varepsilon)\nabla u^\varepsilon \cdot h(u^\varepsilon)\chi_{\{u^\varepsilon > k\}} dxdt
\leq \left( \int_k^\infty h(s)ds \right) \|\mu^\varepsilon\|_{L^1(Q)} + \int_\Omega B_k(u_0^\varepsilon)dx
\leq C \int_k^\infty h(s)ds,
$$

where $B_k(s) = \int_0^s \rho_k(r)b'(r)dr$.

By assumptions (3.4) and (3.7) we obtain that $\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{|u^\varepsilon| > k\}} dxdt$ is bounded, similarly taking $\rho_k(u^\varepsilon) = \int_0^{u^\varepsilon} h(s)\chi_{\{s < -k\}} ds$ as test function in (4.4) we obtain the same result.

We have

$$
\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p dxdt \leq \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{|u^\varepsilon| < k\}} dxdt + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{|u^\varepsilon| > k\}} dxdt,
$$
due to (3.7), (4.7) and (4.18) we get (4.11). By (3.2), (4.8), (4.11) and since $\mu^\varepsilon$ is bounded in $L^1(Q)$, one obtain that $\frac{\partial h}{\partial \varepsilon}$ is bounded in $L^1(0, T; W^{-1,1}(\Omega))$, using compactness arguments (see [41]) yield (4.12) and (4.13).

Let us introduce for $k \geq 0$ fixed, the time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes. More recently, it has been exploited to solve a few nonlinear evolution problems with $L^1$ or measure data. This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(v^\nu_0)_\nu$ in $L^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $\|v^\nu_0\|_{L^\infty(\Omega)} \leq k$, for all $\nu > 0$, and $v^\nu_0 \rightarrow T_k(u_0)$ a.e. in $\Omega$ with $\frac{1}{\nu}\|v^\nu_0\|_{L^p(\Omega)} \rightarrow 0$ as $\nu \rightarrow +\infty$.

For fixed $k \geq 0$ and $\nu > 0$, let us consider the unique solution $T_k(u)_\nu \in L^\infty(Q) \cap L^p(0, T; W^{1,p}_0(\Omega))$ of the monotone problem:

$$
\frac{\partial T_k(u)_\nu}{\partial t} + \nu(T_k(u)_\nu - T_k(u)) = 0 \text{ in } D'(Q),
$$
The behavior of \( T_k(u)_\nu(t = 0) = v_0^\nu \) in \( \Omega \).

The behavior of \( T_k(u)_\nu \) as \( \nu \to +\infty \) is investigated in [29] (see also [24]) and we just recall here that:

\[
T_k(u)_\nu \to T_k(u) \text{ strongly in } L^p(0,T,W^{1,p}_0(\Omega)) \text{ a.e. in } Q \text{ as } \nu \to +\infty
\]

with \( \|T_k(u)_\nu\|_{L^\infty(\Omega)} \leq k \) for any \( \nu > 0 \), and \( \frac{\partial T_k(u)_\nu}{\partial \nu} \in L^p(0,T,W^{1,p}_0(\Omega)) \).

Here and in the rest of paper \( \omega(\varepsilon,n,\delta,\mu) \) will indicate any quantity that vanishes as the parameters go to their limit point with in the same order in which they appear, that is, for example

\[
\lim_\nu \lim_\delta \lim_n \lim_\varepsilon |\omega(\varepsilon,n,\delta,\mu)| = 0.
\]

Now we prove the following proposition

**Proposition 4.3.** The sequence \((\nabla u^\varepsilon)\) converges to \(\nabla u\) a.e. in \(Q\),

In order to prove the Proposition 4.3 we give the following result .

**Lemma 4.4.** Let \( z^\varepsilon \) be a sequence in \( L^p(0,T;W^{1,p}_0(\Omega)) \cap C_0([0,T];L^2(\Omega)) \) such that \( z^\varepsilon(.0) = 0 \), and \( (z^\varepsilon)_t \in L^p(0,T;W^{-1,p}(\Omega)) \), suppose that \( z^\varepsilon \) converges almost everywhere in \( Q \) to a function \( z \) such that \( T_k(z) \in L^p(0,T;W^{1,p}_0(\Omega)) \) for every \( k > 0 \). then we have

\[
\int_0^T \langle \frac{\partial z^\varepsilon}{\partial t}, T_\sigma(z^\varepsilon - T_k(z)_\nu) \rangle \, ds \, dt \geq \omega(n,\nu,k,\sigma)
\]

(4.19)

**Proof.** See [7] □

**Remark 4.5.** The proof of Lemma 4.4 extends the case of sequences \((u_0^\varepsilon)\) converging in \( L^1 \), the only difference being to include the initial condition \( u_0^\varepsilon \) in Landes approximation. This technical point can be found e.g. in [32].

Now we are ready to prove the Proposition 4.3

**Proof.** We follow the method used in [7], by the monotonicity of \( a(x,t,\xi) \), the result will be proved if, up to subsequences still denoted by \( u^\varepsilon \) (for simplicity of notation, we will omit the dependence of \( a \) on \( x \) and \( t \)),

\[
\left[ (a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u) \right]^\theta \to 0.
\]

(4.20)

Note that (4.20) will be true if we show that

\[
\int_Q \left[ (a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u) \right]^\theta \, dx \, dt = \omega(\varepsilon),
\]

(4.21)

for some \( \theta > 0 \), thanks to Proposition 4.2, the following estimate holds

\[
\text{meas}(\{|b(u)| \geq k\}) = \omega(k),
\]

and for simplicity we set \( w = b(u) \).

We can write

\[
\int_Q \left[ (a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u) \right]^\theta \, dx \, dt
\]
Since \( u^\varepsilon \) is bounded in \( L^q(0,T;W_0^{1,q}(\Omega)) \) for some \( q < p \), we can choose \( \theta < \frac{p}{q} < 1 \), so that using Hölder inequality, we obtain

\[
|I_{\varepsilon,k}| \leq c \left( \int_Q \left( |L|^p/p + |\nabla u| + |\nabla u^\varepsilon| \right)^{\theta p/q} \right) \quad \text{meas} \{ |w| \geq k \}^{1-\theta p/q}
\]

\leq c \text{meas} \{ |w| \geq k \}^{1-\theta p/q},

and so \( I_{\varepsilon,k} = \omega(\varepsilon, k) \). On the other hand,

\[
J_{\varepsilon,k} = \int_{|w|<k} \left[ (a(\nabla u^\varepsilon) - a(\nabla u \chi_{|w|<k})) \left( \nabla u^\varepsilon - \nabla u \right) \right]^\theta \, dx \, dt
\]

\leq \int_Q \left[ (a(\nabla u^\varepsilon) - a(\nabla u \chi_{|w|<k})) \left( \nabla u^\varepsilon - \nabla u \chi_{|w|<k} \right) \right]^\theta \, dx \, dt.

We define

\[
\Psi_{\varepsilon,k} = \left( a(\nabla u^\varepsilon) - a(\nabla u \chi_{|w|<k}) \right) \left( \nabla u^\varepsilon - \nabla u \chi_{|w|<k} \right),
\]

and we have

\[
\int_Q \left[ (a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u) \right]^\theta \, dx \, dt \tag{4.22}
\]

\leq \int_Q \Psi_{\varepsilon,k}^\theta \chi_{|w^\varepsilon-T_k(w)| \leq \sigma} + \int_Q \Psi_{\varepsilon,k}^\theta \chi_{|w^\varepsilon-T_k(w)| > \sigma} + \omega(\varepsilon, k),

since \( \Psi_{\varepsilon,k}^\theta \) is bounded in \( L^{q/\theta p}(Q) \) and since \( \chi_{|w-T_k(w)| > \sigma} \) converges to zero almost everywhere in \( Q \) as \( k \) tends to infinity, we obtain

\[
\int_Q \Psi_{\varepsilon,k}^\theta \chi_{|w^\varepsilon-T_k(w)| > \sigma} = \omega(\varepsilon, \nu, k),
\]

Thus, (4.21) becomes

\[
\int_Q \left[ (a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u) \right]^\theta \, dx \, dt
\]

\leq \int_Q \Psi_{\varepsilon,k}^\theta \chi_{|w^\varepsilon-T_k(w)| \leq \sigma} + \omega(\varepsilon, \nu, k).

Using Hölder inequality (with exponents \( 1/\theta \) and \( 1/1-\theta \)) the last integral is smaller than

\[
\text{meas}(Q)^{1-\theta} \left( \int_Q \Psi_{\varepsilon,k} \chi_{|w^\varepsilon-T_k(w)| \leq \sigma} \right)^\theta,
\]

so that (4.21) will be proved if we can show that

\[
\int_Q \Psi_{\varepsilon,k} \chi_{|w^\varepsilon-T_k(w)| \leq \sigma} = \omega(\varepsilon, \nu, k, \sigma). \tag{4.23}
\]
Now recalling the definition of $\Psi_{\epsilon,k}$ we can write by (3.2) 
\[
\int_Q \Psi_{\epsilon,k} \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}} \leq \frac{1}{2} \left( \int_Q b'(u^\epsilon) a(\nabla u^\epsilon) \left( \nabla u^\epsilon - \nabla u \chi_{\{|w|\leq k\}} \right) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}} \right) \\
- \int_Q b'(u^\epsilon) a(\nabla u \chi_{\{|w|\leq k\}}) \left( \nabla u^\epsilon - \nabla u \chi_{\{|w|\leq k\}} \right) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}
\]
(4.24)

By Proposition 4.2 and since $|T_k(w,\nu)| \leq k$ we obtain
\[
\int_Q b'(u^\epsilon) a(\nabla u \chi_{\{|w|\leq k\}}) \left( \nabla u^\epsilon - \nabla u \chi_{\{|w|\leq k\}} \right) \chi_{\{|w-T_k(w,\nu)\leq \sigma\}} + \omega(\epsilon)
= \omega(\epsilon).
\]

On the other hand we have
\[
\int_Q b'(u^\epsilon) a(\nabla u^\epsilon) \left( \nabla u^\epsilon - \nabla u \chi_{\{|w|\leq k\}} \right) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}
= \int_Q a(\nabla u^\epsilon) \nabla (w^\epsilon - T_k(w,\nu)) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}
+ \int_Q b'(u^\epsilon) a(\nabla u^\epsilon) (b'(u^\epsilon)^{-1} \nabla T_k(w,\nu) - b'(u)^{-1} \nabla T_k(w)) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}.
\]
(4.26)

We have
\[
\int_Q b'(u^\epsilon) a(\nabla u^\epsilon) (b'(u^\epsilon)^{-1} \nabla T_k(w,\nu) - b'(u)^{-1} \nabla T_k(w)) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}
= \int_Q a(b'(u^\epsilon)^{-1} \nabla T_{k+\sigma}(w^\epsilon) \nabla T_k(w,\nu) \chi_{\{|w-T_k(w,\nu)\leq \sigma\}}
- \int_Q b'(u^\epsilon) b'(u)^{-1} a(b'(u)^{-1} \nabla T_{k+\sigma}(w^\epsilon) \nabla T_k(w,\nu) \chi_{\{|w-T_k(w,\nu)\leq \sigma\}},
\]
since $a(b'(u^\epsilon)^{-1} \nabla T_{k+\sigma}(w^\epsilon))$ converge weakly to $\Gamma_{k,\sigma}$ in $L^b(Q)$, $b'(u^\epsilon)$ converges to $b'(u)$ $\ast$-weakly in $L^\infty(Q)$, and almost everywhere in $Q$ by Proposition 2.2 we obtain
\[
\int_Q b'(u^\epsilon) a(\nabla u^\epsilon) (b'(u^\epsilon)^{-1} \nabla T_k(w,\nu) - b'(u)^{-1} \nabla T_k(w)) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}}
= \int_Q \Gamma_{k,\sigma}(\nabla T_k(w,\nu) - \nabla T_k(w)) \chi_{\{|w-T_k(w,\nu)\leq \sigma\}} + \omega(\epsilon)
= \omega(\epsilon,\nu).
\]

Hence (4.24), (4.25) and (4.26) imply that
\[
\int_Q \Psi_{\epsilon,k} \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}} \leq \int_Q a(\nabla u^\epsilon) \nabla (w^\epsilon - T_k(w,\nu)) \chi_{\{|w^\epsilon-T_k(w,\nu)\leq \sigma\}} + \omega(\epsilon,\nu).
\]
Now we use the equation solved by $u^\varepsilon$. Taking $T_\sigma(w^\varepsilon - T_k(w)_\nu)^+$ in (4.4) we obtain
\[
\int_0^T \langle \frac{\partial u^\varepsilon}{\partial t}, T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \rangle \, dt + \int_Q a(\nabla u^\varepsilon) \nabla T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt \\
+ \int_Q h(u^\varepsilon) |\nabla u^\varepsilon|T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt = \int_Q \mu^\varepsilon T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt.
\]

By property of $\mu^\varepsilon$ we have
\[
\left| \int_Q \mu^\varepsilon T_\sigma(w^\varepsilon - T_k(w)_\nu) \, dxdt \right| \leq \sigma \|\mu^\varepsilon\|_{L^1(Q)} \leq \sigma |\mu|_{\mathcal{M}(Q)}.
\]

By Lemma 4.4 and assumption (3.7) we obtain
\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt \leq \omega(\varepsilon, \nu, \sigma).
\]

Similarly we take $T_\sigma(w^\varepsilon - T_k(w)_\nu)^-$ as test function in (4.4) we deduce
\[
\int_Q a(\nabla u^\varepsilon) \nabla T_\sigma(w^\varepsilon - T_k(w)_\nu) \, dxdt \leq \omega(\varepsilon, \nu, \sigma). \tag{4.27}
\]

Then we obtain (4.27) and therefore (4.21). \qed

Now we give the basic result about approximate capacitary potential.

Lemma 4.6. Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}(Q)$ where $\mu_s^+$ and $\mu_s^-$ are concentrated respectively, on two disjoint $E^+$ and $E^-$ of zero $p$-capacity. Then, for every $\delta > 0$, there exist two compact sets $K^+_\delta \subseteq E^+$ and $K^-\delta \subseteq E^-$ such that
\[
\mu_s^+(E^+ \setminus K^+\delta) \leq \delta, \quad \mu_s^-(E^- \setminus K^-\delta) \leq \delta, \tag{4.28}
\]
and there exist $\psi^+_\delta$, $\psi^-\delta \in C^1_0(Q)$, such that
\[
\psi^+_\delta \equiv 1 \text{ and } \psi^-\delta \equiv 1 \text{ respectively on } K^+_\delta \text{ and } K^-\delta, \tag{4.29}
\]
\[
0 \leq \psi^+_\delta, \quad \psi^-\delta \leq 1, \tag{4.30}
\]
\[
supp(\psi^+_\delta) \cap supp(\psi^-\delta) \equiv \emptyset. \tag{4.31}
\]

Moreover
\[
\|\psi^+_\delta\|_s \leq \delta, \quad \|\psi^-\delta\|_s \leq \delta, \tag{4.32}
\]
and in particular, there exists a decomposition of $(\psi^+_\delta)_t$ and a decomposition of $(\psi^-\delta)_t$ such that
\[
\|(\psi^+_\delta)^1_t\|_{L^p'(\Omega; W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi^+_\delta)^2_t\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.33}
\]
\[
\|(\psi^-\delta)^1_t\|_{L^p'(\Omega; W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi^-\delta)^2_t\|_{L^1(Q)} \leq \frac{\delta}{3}. \tag{4.34}
\]

Both $\psi^+_\delta$ and $\psi^-\delta$ converges to zero $\ast-$weakly in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as $\delta$ vanishes. Moreover, if $\lambda^+_\delta$ and $\lambda^-\delta$ are as in (4.1) we have
\[
\int_Q \psi^-\delta \, d\lambda^+_\delta = \omega(\varepsilon, \delta), \quad \int_Q \psi^-\delta \, d\mu_s^+ \leq \delta, \tag{4.35}
\]
Let also introduce another auxiliary function in terms of Theorem 4.7.

\[ \int_Q \psi_\delta^+ d\lambda_-^\varepsilon = \omega(\varepsilon, \delta), \quad \int_Q \psi_\delta^- d\mu^- \leq \delta, \quad (4.36) \]

\[ \int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^- \psi_\eta^-) d\mu^+ \leq \delta + \eta, \quad (4.37) \]

\[ \int_Q (1 - \psi_\delta^- \psi_\eta^-) d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^- \psi_\eta^-) d\mu_- \leq \delta + \eta. \quad (4.38) \]

**Proof.** See [34], Lemma 5.

In what follows we will always refer to subsequences of both \( \psi_\delta^+ \) and \( \psi_\delta^- \) that satisfy all the convergence results stated in Lemma 4.6.

Now we will prove the following theorem

**Theorem 4.7.** Let \( \psi^\varepsilon \) and \( \psi \) be as before. Then, for every \( k > 0 \)

\[ T_k(\psi^\varepsilon) \to T_k(\psi) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)). \]

**Proof.** Let us first introduce the following function that we will use in the proof of Theorem 4.7.

\[ H_n(s) = \begin{cases} 
1 & \text{if } |s| \leq n, \\
2n - s & \text{if } n < s \leq 2n, \\
\frac{2n - s}{n} & \text{if } -2n < s \leq -n, \\
0 & \text{if } |s| > 2n.
\end{cases} \]

Let also introduce another auxiliary function in terms of \( H_n \) by \( B_n(s) = 1 - H_n(s) \).

Our aim is to prove the following asymptotic estimate:

\[ \lim_{\varepsilon \to 0} \int_Q a(x,t,\nabla \psi^\varepsilon) \nabla T_k(\psi^\varepsilon) \, dx \, dt \leq \int_Q a(x,t,\nabla \psi) \nabla T_k(\psi) \, dx \, dt. \quad (4.39) \]

In order to prove (4.39), we shall follow several steps.

**Step 1.**

For every \( \delta, \eta > 0 \), let \( \psi_\delta^+, \psi_\eta^+ \), \( \psi_\delta^- \) and \( \psi_\eta^- \) as in Lemma 4.6 and let \( E^+ \) and \( E^- \) be the sets where, respectively, \( \mu^+ \) and \( \mu^- \) are concentrated. Setting \( \Phi_{\delta,\eta} = \psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^- \), we can write

\[ \int_Q a(x,t,\nabla \psi^\varepsilon) \nabla (T_k(\psi^\varepsilon) - T_k(\psi)_\nu) H_n(\psi^\varepsilon) \, dx \, dt \quad (4.40) \]

\[ = \int_Q a(x,t,\nabla \psi^\varepsilon) \nabla (T_k(\psi^\varepsilon) - T_k(\psi)_\nu) H_n(\psi^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \]

\[ + \int_Q a(x,t,\nabla \psi^\varepsilon) \nabla (T_k(\psi^\varepsilon) - T_k(\psi)_\nu) H_n(\psi^\varepsilon)(1 - \Phi_{\delta,\eta}) \, dx \, dt. \]

Now, if \( n > k \), since \( a(x,t,\nabla \psi^\varepsilon \chi_{\{|\psi^\varepsilon| \leq 2n\}}) \nabla T_k(\psi)_\nu \) is weakly compact in \( L^1(\Omega) \) as \( \varepsilon \) goes to zero, \( H_n(\psi^\varepsilon) \) converges to \( H_n(\psi) \) \( * \)-weakly in \( L^\infty(\Omega) \), and almost everywhere in \( Q \), by Proposition 2.2 we have

\[ \lim_{\varepsilon \to 0} \int_Q a(x,t,\nabla \psi^\varepsilon) \nabla (T_k(\psi^\varepsilon) - T_k(\psi)_\nu) H_n(\psi^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \quad (4.41) \]
\[ \lim_{\varepsilon \to 0} \left[ \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta, \eta} \, dx \, dt \right] - \int_Q a(x, t, \nabla u) \nabla T_k(v) H_n(v) \Phi_{\delta, \eta} \, dx \, dt = \lim_{\varepsilon \to 0} \left[ \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta, \eta} \, dx \, dt \right] - \int_Q a(x, t, \nabla u) \nabla T_k(v) \Phi_{\delta, \eta} \, dx \, dt + \omega(\varepsilon). \]

Since \( \Phi_{\delta, \eta} \) converges to zero *- weakly in \( L^\infty(Q) \) as \( \delta \) goes to zero,

\[ \int_Q a(x, t, \nabla u) \nabla T_k(v) \Phi_{\delta, \eta} \, dx \, dt = \omega(\delta). \]

Therefore, if we prove that

\[ \lim_{\eta \to 0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta, \eta} \, dx \, dt \leq 0, \tag{4.42} \]

then we can conclude

\[ \lim_{\eta \to 0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v) H_n(v^\varepsilon)) \Phi_{\delta, \eta} \, dx \, dt \leq 0. \tag{4.43} \]

**Step 2.** Near to \( E \).

Before proving (4.41), we first show the following result

**Lemma 4.8.** Let \( u^\varepsilon \) be a solution of (4.3)-(4.5). Let \( \eta \) be a positive real number, and let \( \varphi^\eta_+ \) and \( \varphi^\eta_- \) be two non negative functions in \( C^\infty_c(Q) \) such that

\[ 0 \leq \varphi^\eta_+ \leq 1, \quad 0 \leq \varphi^\eta_- \leq 1, \]

and

\[ 0 \leq \int_Q \varphi^\eta_- \, d\mu^+_s \leq \eta, \quad 0 \leq \int_Q \varphi^\eta_+ \, d\mu^-_s \leq \eta. \tag{4.44} \]

we then have

\[ \frac{1}{n} \int_{\{2n < u^\varepsilon \leq -n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi^\eta_+ \, dx \, dt = \omega(\varepsilon, n, \eta), \tag{4.45} \]

\[ \frac{1}{n} \int_{\{n \leq u^\varepsilon < 2n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi^\eta_- \, dx \, dt = \omega(\varepsilon, n, \eta). \tag{4.46} \]

**Proof.** Let us prove (4.46); let \( \beta_n(s) = B_n(s^+) \), we can choose \( \beta_n(v^\varepsilon) \varphi^\eta_- \) as test function in (4.4) and rearranging conveniently all terms we have

\[ \frac{1}{n} \int_{\{n \leq u^\varepsilon < 2n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi^\eta_- \, dx \, dt + \int_Q h(u^\varepsilon) |\nabla u^\varepsilon| \beta_n(v^\varepsilon) \varphi^\eta_- \, dx \, dt + \int_Q \beta_n(v^\varepsilon) \varphi^\eta_- \, d\lambda^-_s \]

\[ = \frac{1}{n} \int_{\{n \leq u^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla g^\varepsilon \varphi^\eta_- \, dx \, dt \]

\[ + \int_Q \beta_n(v^\varepsilon) \frac{d\varphi^\eta_-}{dt} \, dx \, dt - \int_Q a(x, t, \nabla u^\varepsilon) \nabla \varphi^\eta_- \beta_n(v^\varepsilon) \, dx \, dt + \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi^\eta_- \, dx \, dt \]

\[ - \int_0^T < \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi^\eta_- > \, dt + \int_Q \beta_n(v^\varepsilon) \varphi^\eta_- \, d\lambda^\varepsilon_+, \]
where \( \overline{\beta_n}(s) = \int_0^s \beta_n(r) \, dr \). Using the fact that \( \int_Q \beta_n(v^\varepsilon) \varphi_-^n \, d\lambda^\varepsilon \geq 0 \) and by assumptions (3.2), (3.4), (3.5), (3.7) and Young’s inequality we obtain

\[
\frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} b'(u^\varepsilon) a(x,t,\nabla u^\varepsilon) \nabla u^\varepsilon \varphi_-^n \, dx \, dt \leq \frac{C}{n} \int_Q \left( |\nabla g^\varepsilon|^p + |L|^p \right) \, dx \, dt
\]

\[
+ \int_Q \overline{\beta_n}(v^\varepsilon) \frac{d\varphi_-^n}{dt} \, dx \, dt - \int_Q a(x,t,\nabla u^\varepsilon) \cdot \nabla \varphi_-^n \beta_n(v^\varepsilon) \, dx \, dt + \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi_-^n \, dx \, dt
\]

\[- \int_0^T < \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi_-^n > \, dt + \int_Q \beta_n(v^\varepsilon) \varphi_-^n \, d\lambda^\varepsilon_+.
\]

By Proposition 4.2 and 4.3 we have \( a(x,t,\nabla u^\varepsilon) \) converges weakly to \( a(x,t,\nabla u) \) in \( (L^q(Q))^N \) as \( \varepsilon \) goes to 0 for every \( q' < 1 + \frac{1}{(N+1)(p-1)} \), since \( \varphi_-^n \) belongs to \( C^\infty_c(Q) \) and \( \beta_n(v^\varepsilon) \) converges to \( \beta_n(v) \) a.e. in \( Q \) and \( * \)-weakly in \( L^\infty(Q) \) as \( \varepsilon \) goes to zero and \( \beta_n(v) \) converges to 0 a.e. in \( Q \) and \( * \)-weakly in \( L^\infty(Q) \) as \( n \) goes to \( +\infty \), thanks to Proposition 2.2, we obtain

\[
\int_Q a(x,t,\nabla u^\varepsilon) \nabla \varphi_-^n \beta_n(v^\varepsilon) \, dx \, dt = \omega(\varepsilon,n).
\]

Since \( \overline{\beta_n}(v^\varepsilon) \) converges to \( \overline{\beta_n}(v) \) in \( L^1(Q) \) as \( \varepsilon \) goes to 0, and \( \overline{\beta_n}(v) \) converges to 0 in \( L^1(Q) \) as \( n \) goes to \( +\infty \), we obtain

\[
\int_Q \overline{\beta_n}(v^\varepsilon) \frac{d\varphi_-^n}{dt} \, dx \, dt = \omega(\varepsilon,n).
\]

Moreover, the weak \( L^1(Q) \) convergence of \( f^\varepsilon \) to \( f \) and thanks to Proposition 2.2 we obtain

\[
\int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi_-^n \, dx \, dt = \omega(\varepsilon,n).
\]

Due the strong convergence of \( \text{div}(G^\varepsilon) \) to \( \text{div}(G) \) in \( L^{q'}(0,T,W^{-1,p'}(\Omega)) \) and the weak convergence in \( L^p(0,T,W_0^{1,p}(\Omega)) \) of \( \beta_n(v^\varepsilon) \) to \( \beta_n(v) \) and \( \beta_n(v) \) to 0 strongly in \( L^p(0,T,W_0^{1,p}(\Omega)) \) (this facts is an easy consequence of the estimate on the truncates of \( u^\varepsilon \) in Proposition 4.2), we obtain

\[
\int_0^T < \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi_-^n > \, dx \, dt = \omega(\varepsilon,n).
\]

Finally, by (4.44) and since \( \beta_n(v^\varepsilon) \) is non negative and bounded and \( \varphi_-^n \) is continuous, we have

\[
\int_Q \beta_n(v^\varepsilon) \varphi_-^n \, d\lambda^\varepsilon_+ \leq \int_Q \varphi_-^n \, d\mu^+_s + \omega(\varepsilon) = \omega(\varepsilon,\eta).
\]

Putting together all these facts lead to (4.46), while (4.45) can be obtained in the same way choosing \( \beta_n(s) = B_n(s^-) \) and \( \beta_n(v^\varepsilon) \varphi_-^n \) as test function in (4.4). \( \square \)
Now let us check (4.42). For fixed \( k > 0 \), we choose \((k-T_k(v^\varepsilon))H_n(v^\varepsilon)\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\) as test function in (4.4), defining \( \Gamma_{n,k}(s) = \int_0^s (k-T_k(r))H_n(r)\exp(-H(r))\,dr \), and \( H(s) = \int_0^s h(r) \max_{|r| \leq 2n} h(r) \,dr \).

Integrating by parts, we obtain

\[
\int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt}(\psi_\delta^+\psi_\eta^+) \,dxdt + \int_Q (k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+) \,dxdt \\
+ \int_Q a(x,t,\nabla u^\varepsilon)\nabla H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt \\
- \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ \,dxdt \\
+ \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt \\
- \frac{\max_{|s| \leq 2n} h(s)}{\gamma\alpha \min_{|s| \leq 2n} h(s)} \int_Q h(v^\varepsilon)\nabla u^\varepsilon \nabla v^\varepsilon \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt \\
= \int_Q f^\varepsilon \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt \\
- \int_0^T <\text{div}(G^\varepsilon), \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ > \,dt \\
+ \int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt - \int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,d\lambda^\varepsilon \cdot
\]

For \( n > k \), we have

\[
H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon| \leq k\}} = a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon| \leq k\}} \text{ a.e. in } Q,
\]

then rearranging all terms of (4.47) and using assumptions (3.4) and (3.7), we obtain

\[
\int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq k\}} \exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt + \int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,d\lambda^\varepsilon \\
+ \max_{|s| \leq 2n} h(s) \int_Q |\nabla u^\varepsilon|^p \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dxdt
\]

\[
\leq - \int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt}(\psi_\delta^+\psi_\eta^+) \,dxdt + \frac{2k}{n} \int_{\{-2n < v^\varepsilon \leq -n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon \exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \,dx \,dt \\
+ \int_Q \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+) \,dxdt \\
- \int_Q f^\varepsilon \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ \,dxdt.
\]
\begin{equation}
\begin{aligned}
&+ \max_{\{n \leq 2n\}} h(s) \int_Q \left| \nabla u^\varepsilon \right|^p H_n(v^\varepsilon) \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon)) \psi_\delta^+ \psi_\eta^+ \, dx dt \\
&+ \int_{\{|u^\varepsilon| \geq 2n\}} h(u^\varepsilon) \nabla u^\varepsilon \cdot \mathbf{P} H_n(v^\varepsilon) \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon)) \psi_\delta^+ \psi_\eta^+ \, dx dt \\
&- \int_0^T < \text{div}(G^\varepsilon), \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi_\delta^+ \psi_\eta^+ > \, dt + \int_Q (k - T_k(v^\varepsilon)) \exp(-H(v^\varepsilon))H_n(v^\varepsilon) \psi_\delta^+ \psi_\eta^+ \, dx dt \\
&+ \int_Q h(v^\varepsilon)H_n(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla g^\varepsilon \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon)) \psi_\delta^+ \psi_\eta^+ \, dx dt \\
&\quad + \int_Q a(x, t, \nabla u^\varepsilon) \chi_{\{|u^\varepsilon| \leq k\}} \nabla g^\varepsilon \exp(-H(v^\varepsilon)) \psi_\delta^+ \psi_\eta^+ \, dx dt.
\end{aligned}
\end{equation}

Let us analyze term by term the right hand side of (4.48). Due to Proposition 4.2 we have $\Gamma_{n,k}(v^\varepsilon)$ converges to $\Gamma_{n,k}(v)$ weakly in $L^p(0, T; W^{1,p}_0(\Omega))$, and since $\Gamma_{n,k}(v) \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q)$, we deduce

\begin{equation}
\begin{aligned}
&\int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt} \left( \psi_\delta^+ \psi_\eta^+ \right) \, dx dt \\
&= \int_Q \Gamma_{n,k}(v) \frac{d}{dt} \psi_\delta^+ \psi_\eta^+ \, dx dt + \int_Q \Gamma_{n,k}(v) \frac{d}{dt} \psi_\delta^+ \psi_\eta^+ \, dx dt + \omega(\varepsilon) = \omega(\varepsilon, \delta).
\end{aligned}
\end{equation}
Since \((k - T_k(v^\varepsilon))H_n(v^\varepsilon)\) converges to \((k - T_k(v))H_n(v)\) a.e. and \(*\)-weakly in \(L^\infty(Q)\), thanks to Proposition 2.2, Proposition 4.2 and Lemma 4.6, we deduce

\[
\int_Q \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)a(x, t, \nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^-) \, dx \, dt \\
= \int_Q \exp(-H(v^\varepsilon))(k - T_k(v))H_n(v)a(x, t, \nabla u)\nabla(\psi_\delta^+\psi_\eta^-) \, dx \, dt + \omega(\varepsilon) \\
= \omega(\varepsilon, \delta).
\]

Moreover, \((k - T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\) weakly converges to \((k - T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+\) in \(L^p(0, T; W_0^{1,p}(\Omega))\), and \(*\)-weakly in \(L^\infty(Q)\), thanks again to Lemma 4.6, we have

\[
\int_0^T <\text{div}(G^\varepsilon), \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ > \, dt = \omega(\varepsilon, \delta),
\]

and

\[
\int_Q f^\varepsilon \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ \, dx \, dt = \omega(\varepsilon, \delta).
\]

Using assumptions (3.7), Proposition 2.2, Proposition 4.2 and Lemma 4.6, we deduce

\[
\int_Q h(v^\varepsilon)a(x, t, \nabla u^\varepsilon)\nabla g^\varepsilon \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \, dx \, dt = \omega(\varepsilon, \delta),
\]

\[
\int_Q a(x, t, \nabla u^\varepsilon)\chi_{\{|u^\varepsilon| \leq k\}} \nabla g^\varepsilon \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ \, dx \, dt = \omega(\varepsilon, \delta).
\]

Thanks to (4.18) we obtain

\[
\int_{\{|u^\varepsilon| \geq 2n\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon) \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \, dx \, dt = \omega(\varepsilon, n).
\]

Using assumptions (3.2), (3.5), Young’s inequality, since \(0 \leq \psi_\delta^+ \leq 1\) and \(\exp(-H(v^\varepsilon)) \leq \exp(C\|h\|_{L^1(\mathbb{R})})\) we obtain

\[
\left| \frac{1}{n} \int_{\{-2n < u^\varepsilon \leq -n\}} \exp(-H(v^\varepsilon))a(x, t, \nabla u^\varepsilon)\nabla v^\varepsilon \psi_\delta^+\psi_\eta^- \, dx \, dt \right| \\
\leq \frac{1}{n} \int_{\{-2n < u^\varepsilon \leq -n\}} b'(u^\varepsilon)a(x, t, \nabla u^\varepsilon)\nabla u^\varepsilon \psi_\delta^+ \psi_\eta^- \, dx \, dt + \frac{C}{n} \int_Q \left( |\nabla g^\varepsilon|^p + |L|^p \right) \, dx \, dt,
\]

applying Lemma 4.8 for \(\varphi_\eta^+ = \psi_\eta^+\), we obtain

\[
\frac{1}{n} \int_{\{-2n < u^\varepsilon \leq -n\}} a(x, t, \nabla u^\varepsilon)\nabla v^\varepsilon \psi_\delta^+ \psi_\eta^- \, dx \, dt = \omega(\varepsilon, n, \eta).
\]

Using (4.38) in Lemma 4.6, we have

\[
\left| \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^- \, d\lambda^- \right| \leq 2k \int_Q \psi_\delta^+\psi_\eta^- \, d\lambda^- = 2k \int_Q \psi_\delta^+\psi_\eta^- \, d\mu^- + \omega(\varepsilon) = \omega(\varepsilon, \delta).
\]

Collecting all we have shown above, we get

\[
\int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+ \, dx \, dt \leq \omega(\varepsilon, \delta, n, \eta).
\]
Since \( \int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi^+_\delta\psi^+_\eta \, d\lambda^\varepsilon_+ \geq 0 \) and \( \exp(-H(+\infty)) \leq \exp(-H(v^\varepsilon)) \leq \exp(-H(-\infty)) \) we obtain

\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon)\psi^+_\delta\psi^+_\eta \, dxdt \leq \omega(\varepsilon, \delta, \eta).
\]

On the other hand, reasoning as before with \( (k+T_k(v^\varepsilon))H_n(v^\varepsilon)\exp(H(v^\varepsilon))\psi^-\psi^-_\eta \) as test function we can obtain

\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon)\psi^-\psi^-_\eta \, dxdt \leq \omega(\varepsilon, \delta, \eta),
\]

therefore, we obtain (4.42) which yields (4.43).

**Remark 4.9.** As we have shown above we have

\[
\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi^+_\delta\psi^+_\eta \, d\lambda^\varepsilon_+ + \int_Q b'(u^\varepsilon)a(x, t, \nabla u^\varepsilon)\chi_{\{|v^\varepsilon| \leq k\}} \nabla u^\varepsilon\psi^+_\delta\psi^+_\eta \, dxdt \leq \omega(\varepsilon, \delta, n, \eta),
\]

by assumptions (3.2), (3.4), thanks to Proposition 4.2, 4.3 and Lemma 4.6 one obtains

\[
\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi^+_\delta\psi^+_\eta \, d\lambda^\varepsilon_+ = \omega(\varepsilon, \delta, n, \eta).
\]

Analogously we obtain

\[
\int_Q H_n(v^\varepsilon)(k+T_k(v^\varepsilon))\psi^-\psi^-_\eta \, d\lambda^\varepsilon_- = \omega(\varepsilon, \delta, n, \eta).
\]

The two last results above show an interesting property of approximating renormalized, they express the fact that \( v^\varepsilon \) (and so the solution \( u^\varepsilon \)) is very large (greater than any \( k > 0 \) ) on the set where the singular measure \( \mu^+_\delta \) is concentrated, and small (smaller than any \( k < 0 \) ) on the set where the singular measure \( \mu^-_\eta \) is concentrated.

**Step 3.** Far from \( E \).

We first prove a result that will be essential to deal with the second term on the right hand side of (4.40).

**Lemma 4.10.** Let \( k \geq 0 \) be fixed. Let \( S \) be an increasing \( C^\infty(\mathbb{R}) \)-function such that \( S(r) = r \) for \( |r| \leq k \) and supp \( S' \) is compact. Then

\[
\int_0^T \int_0^t \nabla (T_k(v^\varepsilon) - T_k(v)\nu)(1 - \Phi_{\delta, \eta}) \, dsdt \geq \omega(\varepsilon, \nu).
\]

**Proof.** The proof of the above Lemma follows the arguments in [14], Lemma 1 and we just sketch the proof of it.

Let \( k \geq 0 \) be fixed. Since \( S \) is increasing and \( S(r) = r \) for \( |r| \leq k \),

\[
T_k(S(v^\varepsilon)) = T_k(v^\varepsilon) \quad \text{and} \quad T_k(S(v)) = T_k(v) \quad \text{a.e. in } Q.
\]

As a consequence \( T_k(S(v))\nu = T_k(v)\nu \) a.e. in \( Q \), for any \( \nu > 0 \).

It follows that under the notation \( z^\varepsilon = S(v^\varepsilon) \) and \( z = S(v) \), and thanks to
properties of $T_k(z)$ we have
\[
\int_0^T \int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt \quad (4.50)
\]
\[
= \int_0^T \int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (T_k(z^\varepsilon) - T_k(z_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt
\]
\[
= \int_0^T \int_0^t \langle \frac{\partial (z^\varepsilon - T_k(z_\nu))}{\partial t}, (z^\varepsilon - T_k(z_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt
\]
\[
- \int_0^T \int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (z^\varepsilon - T_k(z^\varepsilon))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt
\]
\[
+ \int_0^T \int_0^t \langle \frac{\partial T_k(z_\nu)}{\partial t}, (z^\varepsilon - T_k(z_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt,
\]
integrating by parts we have
\[
\int_0^T \int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt \quad (4.51)
\]
\[
= \frac{1}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z_\nu))^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt - \frac{1}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z^\varepsilon))^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt
\]
\[
+ \frac{1}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z)_\nu)^2 \, dx \, dt - \frac{T}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z)_\nu)^2(t = 0) \, dx
\]
\[
- \frac{1}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z))^2 \, dx \, dt + \frac{T}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z))^2(t = 0) \, dx
\]
\[
+ \int_0^T \int_0^t \int_0^t \int_0^t \frac{\partial T_k(z_\nu)}{\partial t}(z^\varepsilon - T_k(z_\nu))(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt \, ds \, dt,
\]
since $\int_0^r (s - T_k(s)) \, ds = \frac{1}{2}(r - T_k(r))^2$.

Using the definition of $z^\varepsilon$ and $z$, the fact that $S$ is bounded and $v^\varepsilon$ converges to $v$ a.e. on $Q$, we have $z^\varepsilon$ converges to $z$ strongly in $L^2(Q)$ and in $L^\infty(Q)$, the strong convergence of $b(u_0^\varepsilon)$ to $b(u_0)$ in $L^1(\Omega)$ implies that $z^\varepsilon(t = 0)$ converges to $S(b(u_0))$ strongly in $L^2(\Omega)$.

Passing to the limit as $\varepsilon$ tends to zero in (4.51) leads to
\[
\int_0^T \int_0^t \langle \frac{\partial S(v)}{\partial t}, (T_k(v)^\varepsilon - T_k(v_\nu))(1 - \Phi_{\delta,\eta}) \rangle \, ds \, dt \quad (4.52)
\]
\[
= \frac{1}{2} \int_0^T \int_0^t (z - T_k(z)_\nu)^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, dt - \frac{1}{2} \int_0^T \int_0^t (z - T_k(z))^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, dt
\]
\[
+ \frac{1}{2} \int_0^T \int_0^t (z - T_k(z)_\nu)^2 \, dx \, dt - \frac{T}{2} \int_0^T \int_0^t (z - T_k(z)_\nu)^2(t = 0) \, dx
\]
\[
- \frac{1}{2} \int_0^T \int_0^t (z - T_k(z))^2 \, dx \, dt + \frac{T}{2} \int_0^T \int_0^t (z - T_k(z))^2(t = 0) \, dx
\]
\[
+ \int_0^T \int_0^t \int_0^t \int_0^t \frac{\partial T_k(z_\nu}){\partial t}(z - T_k(z)_\nu)(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt \, ds \, dt + o(\varepsilon),
\]
by rewriting the definition of $T_k(u)_{\nu}$ in terms of $T_k(z)$ we have
\[
\frac{\partial T_k(z)_{\nu}}{\partial t} + \nu(T_k(z)_{\nu} - T_k(z)) = 0 \text{ in } D'(Q),
\]
\[
T_k(z)_{\nu}(t = 0) = u_0' \text{ in } \Omega.
\]

By properties of $T_k(z)_{\nu}$ we obtain that $T_k(z)_{\nu}$ converges to $T_k(z)$ strongly in $L^2(Q)$ and $T_k(z)_{\nu}(t = 0)$ converges to $T_k(S(b(u_0)))$ strongly in $L^2(\Omega)$ as $\nu$ tends to $\infty$.

Passing to the limit-inf as $\nu$ tends to $\infty$ in (4.52) leads to
\[
\lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_0^T \int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_{\nu})(1 - \Phi_{\delta,\eta}) \rangle \, dsdt
\]
\[
= \nu \int_0^T \int_0^t \int_\Omega (T_k(z) - T_k(z)_{\nu})(z - T_k(z)_{\nu})(1 - \Phi_{\delta,\eta}) \, dxdsdt.
\]

Thanks to definition of $T_k(z)_{\nu}$ we have
\[
\int_0^T \int_0^t \int_\Omega (T_k(z) - T_k(z)_{\nu})(z - T_k(z)_{\nu})(1 - \Phi_{\delta,\eta}) \, dxdsdt
\]
\[
= \int_{\{|z| \leq k\}} (z - T_k(z)_{\nu})(z - T_k(z)_{\nu})(1 - \Phi_{\delta,\eta}) \, dxdsdt
\]
\[
+ \int_{\{|z| > k\}} (k - T_k(z)_{\nu})(z - T_k(z)_{\nu})(1 - \Phi_{\delta,\eta}) \, dxdsdt
\]
\[
+ \int_{\{|z| < -k\}} (-k - T_k(z)_{\nu})(z - T_k(z)_{\nu})(1 - \Phi_{\delta,\eta}) \, dxdsdt,
\]

and the three terms are all non negatives, then
\[
\int_0^T \int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_{\nu})(1 - \Phi_{\delta,\eta}) \rangle \, dt \geq \omega(\varepsilon, \nu)
\]

Now, let us multiply by $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})^+(1 - \Phi_{\delta,\eta})$ the equation solved by $u^\varepsilon$ and integrate to obtain
\[
\int_0^T \langle \frac{\partial u^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})^+(1 - \Phi_{\delta,\eta}) \rangle \, dt
\]
\begin{align*}
+ & \int_Q a(x, t, \nabla u^\varepsilon).\nabla(T_k(v^\varepsilon) - T_k(v)_{\nu})^+H_n(v^\varepsilon)(1 - \Phi_{\delta,\eta}) \, dxdt \\
+ & \int_Q a(x, t, \nabla u^\varepsilon).\nabla H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})^+(1 - \Phi_{\delta,\eta}) \, dxdt \\
- & \int_Q a(x, t, \nabla u^\varepsilon).\nabla \Phi_{\delta,\eta}H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})^+ \, dxdt \\
+ & \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})^+(1 - \Phi_{\delta,\eta}) \, dxdt \\
= & \int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_{\nu})(1 - \Phi_{\delta,\eta})^+ \, dxdt
\end{align*}
By assumption (3.7) we have
\[ -\int_0^T \langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+ \rangle dt > 0 \]
\[ + \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) d\lambda_+^\varepsilon \]
\[ - \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) d\lambda_-^\varepsilon. \]

Let us analyze term by term the identity (4.53), by Lemma 4.8 we have
\[ \int_0^T \langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) \rangle dt \geq \omega(\varepsilon, \nu). \]

By assumption (3.7) we have
\[ \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) dxdt \geq 0 \]

The almost everywhere and \(*-\)weak convergence of \(H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+\) to \(H_n(v)(T_k(v) - T_k(v)\nu)^+\) in \(L^\infty(Q)\), the properties of \(T_k(v)\nu\) and thanks to Propositions 2.2 and 4.2 we have
\[ \int_Q a(x, t, \nabla u^\varepsilon)\cdot \nabla \Phi_{\delta,\eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+ dxdt = \omega(\varepsilon, \nu). \]

Due the strong convergence of \(\text{div}(G^\varepsilon)\) to \(\text{div}(G)\) in \(L^{p'}(0, T, W^{-1}p'(\Omega))\), Proposition 4.2 and the properties of \(T_k(v)\nu\) one obtains
\[ \int_0^T < \text{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) > dt = \omega(\varepsilon, \nu). \]

The weak convergence of \(f^\varepsilon\) to \(f\) in \(L^1(Q)\), the almost everywhere and \(*-\)weak convergence of \(H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+\) to \(H_n(v)(T_k(v) - T_k(v)\nu)^+\) in \(L^\infty(Q)\), Propositions 2.2, the properties of \(T_k(v)\nu\) and the Lebesgue’s dominated convergence theorem leads to
\[ \int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) dxdt = \omega(\varepsilon, \nu). \]

By Lemma 4.6 and the fact that \(|H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+| \leq 2k\) we obtain
\[ \left| \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) d\lambda_+^\varepsilon \right| \leq 2k \int_Q (1 - \psi^+_{\delta,\eta}) d\lambda_+^\varepsilon + 2k \int_Q \psi^-_{\delta,\eta} d\lambda_-^\varepsilon, \]

and
\[ \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta), \]

and similarly we get
\[ \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)\nu)^+(1 - \Phi_{\delta,\eta}) d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta). \]
It remains to prove that
\[ \int_Q a(x, t, \nabla u^\varepsilon) \nabla H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta, \eta}) \, dx \, dt = \omega(\varepsilon, n, \delta, \eta). \]

We have
\[
\left| \frac{1}{n} \int_{\{n \leq |v^\varepsilon| < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon (T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta, \eta}) \, dx \, dt \right|
\leq \frac{2k}{n} \int_{\{n \leq |v^\varepsilon| < 2n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon (1 - \psi^+_\delta \psi^+_\eta) \, dx \, dt,
\]
\[
+ \frac{2k}{n} \int_{\{-2n < v^\varepsilon \leq -n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon (1 - \psi^-_\delta \psi^-_\eta) \, dx \, dt,
\]
\[
- \frac{2k}{n} \int_{\{-2n < v^\varepsilon \leq -n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \psi^+_\delta \psi^+_\eta \, dx \, dt.
\]

We can apply Lemma 4.8 for every term above. Indeed, if we define \( \varphi^\pm_{\delta, \eta} = 1 - \psi^\pm_\delta \psi^\pm_\eta \), we have by Lemma 4.6,
\[
\int_Q \varphi^\pm_{\delta, \eta} \, d\mu^+_\delta \leq \eta + \delta,
\]
then \( \varphi^\pm_{\delta, \eta} \) satisfies (4.44), thanks to Lemma 4.8 we obtain
\[
\frac{2k}{n} \int_{\{n \leq |v^\varepsilon| < 2n\}} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon (1 - \psi^+_\delta \psi^+_\eta) \, dx \, dt \leq \omega(\varepsilon, n) + \delta + \eta = \omega(\varepsilon, n, \delta, \eta).
\]

In analogous way we obtain the same result for the others terms. Therefore, we obtain our estimate far from \( E \)
\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu)^+ H_n(v^\varepsilon)(1 - \Phi_{\delta, \eta}) \, dx \, dt \leq \omega(\varepsilon, \nu, n, \delta, \eta). (4.54)
\]

Similarly to (4.54), we take \( (T_k(v^\varepsilon) - T_k(v)_\nu)^- H_n(v^\varepsilon)(1 - \Phi_{\delta, \eta}) \) as test function in (4.4) we deduce
\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon)(1 - \Phi_{\delta, \eta}) \, dx \, dt \leq \omega(\varepsilon, \nu, n, \delta, \eta). (4.55)
\]

\( \square \)
**Step 4.** Strong convergence of truncates.
Collecting together (4.40), (4.41) and (4.55), we have by taking again $n > k$,
\[
\lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \, dx \, dt \leq \int_{Q} a(x, t, \nabla u) \nabla T_k(v) \, dx \, dt. \tag{4.56}
\]

Now, we prove that
\[
\lim_{\varepsilon \to 0} \int_{Q} b'(u^\varepsilon) \left[ a(x, t, \nabla u^\varepsilon \chi_{\{|v|^\leq k\}}) - a(x, t, \nabla u \chi_{\{|v|^\leq k\}}) \right] \left[ \nabla u^\varepsilon \chi_{\{|v|^\leq k\}} - \nabla u \chi_{\{|v|^\leq k\}} \right] \, dx \, dt = 0. \tag{4.57}
\]

We set
\[
A^\varepsilon = \int_{Q} b'(u^\varepsilon) \left[ a(x, t, \nabla u^\varepsilon \chi_{\{|v|^\leq k\}}) - a(x, t, \nabla u) \chi_{\{|v|^\leq k\}} \right] \left[ \nabla u^\varepsilon \chi_{\{|v|^\leq k\}} - \nabla u \chi_{\{|v|^\leq k\}} \right] \, dx \, dt.
\]

We split (4.57), into $A^\varepsilon = A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon$, where
\[
A_1^\varepsilon = \int_{Q} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \chi_{\{|v|^\leq k\}} \, dx \, dt,
\]
\[
A_2^\varepsilon = - \int_{Q} b'(u^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla u \chi_{\{|v|^\leq k\}} \chi_{\{|v|^\leq k\}} \, dx \, dt,
\]
\[
A_3^\varepsilon = - \int_{Q} b'(u^\varepsilon) a(x, t, \nabla u) (\nabla u^\varepsilon \chi_{\{|v|^\leq k\}}) - \nabla u \chi_{\{|v|^\leq k\}} \right) \, dx \, dt.
\]

We pass to the limit as $\varepsilon$ tends to 0 in $A_1^\varepsilon$, $A_2^\varepsilon$ and $A_3^\varepsilon$. Let us remark that we have
\[
b'(u^\varepsilon) \nabla u^\varepsilon \chi_{\{|v|^\leq k\}} = \nabla T_k(v^\varepsilon) + \nabla g^\varepsilon \chi_{\{|v|^\leq k\}} \text{ a.e in } Q,
\]
and we have also $\chi_{\{|v|^\leq k\}}$ almost everywhere converges to $\chi_{\{|v|^\leq k\}}$ in $Q$ (see [7]), we obtain:
\[
\lim_{\varepsilon \to 0} A_1^\varepsilon = \lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \, dx \, dt + \lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \chi_{\{|v|^\leq k\}} \nabla g^\varepsilon \, dx \tag{4.58}
\]
\[
\leq \int_{Q} a(x, t, \nabla u) \nabla T_k(v) \, dx \, dt + \int_{Q} a(x, t, \nabla u) \nabla g \chi_{\{|v|^\leq k\}} \, dx \, dt.
\]

As a consequence of Proposition 4.2, we deduce that
\[
\lim_{\varepsilon \to 0} A_2^\varepsilon = - \int_{Q} a(x, t, \nabla u) (\nabla T_k(v) + \nabla g) \, dx \, dt, \tag{4.59}
\]
and
\[
\lim_{\varepsilon \to 0} A_3^\varepsilon = \lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u) \left( \nabla T_k(v^\varepsilon) + \nabla g^\varepsilon \right) \chi_{\{|v|^\leq k\}} \tag{4.60}
\]
\[
- b'(u^\varepsilon) b'(u)^{-1} \left( \nabla T_k(v) + \nabla g \chi_{\{|v|^\leq k\}} \right) \, dx \, dt = 0.
\]

Therefore collecting (4.58), (4.59) and (4.60) yield (4.57). Through the monoto-
nicity argument which relies on (3.6) (see [12], Lemma 5), we can deduce from
(4.56) and Proposition 4.3, that $a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \chi_{\{|v|^\leq k\}}$ converges to $a(x, t, \nabla u) \nabla u \chi_{\{|v|^\leq k\}}$ weakly in $L^1(Q)$, by coercivity argument we have that $|\nabla u^\varepsilon|^p \chi_{\{|v|^\leq k\}}$ is equi-
integrable, as a consequence of Vitali’s theorem and since $g^\varepsilon$ strongly converges in
$L^p(0, T; W^{1,p}_0(\Omega))$, $T_k(v^\varepsilon) \to T_k(v)$ strongly in $L^p(0, T; W^{1,p}_0(\Omega))$. 

the proof of Theorem 4.7 is complete.

\begin{proof}

\* Equi-integrability of the nonlinearity sequence. We shall prove now that \( h(u^\varepsilon)|\nabla u^\varepsilon|^p \) converge strongly to \( h(u)|\nabla u|^p \) in \( L^1(Q) \), by Proposition 4.3 we have \( h(u^\varepsilon)|\nabla u^\varepsilon|^p \) converge to \( h(u)|\nabla u|^p \) a.e. in \( Q \) and (4.18) implies that

\[
\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p |\chi_{\{|u^\varepsilon|>k\}}| \, dx \, dt = \omega(k). \tag{4.61}
\]

Let \( E \) be a subset of \( Q \) we have

\[
\int_E h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt = \int_{E \cap \{|u^\varepsilon|<k+1, |u^\varepsilon|<k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt + \int_{E \cap \{|u^\varepsilon|<k+1, |u^\varepsilon|>k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt
\]
\[
+ \int_{E \cap \{|u^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \Phi_{\delta,\eta} \, dx \, dt, \tag{4.62}
\]
\[
+ \int_{E \cap \{|u^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1 - \Phi_{\delta,\eta}) \, dx \, dt.
\]

Let \( \beta_k(s) = B_{k,k+1}(s^+) \), we can choose \( \beta_k(v^\varepsilon)\psi^\varepsilon_\eta \) as test function in (4.4) and rearranging conveniently all terms we obtain

\[
\int_{\{v^\varepsilon> k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \psi^\varepsilon_\eta \, dx \, dt + \int_{\{k \leq v^\varepsilon<k+1\}} b'(u^\varepsilon) a(x,t, \nabla u^\varepsilon) \nabla u^\varepsilon \psi^\varepsilon_\eta \, dx \, dt + \int_Q \beta_k(v^\varepsilon) \psi^\varepsilon_\eta \, d\lambda^{-}_\varepsilon
\]
\[
\leq C \int_{\{v^\varepsilon>k\}} \left( |\nabla g^\varepsilon|^p + |L|^p + |G^\varepsilon|^p + |f^\varepsilon| \right) \, dx \, dt
\]
\[
+ \int_Q \overline{\beta_k(v^\varepsilon)} \frac{d\psi^\varepsilon_\eta}{dt} \, dx \, dt - \int_Q a(x,t, \nabla u^\varepsilon) \nabla \psi^\varepsilon_\eta \beta_k(v^\varepsilon) \, dx \, dt
\]
\[
+ \int_Q G^\varepsilon \cdot \nabla \psi^\varepsilon_\eta \beta_k(v^\varepsilon) \, dx \, dt + \int_Q \beta_k(v^\varepsilon) \psi^\varepsilon_\eta \, d\lambda^+\varepsilon.
\]

By following the same proof as in the Lemma 4.8 we obtain

\[
\int_{\{v^\varepsilon> k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \psi^\varepsilon_\eta \, dx \, dt = \omega(\varepsilon, k, \eta).
\]

Similarly as above we can choose \( \beta_k(s) = B_{k,k+1}(s^-) \) and \( \beta_k(v^\varepsilon)\psi^\varepsilon_\eta^\varepsilon \) as test function in (4.4) imply that

\[
\int_{\{v^\varepsilon> k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \Phi_{\delta,\eta} \, dx \, dt = \omega(\varepsilon, k, \eta). \tag{4.62}
\]

By taking \( T_1(v^\varepsilon - T_k(v^\varepsilon))^+ (1 - \Phi_{\delta,\eta}) \) as test function in (4.4) we obtain

\[
\int_Q \overline{\Theta_k(v^\varepsilon)} \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, dt + \int_{\{v^\varepsilon> k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1 - \Phi_{\delta,\eta}) \, dx \, dt + \int_{\{k \leq v^\varepsilon<k+1\}} b'(u^\varepsilon) a(x,t, \nabla u^\varepsilon) \nabla u^\varepsilon \Phi_{\delta,\eta} \, dx \, dt
\]
\[
\leq C \int_{\{v^\varepsilon>k\}} \left( |\nabla g^\varepsilon|^p + |L|^p + |G^\varepsilon|^p + |f^\varepsilon| \right) \, dx \, dt - \int_Q G^\varepsilon \cdot \nabla \psi^\varepsilon_\eta T_1(v^\varepsilon - T_k(v^\varepsilon))^+ \, dx \, dt
\]
\[
+ \int_Q a(x,t, \nabla u^\varepsilon) \nabla \psi^\varepsilon_\eta T_1(v^\varepsilon - T_k(v^\varepsilon))^+ \, dx \, dt + \int_Q T_1(v^\varepsilon - T_k(v^\varepsilon))^+ (1 - \Phi_{\delta,\eta}) \, d\lambda^+\varepsilon.
\]
\end{proof}
Then we obtain \( \int_{t^\varepsilon > k+1} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1 - \Phi_{\delta,\eta}) \, dx \, dt = \omega(\varepsilon, k, \delta, \eta) \), and similarly by using \( T_1(v^\varepsilon - T_k(v^\varepsilon))^- \) as test function in (4.4),

\[
\int_{\{|\varepsilon| > k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1 - \Phi_{\delta,\eta}) \, dx \, dt = \omega(\varepsilon, k, \delta, \eta).
\]

By assumption (3.2) and (3.7) we have

\[
\int_{E \cap \{|\varepsilon| < k+1, |\varepsilon| < k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt \leq \max_{\{|\varepsilon| \leq k\}} h(s) \int_{E \cap \{|\varepsilon| < k+1\}} |\nabla u^\varepsilon|^p \, dx \, dt
\]

\[
\leq \max_{\{|\varepsilon| \leq k\}} h(s) \frac{1}{\gamma} \left( \int_E |\nabla T_k(v^\varepsilon)|^p \, dx \, dt + \int_E |\nabla g^\varepsilon|^p \, dx \, dt \right)
\]

and

\[
\int_{E \cap \{|\varepsilon| < k+1, |\varepsilon| > k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt \leq \int_{E \cap \{|\varepsilon| > k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p \, dx \, dt
\]

Then (4.18), (4.61), (4.62), (4.63) and thanks to Theorem 4.7 we deduce that \( h(u^\varepsilon)|\nabla u^\varepsilon|^p \) is equi-integrable and by Vitali’s Theorem we deduce that

\[
h(u^\varepsilon)|\nabla u^\varepsilon|^p \to h(u)|\nabla u|^p \text{ in } L^1(Q).
\]

**Proof.** *(Proof of Theorem 4.1).* Now we are able to prove that Problem (1.1)-(1.3) has a renormalized solutions.

Let \( S \in W^{2,\infty}(\mathbb{R}) \), such that \( S' \) has a compact support as in Definition 3.1, and let \( \varphi \in C_\infty^\varepsilon(Q) \), then the approximating solutions \( u^\varepsilon \) (and \( v^\varepsilon \)) satisfy

\[
-\int_0^T \langle \varphi, S(v^\varepsilon) \rangle \, dt + \int_Q S'(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla \varphi \, dx \, dt + \int_Q S''(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dx \, dt
\]

\[
+ \int_Q S'(v^\varepsilon) h(u^\varepsilon)|\nabla u^\varepsilon|^p \varphi \, dx \, dt
\]

\[
= \int_Q f^\varepsilon S'(v^\varepsilon) \varphi \, dx \, dt + \int_Q G^\varepsilon S'(v^\varepsilon) \nabla \varphi \, dx \, dt + \int_Q S''(v^\varepsilon) G^\varepsilon \nabla v^\varepsilon \varphi \, dx \, dt
\]

\[
+ \int_Q S'(v^\varepsilon) \varphi \, d\lambda_-^\varepsilon - \int_Q S'(v^\varepsilon) \varphi \, d\lambda_+^\varepsilon.
\]

Thanks to Theorem 4.7 and (4.64), all terms in (4.65) easily pass to the limit on \( \varepsilon \) except the last two terms that give some problem. We can write following the arguments in [34]

\[
\int_Q S'(v^\varepsilon) \varphi \, d\lambda_+^\varepsilon = \int_Q S'(v^\varepsilon) \varphi \psi_\delta^+ \, d\lambda_+^\varepsilon + \int_Q S'(v^\varepsilon) \varphi (1 - \psi_\delta^+) \, d\lambda_+^\varepsilon.
\]
Let \( \psi_\delta^+ \) be defined as in Lemma 4.6, then we have
\[
\left| \int_Q S'(v^\varepsilon) \varphi (1 - \psi_\delta^+) \, d\lambda_+ \right| \leq C \int_Q (1 - \psi_\delta^+) \, d\lambda_+ = \omega(\varepsilon, \delta),
\]
while choosing \( S'(v^\varepsilon) \varphi \psi_\delta^+ \) in (4.4) one gets,
\[
\int_Q S'(v^\varepsilon) \varphi \psi_\delta^+ \, d\lambda_+ = - \int_Q f^\varepsilon S'(v^\varepsilon) \varphi \psi_\delta^+ \, dx dt - \int_Q G^\varepsilon S'(v^\varepsilon) \nabla (\varphi \psi_\delta^+) \, dx dt \tag{4.67}
\]
\[- \int_Q G^\varepsilon S''(v^\varepsilon) \nabla v^\varepsilon \varphi \psi_\delta^+ \, dx dt + \int_Q S'(v^\varepsilon) \varphi \psi_\delta^+ \, d\lambda_- - \int_Q (\varphi \psi_\delta^+)_t \, dx dt
\]
\[+ \int_Q S'(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla (\psi_\delta^+ \varphi) \, dx dt + \int_Q S''(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \psi_\delta^+ \varphi \, dx dt.
\]

Now, thanks to Proposition 4.2 and the properties of \( \psi_\delta^+ \), we have
\[
\int_Q f^\varepsilon S'(v^\varepsilon) \varphi \psi_\delta^+ \, dx dt = \omega(\varepsilon, \delta) \text{ and } \int_Q G^\varepsilon S'(v^\varepsilon) \nabla (\varphi \psi_\delta^+) \, dx dt = \omega(\varepsilon, \delta).
\]

By Lemma 4.6, we deduce
\[
\left| \int_Q S'(v^\varepsilon) \varphi \psi_\delta^+ \, d\lambda_- \right| \leq C \int_Q \psi_\delta^+ \, d\lambda_- = \omega(\varepsilon, \delta).
\]

Again by Lemma 4.4, and since \( S(v) \in L^p(0, T; W_0^{1, p}(\Omega)) \cap L^\infty(Q) \),
\[
\int_Q S(v^\varepsilon)(\varphi \psi_\delta^+)_t \, dx dt = \omega(\varepsilon, \delta).
\]

By Theorem 4.7 and Lemma 4.6, we have
\[
\int_Q S'(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla (\psi_\delta^+ \varphi) \, dx dt = \omega(\varepsilon, \delta),
\]
and
\[
\int_Q S''(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \psi_\delta^+ \varphi \, dx dt = \omega(\varepsilon, \delta).
\]

and again by Lemma 4.6 and (4.64) we obtain
\[
\int_Q S'(v^\varepsilon) h(u^\varepsilon) |\nabla u^\varepsilon|^p \psi_\delta^+ \varphi \, dx dt = \omega(\varepsilon, \delta)
\]
Therefore, from (4.66) we deduce
\[
\int_Q S'(v^\varepsilon) \varphi \, d\lambda_+ = \omega(\varepsilon). \tag{4.68}
\]

Similarly, we can prove that
\[
\int_Q S'(v^\varepsilon) \varphi \, d\lambda_- = \omega(\varepsilon). \tag{4.69}
\]

As a consequence of the above convergence results, we are in a position to pass to the limit as \( \varepsilon \) tends to 0 in (4.65) and to conclude that \( u \) satisfies (3.11).
It remains to show that \( S(v) \) satisfies the initial condition (3.12). To this end, firstly remark that \( S(v^\varepsilon) \) being bounded in \( L^\infty(Q) \), secondly, (4.65) and the above considerations on the behavior of the terms of this equation show that \( \frac{\partial S(v^\varepsilon)}{\partial t} \) is bounded in \( L^1(Q) + L^r(0,T;W^{-1,\varepsilon}(\Omega)) \).

As a consequence, an Aubin’s type lemma (see e.g., [41], Corollary 4) implies that \( S(v^\varepsilon) \) lies in a compact set of \( C([0,T];W^{-1,\varepsilon}(\Omega)) \) for any \( s < \inf(p', \frac{N}{N-1}) \). It follows that, on one hand, \( S(v^\varepsilon)(t=0) \) converges to \( S(v)(t=0) \) strongly in \( W^{-1,\varepsilon}(\Omega) \). On the other hand, the smoothness of \( S \) imply that \( S(v^\varepsilon)(t=0) \) converges to \( S(b(u))(t=0) \) strongly in \( L^r(\Omega) \) for all \( r < \infty \). Due to (4.2), we conclude that \( S(v^\varepsilon)(t=0) = S(b(u^\varepsilon)) \) converges to \( S(b(u))(t=0) \) strongly in \( L^r(\Omega) \). Then \( v \) satisfies (3.12).

Now choosing \( \beta_n(v^\varepsilon) \) as test function in (4.4) where \( \varphi \in C^\infty_c(Q) \), we obtain

\[
-\int_0^T \langle \varphi_t, \beta_n(v^\varepsilon) \rangle \, dt + \int_Q \beta_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon) \nabla \varphi \, dxdt + \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x,t,\nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dxdt
\]

\[
+ \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p \beta_n(v^\varepsilon) \varphi \, dxdt
\]

\[
= \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi \, dxdt - \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi \rangle \, dxdt + \int_Q \beta_n(v^\varepsilon) \varphi \, dx + \int_Q \beta_n(v^\varepsilon) \varphi \, dx - \int_Q \beta_n(v^\varepsilon) \varphi \, dx.
\]

Reasoning as before (in particular as in the proof of Lemma 4.8) we obtain

\[
\int_0^T \langle \varphi_t, \beta_n(v^\varepsilon) \rangle \, dt = \omega(\varepsilon,n), \quad \int_Q \beta_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon) \nabla \varphi \, dxdt = \omega(\varepsilon,n),
\]

\[
\int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi \, dxdt = \omega(\varepsilon,n), \quad \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi \rangle \, dxdt = \omega(\varepsilon,n),
\]

thanks to Theorem 4.7 we have

\[
\frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x,t,\nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dxdt = \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x,t,\nabla v) \nabla v \varphi \, dxdt + \omega(\varepsilon),
\]

and (4.64) gives

\[
\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p \beta_n(v^\varepsilon) \varphi \, dxdt = \omega(\varepsilon,n).
\]

Now we deal with the two last terms in the right hand side of (4.70) we can write

\[
\int_Q \beta_n(v^\varepsilon) \varphi \, dx + \int_Q \varphi \, dx = \int_Q h_n(v^\varepsilon) \varphi \, dx + \int_Q \varphi \, dx + \omega(\varepsilon),
\]

where \( h_n(s) = H_n(s^+) \). By construction of \( \lambda_+^\varepsilon \) we have

\[
\int_Q \varphi \, d\lambda_+^\varepsilon = \int_Q \varphi \, d\mu^+_s + \omega(\varepsilon).
\]

Following the same argument as in (4.64) and (4.65) by taking \( h_n(v^\varepsilon) = S'(v^\varepsilon) \) we obtain

\[
\int_Q h_n(v^\varepsilon) \varphi \, d\lambda_+^\varepsilon = \omega(\varepsilon).
\]
If we prove that
\[ \int_Q \beta_n(v^\varepsilon) \phi \, d\lambda_\varepsilon = \omega(\varepsilon), \]  
(4.71)
then, we obtain for every \( \varphi \in C^\infty_c(Q) \)
\[ \lim_{n \to \infty} \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi \, dx \, dt = \int_Q \varphi \, d\mu^+_s \]  
(4.72)
We can write
\[ \int_Q \beta_n(v^\varepsilon) \phi \, d\lambda^-_\varepsilon = \int_Q \beta_n(v^\varepsilon) \phi \psi^-_\varepsilon \, d\lambda^-_\varepsilon + \int_Q \beta_n(v^\varepsilon) \varphi(1 - \psi^-_\varepsilon) \, d\lambda^-_\varepsilon, \]
by Lemma 4.6, we obtain
\[ \int_Q \beta_n(v^\varepsilon) \varphi(1 - \psi^-_\varepsilon) \, d\lambda^-_\varepsilon = \omega(\varepsilon, \delta). \]
Choosing \( \beta_n(v^\varepsilon) \phi \psi^-_\varepsilon \) as a test function in the formulation of \( u^\varepsilon \)
\[ \int_Q \beta_n(v^\varepsilon) \varphi \psi^-_\varepsilon \, d\lambda^-_\varepsilon = \int_0^T \langle (\varphi \psi^-_\varepsilon)_t, \beta_n(v^\varepsilon) \rangle \, dt - \int_Q \beta_n(v^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla (\varphi \psi^-_\varepsilon) \, dx \, dt \]
\[ - \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla \varphi \psi^-_\varepsilon \, dx \, dt \]
\[ - \int_Q h(u^\varepsilon) |\nabla u^\varepsilon|^p \beta_n(v^\varepsilon) \psi^-_\varepsilon \phi \, dx \, dt + \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi \psi^-_\varepsilon \, dx \, dt + \int_Q G^\varepsilon \beta_n(v^\varepsilon) \nabla (\varphi \psi^-_\varepsilon) \, dx \, dt \]
\[ + \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} G^\varepsilon \nabla v^\varepsilon \psi^-_\varepsilon \, dx \, dt + \int_Q \beta_n(v^\varepsilon) \varphi \psi^-_\varepsilon \, d\lambda^+_\varepsilon. \]
Using again Proposition 2.2, Proposition 4.2, Lemma 4.6, Lemma 4.8 and (4.64) yields (4.71), and therefore we obtain (4.72) for every \( \varphi \in C^\infty_c(Q) \). Now if \( \varphi \in C^\infty(\overline{Q}) \), we can split
\[ \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi \, dx \, dt = \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi \psi^+_\varepsilon \, dx \, dt \]
(4.73)
\[ + \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi(1 - \psi^+_\varepsilon) \, dx \, dt. \]
Thanks to (4.72), we have
\[ \lim_{n \to \infty} \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi \psi^+_\varepsilon \, dx \, dt = \int_Q \varphi \, d\mu^+_s + \omega(\delta), \]
By Lemma 4.8, we obtain
\[ \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi(1 - \psi^+_\varepsilon) \, dx \, dt = \omega(\varepsilon, n, \delta). \]
Thanks to Theorem 4.7, we deduce
\[ \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla \varphi(1 - \psi^+_\varepsilon) \, dx \, dt = \omega(n, \delta). \]
Putting together all these facts above, from (4.73) we get (3.13) for every $\varphi \in C^\infty(Q)$, and by density argument (3.13) holds for every $\varphi \in C(Q)$. To obtain (3.14) we can reason as before using $\psi^\delta_+ \psi^\delta_-$ in the place of $\psi^\delta_+$ and viceversa, and this conclude the proof of Theorem 4.1. □

References


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