Gulf Journal of Mathematics
Vol 19, Issue 1 (2025) 156-168

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v19i1.2524

A FINITE DIFFERENCE METHOD ON UNIFORM MESHES FOR SOLVING THE TIME-SPACE FRACTIONAL ADVECTION-DIFFUSION EQUATION

ALLAOUA BOUDJEDOUR[1], IQBAL M. BATIHA[2,3*], SELMA BOUCETTA[4], MOHAMED DALAH[5], KHALED ZENNIR[6] and ADEL OUANNAS[7]

Abstract. In order to investigate linear time and spatial fractional advection equations, we present a finite difference scheme (FDS) in this paper. The fractional Taylor series method for u at t_{j+1} and x_{i+1} is used to approximate the fractional derivatives. First, we construct our numerical scheme (NS) for the mathematical model. In the second part, we study the stability and convergence of our numerical scheme. Finally, the numerical simulations of the fractional advection equation, using the FDM, is plotted for several values of fractional parameters α and ν. It will be shown that the convergence is achieved properly which confirms the effectiveness of the proposed algorithm.

1. Introduction and position of problem

In recent years, mathematical models of dynamic systems with fractional characteristics have garnered special attention, see [1, 4, 8, 9, 10, 11, 12, 13, 14]. As a rule, the fractional property of modeled systems can be considered with respect to space and time, see [30]. The fractional derivatives in relation to the time and space factor is often expressed by such an important property in applied sciences and simulated systems. Mathematically, the transition of a dynamic model to its fractional description can be carried out using the apparatus of fractional derivatives. In particular, to mathematically formalize the characteristics of fractional derivatives with respect to spatial coordinates are used, and to represent the hereditary properties of a system or the memory effect, a fractional derivative with respect to time. There is a known practice of using the fractional differential approach in mathematical modeling of filtration in complex not homogeneous porous media, processes of transformation of temperature and humidity fields in the surface layer of the atmosphere [21], kinetics of dispersive transfer of charge carriers in semiconductor structures [34], processes of anomalous diffusion and diffusion of particles in not homogeneous media [20], processes of thermal conductivity [31, 38]. Due to the fact that analytical solutions of fractional differential equations with partial derivatives cause particular difficulties,

Date: Received: Dec 13, 2024; Accepted: Dec 26, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 35A24.

Key words and phrases. Fractional derivatives, finite difference scheme, fractional advection equation, stability, consistence, convergence, fractional Taylor series method.

numerical methods, including finite-difference ones, are widely used in the practice of mathematical modeling [2, 23, 24, 37]. The corresponding derivative is calculated using Lagrange's rule for differential operators. Computing n^{th}-order derivative over the integral of order $p = n - q$, the q order derivative is obtained. It is important to remark that n is the smallest integer greater than q. The Riemann–Liouville fractional derivative and integral has multiple applications such as in case of solutions to the equation in the case of multiple systems, and Variable order fractional parameter [7, 26, 29].

To a certain degree, some FDMs for FDEs can be thought of as extensions of the analogous techniques for classical equations [6, 17, 19, 23, 27, 28, 32, 37]. In [33], difference methods for solving boundary value problems for differential equations of fractional order are considered. The third boundary value problem for a fractional order diffusion equation is examined in [22], as are local one-dimensional schemes for fractional order differential diffusion equations with first-kind boundary conditions. The L^1/L^2-formulas are very effective for approximating time/space fractional PDEs and are more stable as$\alpha \to 0^+$, see [3, 5, 16, 25]. Many different techniques are used by the authors in [15], that proposes well established methods for fractional time-space PDE using fractional Laplacian with less regular forcing terms, which is very interesting.

In order to examine the linear time and space fractional advection equation with boundary conditions, we first introduce the finite difference scheme (FDS). Then, we construct our numerical scheme (NS) for the mathematical model. After this, we prove the stability, consistence and convergence of the fractional method. We also give some numerical simulations of the fractional advection equation using the FDM which is plotted for different values of fractional parameters α and ν.

To begin with, let $\Omega =]0, L[$ be a domain of $\mathbb{R}$, where L is any given finite constant. For

$$x \in \Omega,\ t \in]0, T[,\ T > 0,$$

we consider the time-space fractional advection-diffusion equation (TSFADE) of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\nu u}{\partial x^\nu} = f, \quad \alpha, \nu \in]0, 1[, \quad \alpha \neq \nu, \tag{1.1}$$

together with the corresponding boundary and initial conditions

$$\begin{aligned} \frac{\partial u}{\partial x}\Big|_{x=0,\ x=L} &= 0, \\ u(x, t=0) &= u_0(x), \\ u(x, t=T) &= u_1(x). \end{aligned} \tag{1.2}$$

In (1.1)-(1.2), we have $u = u(x, t)$ represents the variable concentration field of one space variable x at time t and which time and space-fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$, $\frac{\partial^\nu u}{\partial x^\nu}$ are given by means of the α and ν order Caputo fractional derivative, here $f = f(x, t)$ is an external forces. The main aim of the paper is to present a fractional model of the evolution of the thermal field in diffusion materials. The mathematical model is formalized as an initial boundary value problem for a partial differential equation containing a fractional derivative with respect to time.

The computational algorithm for solving the problem is based on an analogue of the finite difference scheme to approximate the fractional order derivative together with the fractional Taylor series method (It was named after a person who, as it is believed, made the biggest contribution into the study of the principle). An application program has been developed that allows computer simulation of the model. The operation of the algorithm was verified on a test problems. The results of computational experiments are presented using the example.

The structure of this paper is summarized as follows. Section 1 describes introduction and position of problem for the mathematical model of the time-space fractional advection-diffusion equation (TSFADE) with boundary conditions and initial condition. In Section 2, we apply the finite difference method (FDM) to impend the new dicretize model. In Section 3, we also study the consistence, stability and convergence of the model in different steps. Finally, in the section 4, we finish the paper by given some numerical simulations of the fractional advection equation using the FDM which is plotted for several values of fractional parameters α and ν using MATLAB program.

2. Discretization

Let $[0, L]$ be the domain of interest, and we dicretize the domain first using Taylor's theorem and the Taylor series expansion for an n-times differentiable function. Then, we define

$$x_i = ih, \forall i = \overline{0, N} \ \text{ and } \ t_j = jk, \forall j = \overline{0, M}, \tag{2.1}$$

where k represent the time step size and h represents the space step length. Let us assume that

$$u(x_i, t_j) = u_i^j \ \text{ and } \ f(x_i, t_j) = f_i^j, \forall i = \overline{0, N} \ \text{ and } \ \forall j = \overline{0, M}, \tag{2.2}$$

here u_i^j is the numerical approximation of $u(x_i, t_j)$. Then, the equation (1.1) can be written at nodes x_i and t_j as follows

$$\frac{\partial^\alpha u_i^j}{\partial t^\alpha} - \frac{\partial^\nu u_i^j}{\partial x^\nu} = f_i^j, \tag{2.3}$$

$$\forall x_i \in [0, L], \forall i = \overline{1, N}, \quad \forall t_j \in [0, T], \forall j = \overline{1, M},$$
$$\forall \alpha, \nu \in]0, 1[, \quad \alpha \neq \nu.$$

2.1. Development of the Scheme.

The Euler method uses the first order Taylor series to obtain the integration scheme. It is the easiest numerical integration method but the errors are usually very big so is not frequently used. If we suppose that u is a solution of problem (1.1), then according to the fractional Taylor series for u at t_{j+1} and x_{i+1} it can be written

$$u(x_i, t_{j+1}) = u(x_i, t_j) + \frac{\Delta_1^*}{\Gamma(\alpha+1)} \frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_j) + R_1(x_i, t_{j+1}, t_j, 0), \tag{2.4}$$

$$u(x_{i+1}, t_j) = u(x_i, t_j) + \frac{\Delta_1}{\Gamma(\nu+1)} \frac{\partial^\nu u}{\partial x^\nu}(x_i, t_j) + R_2(x_{i+1}, x_i, t_j, 0), \tag{2.5}$$

such that $\Delta_1^* = k^\alpha[(j+1)^\alpha - j^\alpha]$ and $\Delta_1 = h^\nu[(i+1)^\nu - i^\nu]$, then we find

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta_1^*}\Gamma(\alpha+1), \tag{2.6}$$

$$\left(\frac{\partial^\nu u}{\partial x^\nu}\right)_i^j \approx \frac{u_{i+1}^{j} - u_i^j}{\Delta_1}\Gamma(\nu+1). \tag{2.7}$$

Using (2.6) and (2.7) in (1.1), then we get the following scheme

$$u_i^{j+1} = \left(1 - \frac{\Delta_1^*\Gamma(\nu+1)}{\Delta_1\Gamma(\alpha+1)}\right)u_i^j + \frac{\Delta_1^*\Gamma(\nu+1)}{\Delta_1\Gamma(\alpha+1)}u_{i+1}^j + \frac{\Delta_1^*}{\Gamma(\alpha+1)}f_i^j, \tag{2.8}$$

$$\text{for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1},$$

with the boundary conditions

$$\begin{aligned} u_0^j = u_1^j &= 0 \text{ for all } j = \overline{0, M}, \\ u_i^0 &= u_0(x_i) \text{ for all } i = \overline{0, N}, \\ u_i^M &= u_1(x_i) \text{ for all } i = \overline{0, N}. \end{aligned}$$

The fractional Taylor series improves over traditional methods by accurately incorporating fractional-order derivatives, capturing memory effects inherent in fractional systems. Unlike standard approaches, it ensures higher stability and precision, as demonstrated in the convergence analysis, making it more effective for varied fractional parameters and boundary conditions.

3. Consistence, Stability, and Convergence

Here, we treat the questions of consistence, stability and convergence of the implicit finite difference method.

3.1. **Consistence.** By (2.4)–(2.8), the local truncation error R of the Scheme (2.8) is given by

$$\begin{aligned} R_i^j = & \left[\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_j) - \frac{u_i^{j+1} - u_i^j}{\Delta_1^*}\Gamma(\alpha+1)\right] \\ + & \left[\frac{\partial^\nu u}{\partial x^\nu}(x_i, t_j) - \frac{u_{i+1}^j - u_i^j}{\Delta_1}\Gamma(\nu+1)\right]. \end{aligned} \tag{3.1}$$

We put

$$R_i^j = R_1(x_i, t_{j+1}, t_j, 0) + R_2(x_{i+1}, x_i, t_j, 0), \tag{3.2}$$

$$\text{for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1},$$

then we have

$$R_1(x_i, t_{j+1}, t_j, 0) = k^{2\alpha}\left[\frac{A_1\sigma_1 j^\alpha}{\Gamma^2(\alpha+1)} - \frac{C_1 k^\alpha \sigma_1 j^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{(D_1 - B_1)j^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$\text{for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1},$$

such that

$$A_1 = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x_i, 0),$$

$$B_1 = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x_i, \xi_1),$$

$$C_1 = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x_i, \xi_2),$$

and

$$D_1 = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x_i, \xi_3).$$

The error

$$R_2(x_{i+1}, x_i, t_j, 0),$$

can be written as follows

$$R_2(x_{i+1}, x_i, t_j, 0) = h^{2\nu} \left[\frac{A_2 \sigma_2 i^{\nu}}{\Gamma^2(\nu+1)} - \frac{C_2 h^{\nu} \sigma_2 i^{2\nu}}{\Gamma(\nu+1)\Gamma(2\nu+1)} + \frac{(D_2 - B_2) i^{2\alpha}}{\Gamma(2\nu+1)} \right],$$

$$\text{for all } \ i = \overline{0, N-1} \ \text{ and } \ j = \overline{0, M-1},$$

such that

$$A_2 = \frac{\partial^{2\nu} u}{\partial x^{2\nu}}(0, t_j),$$

$$B_2 = \frac{\partial^{2\nu} u}{\partial x^{2\nu}}(\tau_1, t_j),$$

$$C_2 = \frac{\partial^{2\nu} u}{\partial x^{2\nu}}(\tau_2, t_j),$$

and

$$D_2 = \frac{\partial^{2\nu} u}{\partial x^{2\nu}}(\tau_3, t_j).$$

Now, let

$$\sigma_1 = (j+1)^{\alpha} - j^{\alpha} \ \text{ for all } \ j = \overline{0, M-1}, \tag{3.3}$$

and

$$\sigma_2 = (i+1)^{\nu} - i^{\nu} \ \text{ for all } \ i = \overline{0, N-1}. \tag{3.4}$$

3.2. **Stability of the Difference Scheme.** In this part, we will discuss the stability of the difference scheme (2.8) by the Fourier method. Let U_i^k be the approximate solution of (2.8) and define

$$\rho_i^j = u_i^j - U_i^j \ \text{ for all } \ i = \overline{0, N-1} \ \text{ and } \ j = \overline{0, M-1}. \tag{3.5}$$

So, for all $i = \overline{0, N-1}$ and $j = \overline{0, M-1}$, we easily get the following error equations

$$\rho_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right) \rho_i^j + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \rho_{i+1}^j,$$

such that

$$\rho_i^0 = 0 \text{ for all } \ i = \overline{0, N}.$$

Taking in mind that

$$\left\|\rho^j\right\|_2^2 = \sum_{m=-\infty}^{\infty} \left|\xi_j(m)\right|^2 \text{ for all } j = \overline{0, M}.$$

Now, we suppose that the equations (3.3)–(3.4) has the following form

$$\rho_i^j = \xi_j e^{I\lambda i h} \text{ for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1},$$

where $I = \sqrt{-1}$ and $\lambda = 2\pi m$.
Then we obtain

$$\xi_{j+1} = \left[1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e^{I\lambda h} \right] \xi_j, \text{ for all } j = \overline{0, M-1},$$

where h represents the space step length with the range $h = \frac{L}{N}$. For more details, please see [18], for numerical solutions using implicit schemes and [35], for an analysis about conditional and unconditional stability when using explicit or implicit schemes.

Theorem 3.1. *The scheme* (2.8) *is stable if*

$$\left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e^{I\lambda h} \right| \leq 1 \text{ for all } \alpha, \nu \in]0,1[.$$

Proof. The scheme (2.8) is stable if

$$\|\rho^j\|_2 \leq \|\rho^0\|_2 \text{ for all } j = \overline{0, M}. \tag{3.6}$$

Now, for $j = 0$, we get

$$\|\rho^1\|_2 = |\xi_1| = \left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e^{I\lambda h} \right| |\xi_0|,$$

and for $j = 1$, we get

$$\|\rho^2\|_2 = |\xi_2| = \left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e^{I\lambda h} \right|^2 |\xi_0|,$$

$$\vdots$$

Finally, for $j = M$, we get

$$\|\rho^M\|_2 = |\xi_j| = \left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e^{I\lambda h} \right|^M |\xi_0|.$$

So we obtain

$$\left\|\rho^j\right\|_2 \leq \left\|\rho^0\right\|_2 \text{ for all } j = \overline{0, M}.$$

This completes the proof of Theorem 3.1 which means that the difference scheme (2.8) is stable. □

3.3. **Convergence of the Difference Scheme.** Assume that $u(x_i, t_j)$ is the exact solution of equation at (x_i, t_j), we define

$$e_i^j = u_i^j - u(x_i, t_j) \text{ for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1}.$$

so we get

$$e_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right) e_i^j - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e_{i+1}^j + R_i^j,$$
$$\text{for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1}.$$

Theorem 3.2. *The scheme* (2.8) *converges if and only if*

$$\|e^j\|_\infty \le C^* M(k^{2\alpha} + h^{2\nu}) \text{ for all } j = \overline{0, M} \text{ and for all } \alpha, \nu \in]0, 1[.$$

Proof. We have

$$e_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right) e_i^j + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e_{i+1}^j + R_i^j \Bigg),$$
$$\text{for all } i = \overline{0, N-1} \text{ and } j = \overline{0, M-1},$$

and

$$\|e^j\|_\infty = \max_{0 \le i \le N} |e_i^j|.$$

Then, we obtain

$$\begin{aligned} |e_i^{j+1}| &= \left| \left(1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right) e_i^j + \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} e_{i+1}^j + R_i^j \right| \\ &\le \left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right| |e_i^j| + \left| \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right| |e_{i+1}^j| + |R_i^j|, \end{aligned}$$

which implies

$$\|e^{j+1}\|_\infty \le \left(\left| 1 - \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right| + \left| \frac{\Delta_1^* \Gamma(\nu+1)}{\Delta_1 \Gamma(\alpha+1)} \right| \right) \|e^j\|_\infty + \|R^j\|_\infty.$$

After that, we can have

$$\begin{aligned} \|R^j\|_\infty &\le (k^{2\alpha} + h^{2\nu}) \\ &\times \max_{0 \le i \le N} \left(\left\| \frac{A_1 \sigma_1 j^\alpha}{\Gamma^2(\alpha+1)} - \frac{C_1 k^\alpha \sigma_1 j^{2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{(D_1 - B_1) j^{2\alpha}}{\Gamma(2\alpha+1)} \right\| \right. \\ &\times \left. \left\| \frac{A_2 \sigma_2 i^\nu}{\Gamma^2(\nu+1)} - \frac{C_2 h^\nu \sigma_2 i^{2\nu}}{\Gamma(\nu+1)\Gamma(2\nu+1)} + \frac{(D_2 - B_2) i^{2\alpha}}{\Gamma(2\nu+1)} \right\| \right) \\ &= C^*(k^{2\alpha} + h^{2\nu}). \end{aligned}$$

For $j = 0$, we have

$$\|e^1\|_\infty \le C^*(k^{2\alpha} + h^\nu),$$

and for $j = 1$, we have

$$\|e^2\|_\infty \le 2C^*(k^{2\alpha} + h^\nu).$$

$$\vdots$$

Finally, for $j = M$, we get

$$\|e^j\|_\infty \leq C^{*M}(k^{2\alpha} + h^{2\nu}).$$

This completes the proof of Theorem (3.2) which means that the difference scheme (2.8) converge. □

4. Test Problem

In this section, we present a numerical comparison between differential values of different values of α and ν for solving linear time and space fractional advection equation. Two examples are given. The numerical results reveal that differential finite difference method is very efficient and accurate.

Example 4.1. Consider the linear time and space fractional advection equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\nu u}{\partial x^\nu} = x^2 + t^3, \quad 0 < x < 4, \quad 0 < t < 4, \tag{4.1}$$

together with the corresponding boundary and initial conditions

$$\frac{\partial u}{\partial x}\bigg|_{x=0,\ x=4} = 0, \quad 0 < t < 4, \tag{4.2}$$

$$u(x,0) = x, \quad x \in]0,4[, \tag{4.3}$$

$$u(x,4) = \frac{\pi}{x}, \quad x \in\in]0,4[. \tag{4.4}$$

We test our method by taking

$$N = M = 50, 100, 150, 200; \alpha = 0.80, \nu = 0.85, f(x,t) = x^2 + t^3, x \in [0,4],$$

and $t \in [0,4]$. Also, we test our method by taking

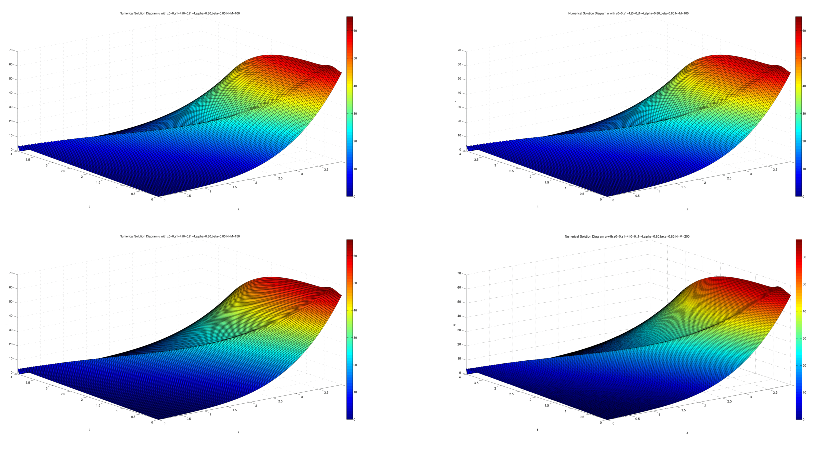

FIGURE 1. Numerical Simulation of the linear time and space fractional advection equation with $z_0 = 0, z_1 = 4, t_0 = 0, t_1 = 4, \alpha = 0.80, \nu = 0.85, N = M = 50, 100, 150, 200, f(x,t) = x^2 + t^3, x \in [0,4]$ and $t \in [0,4]$.

$$N = M = 50, 100, 150, 200; \alpha = 0.88, \nu = 0.89, f(x,t) = x^2 + t^3, x \in [0,4],$$

and $t \in [0,4]$.

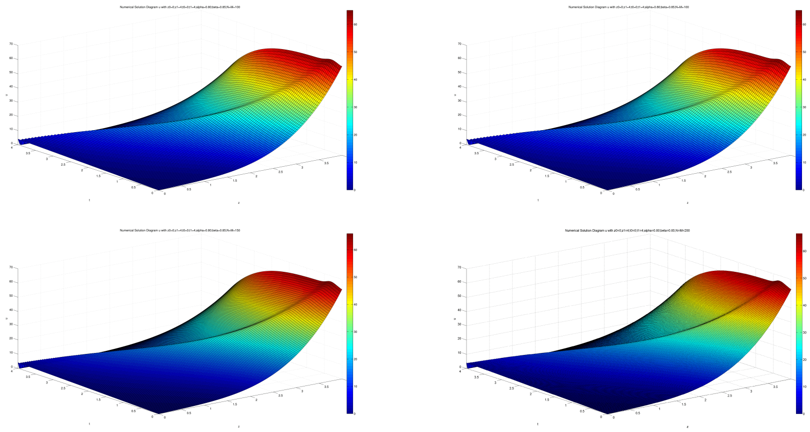

FIGURE 2. Numerical Simulation of the linear time and space fractional advection equation with $z_0 = 0$; $z_1 = 4$; $t_0 = 0$; $t_1 = 4$; α=0.88;ν=0.89; $N = M = 50, 100, 150, 200$; $f(x,t) = x^2 + t^3, x \in [0,4]$ and $t \in [0,4]$.

Example 4.2. Consider the linear time and space fractional advection equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\nu u}{\partial x^\nu} = x^2.t^3, \quad 0 < x < 4, \quad 0 < t < 4, \tag{4.5}$$

together with the corresponding boundary and initial conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,\ x=4} = 0, \quad 0 < t < 4, \tag{4.6}$$

$$u(x,0) = sin(x).cos(x), \quad x \in]0,4[, \tag{4.7}$$

$$u(x,4) = 1 + sin(x), \quad x \in]0,4[. \tag{4.8}$$

We test our method by taking

$$N = M = 50, 100, 150, 200; \alpha = 0.90, \nu = 0.95, f(x,t) = x^2.t^3, x \in [0,4],$$

and $t \in [0,4]$.

Finally, we test our method by taking

$$N = M = 50, 100, 150, 200; \alpha = 0.96, \nu = 0.98, f(x,t) = x^2.t^3, x \in [0,4],$$

and $t \in [0,4]$.

Conclusion and comments

In this paper, a finite difference scheme (FDS) has been proposed. In order to study the linear time and space fractional advection equation, we have constructed a numerical scheme associated to the continuous problem. The stability, consistence and convergence of our method are presented (many applications of such fractional maps are expected to be appear later). Besides, numerical experiments are also presented, which verify the theoretical analysis. If the problem is nonlinear, the regularity on coefficients would be acceptable for our scheme

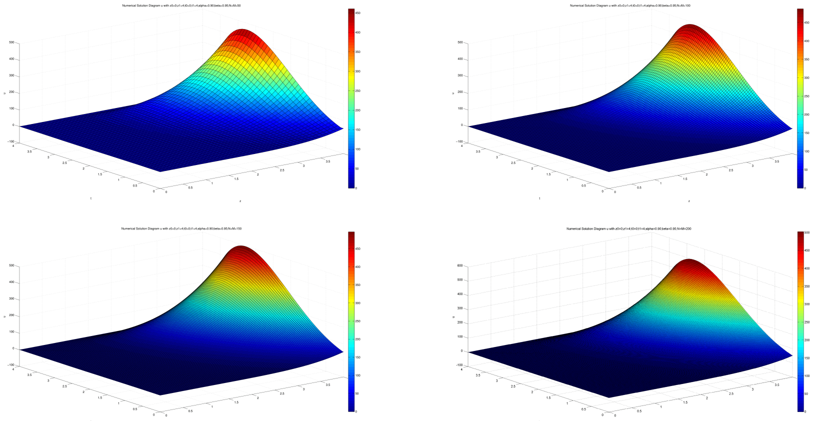

FIGURE 3. Numerical Simulation of the linear time and space fractional advection equation with $z_0 = 0$; $z_1 = 4$; $t_0 = 0$; $t_1 = 4$; α=0.90;ν=0.95; $N = M = 50, 100, 150, 200$; $f(x,t) = x^2.t^3, x \in [0,4]$ and $t \in [0,4]$.

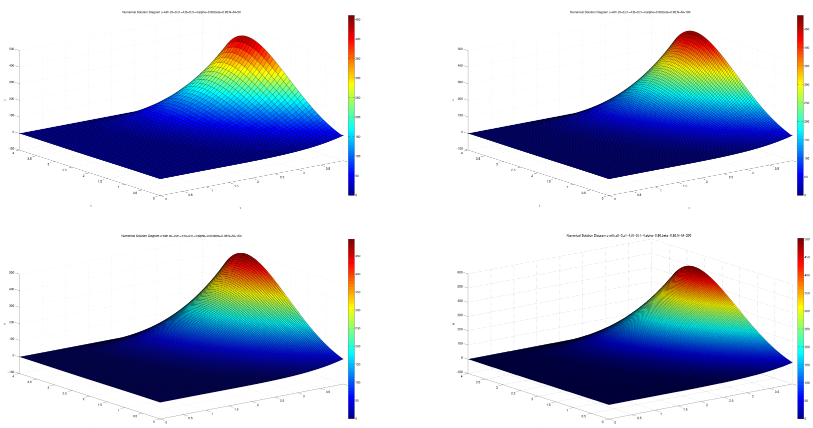

FIGURE 4. Numerical Simulation of the linear time and space fractional advection equation with $z_0 = 0$; $z_1 = 4$; $t_0 = 0$; $t_1 = 4$; α=0.96;ν=0.98; $N = M = 50, 100, 150, 200$; $f(x,t) = x^2.t^3, x \in [0,4]$ and $t \in [0,4]$.

to retain its numerical properties, it will be addressed in our next paper working on. For solving time-fractional diffusion or advection-diffusion equations, we mention that there are some new related works, such as [36], in which the idea of convergence analysis can be referenced. In our next future works, we will consider the error estimation and corresponding convergence rate. Future work will focus on extending the proposed finite difference scheme to nonlinear time-space fractional advection-diffusion equations. Additionally, efforts will be made to improve convergence rates by incorporating higher-order numerical approximations and adaptive mesh refinement techniques.

References

1. Abdelkader, L., Iqbal, J., Zoubir, D., Ahmed, A., Mahdi, R., & Shawkat, A. (2024). Fractional calculus in beam deflection: Analyzing nonlinear systems with Caputo and conformable derivatives. AIMS Mathematics, 9(8), 21609–21627. https://doi.org/10.3934/math.20241050
2. Ali, U., Abdullah, F. A., & Ismail, A. I. (2017). Crank-Nicolson finite difference method for two-dimensional fractional sub-diffusion equation. Journal of Interpolation and Approximation in Scientific Computing, 2, 18–29. https://doi.org/10.5899/2017/jiasc-00117
3. Alikhanov, A. A. (2015). A new difference scheme for the time fractional diffusion equation. Journal of Computational Physics, 280, 424–438. https://doi.org/10.1016/j.jcp.2014.09.031
4. Almatroud, O. A., Hioual, A., Ouannas, A., & Batiha, I. M. (2024). Asymptotic stability results of generalized discrete time reaction diffusion system applied to Lengyel-Epstein and Dagn Harrison models. Computers & Mathematics with Applications, 170, 25–32. https://doi.org/10.1016/j.camwa.2024.05.012
5. Aniley, W. T., & Duressa, G. F. (2024). Nonstandard finite difference method for time-fractional singularly perturbed convection-diffusion problems with a delay in time. Results in Applied Mathematics, 21, 100432. https://doi.org/10.1016/j.rinam.2023.100432
6. Azzaoui, B., Brahim, B., & Zennir, Kh. (2023). Positive solutions for integral nonlinear boundary value problem in fractional Sobolev spaces. Mathematical Methods in the Applied Sciences, 46(3), 3115–3131. https://doi.org/10.1002/mma.8945
7. Baleanu, D., Wu, G. C., Bai, Y. R., & Chen, F. L. (2017). Stability analysis of Caputo-like discrete fractional systems. Communications in Nonlinear Science and Numerical Simulation, 48, 520–530. https://doi.org/10.1016/j.cnsns.2017.01.010
8. Batiha, I. M., Barrouk, N., Ouannas, A., & Farah, A. (2023). A study on invariant regions, existence and uniqueness of the global solution for tridiagonal reaction-diffusion systems. Journal of Applied Mathematics and Informatics, 41(4), 893–906. https://doi.org/10.14317/jami.2023.893
9. Batiha, I. M., Jebril, I. H., Anakira, N., Al-Nana, A. A., Batyha, R., & Momani, S. (2024). Two-dimensional fractional wave equation via a new numerical approach. International Journal of Innovative Computing, Information & Control, 20(4), 1045–1059. https://doi.org/10.24507/ijicic.20.04.1045
10. Batiha, I. M., Oudetallah, J., Ouannas, A., Al-Nana, A. A., & Jebril, I. H. (2021). Tuning the fractional-order PID-controller for blood glucose level of diabetic patients. International Journal of Advances in Soft Computing and its Applications, 13(2), 1–10. https://www.i-csrs.org/Volumes/ijasca/2021.2.1.pdf
11. Batiha, I. M., Njadat, S. A., Batyha, R. M., Zraiqat, A., Dababneh, A., & Momani, S. (2022). Design fractional-order PID controllers for single-joint robot arm model. International Journal of Advances in Soft Computing and its Applications, 14(2), 96–114.
12. Batiha, I. M., Ogilat, O., Bendib, I., Ouannas, A., Jebril, I. H., & Anakira, N. (2024). Finite-time dynamics of the fractional-order epidemic model: Stability, synchronization, and simulations. Chaos, Solitons & Fractals: X, 13, 100118. https://doi.org/10.1016/j.csfx.2024.100118
13. Batiha, I. M., Ogilat, O., Hioual, A., Ouannas, A., Anakira, N., Amourah, A. A., & Momani, S. (2024). On discrete FitzHugh–Nagumo reaction–diffusion model: Stability and simulations. Partial Differential Equations in Applied Mathematics, 11, 100870. https://doi.org/10.1016/j.padiff.2023.100870
14. Benbrahim, A., Batiha, I. M., Jebril, I. H., Bourobta, A., Oussaeif, T.-E., & Alkhazaleh, S. (2024). A priori predictions for a weak solution to time-fractional nonlinear reaction-diffusion equations incorporating an integral condition. Nonlinear Dynamics and Systems Theory, 24(5), 431–441. https://www.e-ndst.kiev.ua/v24n5.htm

15. Difonzo, F. V., & Garrappa, R. (2023). A numerical procedure for fractional-time-space differential equations with the spectral fractional Laplacian. In: Cardone, A., Donatelli, M., Durastante, F., Garrappa, R., Mazza, M., & Popolizio, M. (Eds.), Fractional Differential Equations (pp. 1–14). Springer, Singapore. https://doi.org/10.1007/978-3-031-12243-6
16. Gao, G.-H., Sun, Z.-Z., & Zhang, H.-W. (2014). A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. Journal of Computational Physics, 259, 33–50. https://doi.org/10.1016/j.jcp.2013.11.016
17. Huang, Q., Huang, G., & Zhan, H. (2008). A finite element solution for the fractional advection-dispersion equation. Advances in Water Resources, 31(12), 1578–1589. https://doi.org/10.1016/j.advwatres.2008.07.010
18. Jannelli, A., Ruggieri, M., & Paola, M. (2019). Analytical and numerical solutions of time and space fractional advection-diffusion-reaction equation. Communications in Nonlinear Science and Numerical Simulation, 70, 89–101. https://doi.org/10.1016/j.cnsns.2018.10.009
19. Kelley, W. G., & Peterson, A. C. (2001). Difference Equations: An Introduction with Applications. Academic Press.
20. Kochubej, A. N. (1990). Diffusion of fractional order. Differential Equations, 26(4), 485–492. https://doi.org/10.1007/BF01456257
21. Korchagina, A. N. (2014). Application of fractional order derivatives for solving problems of continuum mechanics. Izvestiya of Altai State University Journal, 1(81), 65–67. https://doi.org/10.14258/izvasu(2014)1.1-81-12
22. Lafisheva, M. M., & Shkhanukov-Lafishev, M. Kh. (2008). Local one-dimensional scheme for the fractional order diffusion equation. Computational Mathematics and Mathematical Physics, 48(10), 1878–1887. https://doi.org/10.1134/S0965542508100116
23. Mickens, R. E. (1994). Nonstandard Finite Difference Models of Differential Equations. World Scientific. https://doi.org/10.1142/2354
24. Mickens, R. E. (2003). A nonstandard finite difference scheme for a Fisher PDE having nonlinear diffusion. Computers & Mathematics with Applications, 45, 429–436. https://doi.org/10.1016/S0898-1221(03)00046-1
25. Mirzaee, F., Sayevand, K., & Rezaei, S. (2021). Finite difference and spline approximation for solving fractional stochastic advection-diffusion equation. Iranian Journal of Science and Technology. Transactions of Science, 617, 1–12. https://doi.org/10.1007/s40995-021-00647-5
26. Miller, K. S., & Ross, B. (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons.
27. Naimi, A., Brahim, B., & Zennir, Kh. (2022). Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions. Tamkang Journal of Mathematics, 53(3), 239–257. https://doi.org/10.5556/j.tjm.2022.53.3.239
28. Naimi, A., Brahim, B., & Zennir, Kh. (2022). Existence and stability results of a nonlinear fractional integro-differential equation with integral boundary conditions. Kragujevac Journal of Mathematics, 46(5), 685–699. https://doi.org/10.46793/KgJMat2205.685N
29. Oldham, K. B., & Spanier, J. (1974). The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press.
30. Podlubny, I. (1999). Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. San Diego: Academic Press.
31. Sierociuk, D., Dzielinski, A., Sarwas, G., Petras, I., Podlubny, I., & Skovranek, T. (2013). Modelling heat transfer in heterogeneous media using fractional calculus. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 371(1990), 20120146. https://doi.org/10.1098/rsta.2012.0146

32. Shabbir, M. S., Din, Q., Safeer, M., Khan, M. A., & Ahmad, K. (2019). A dynamically consistent nonstandard finite difference scheme for a predator-prey model. Advances in Difference Equations, 2019, Article 381. https://doi.org/10.1186/s13662-019-2305-0
33. Taukenova, F. I., & Shkhanukov-Lafishev, M. Kh. (2006). Difference methods for solving boundary value problems for differential equations of fractional order. Computational Mathematics and Mathematical Physics, 46(10), 1871–1881. https://doi.org/10.1134/S0965542506100100
34. Uchaikin, V. V., & Sibatov, R. T. (2007). Fractional differential kinetics of dispersive transport as the consequence of its self-similarity. JETP Letters, 86(8), 512–516. https://doi.org/10.1134/S0021364007200110
35. Zhang, F., Gao, X., & Xie, Z. (2019). Difference numerical solutions for time-space fractional advection-diffusion equation. Boundary Value Problems, 2019, Article 14. https://doi.org/10.1186/s13661-019-1125-y
36. Zhang, X., Feng, Y., Luo, Z., & Liu, J. (2025). A spatial sixth-order numerical scheme for solving fractional partial differential equations. Applied Mathematics Letters, 159, 109265. https://doi.org/10.1016/j.aml.2025.109265
37. Zibaei, S., Zeinadini, M., & Namjoo, M. (2016). Numerical solutions of Burgers-Huxley equation by exact finite difference and NSFD schemes. Journal of Difference Equations and Applications, 22(8), 1098–1113. https://doi.org/10.1080/10236198.2016.1188670
38. Zecova, M., & Terpak, J. (2015). Heat conduction modeling by using fractional-order derivatives. Applied Mathematics and Computation, 257, 365–373. https://doi.org/10.1016/j.amc.2015.01.073

[1]Department of Mathematics, University Constantine 1, Frères Mentouri, Constantine, Algeria
Email address: allaoua.boudjedour@umc.edu.dz

[2]Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, Jordan
Email address: i.batiha@zuj.edu.jo

[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE

[4]Department of Mathematics, University Constantine 1, Frères Mentouri, Constantine, Algeria
Email address: selma.boucetta@doc.umc.edu.dz

[5]Department of Mathematics, University Constantine 1, Frères Mentouri, Constantine, Algeria
Email address: dalah.mohamed@umc.edu.dz

[6]Department of Mathematics, Qassim University, Qassim, Saudi Arabia
Email address: k.zennir@qu.edu.sa

[7]Department of Mathematics and Computer Science, University of Oum EL-Bouaghi, Oum El Bouaghi, Algeria
Email address: ouannas.adel@univ-oeb.dz