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ON λ_t-ISOTYPE SUBMODULES OF $QTAG$-MODULES

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Abstract. Let $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ be an increasing sequence of ordinals. We define the class of λ_t-isotype submodules and study their crucial properties. The paper discusses various contexts in which every λ_t-isotype submodule is an h-neat, an h-pure, an isotype submodule, a direct summand, an absolute direct summand, respectively. Moreover, some comprehensive characterizations in this direction are also established.

1. Introduction and Fundamentals

The theory of modules is concerned with all questions directly or indirectly related to abelian group theory. In 1976, Singh [20] introduced a new class of modules that are now known as torsion abelian groups-like-modules or TAG-modules. This kind of TAG-module has been studied extensively. Many ideas for groups, such as height, divisibility, purity, etc., play an important role in the research of generalizations for these modules. Using these concepts, many authors introduced and studied various types of torsion abelian groups-like-modules (see, for example, [3, 16]). Many intriguing findings have emerged, but there is still much to discover. However, TAG-modules are especially fascinating.

If a module M fulfills the following two criteria related to uniserial modules when placed over an arbitrary (associative, unitary) ring R, it is considered to be a TAG-module.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided that the composition length $d(U_1/N) \leq d(U_2/\phi(N))$ holds.

It has been proven that the modules satisfying the first criterion can be referred to as quasi-torsion abelian groups-like modules, or $QTAG$-modules (see [21]), and that the second criterion can be inferred from the first with only a few additional hypotheses. Several authors have examined various concepts and properties of $QTAG$-modules; they have also characterized the various submodules of these modules, advancing the theory of these modules by presenting a number of ideas

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(see, for instance, [10, 11, 12, 18, 19] and the references therein). Numerous developments in the theory of torsion abelian groups are similar to earlier discoveries, which is not surprising. The current work adds to our understanding of the structure of $QTAG$-modules and transfers some concepts from the theory of torsion abelian groups to the structure of $QTAG$-modules.

We begin with some terminology. Let all the rings R discussed here be associative with unity $(1 \neq 0)$. By the term "module" we mean a unital $QTAG$-module. A module M over a ring R is known as the uniserial module if its submodules are totally ordered by inclusion, that is, for any two submodules N and L of M, either $N \subseteq L$ or $L \subseteq N$. Likewise, we will state that M is uniform if two of its non-zero submodules intersect in a non-zero fashion. In particular, an element u in a module M is called the uniform element if uR is a non-zero uniform (hence uniserial) module. The decomposition length of any module M is indicated by the notation $d(M)$. In addition, if $u \in M$, then $e(u)$ is known as the exponent of u and $e(u) = d(uR)$. As is customary, we use $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$ to determine the height of u in M.

After that, we review the following ideas. The submodule of M generated by the elements of height at least t is shown by $H_t(M) = \{u \in M |\ H_M(u) \geq t\}$ for each non-negative integer t. The submodule of a module M that has elements of infinite height is always symbolized by M^1. The topology of M, which admits as a base of neighborhoods of zero, is known as the h-topology. This topology has the submodules $H_t(M)$ with $t = 0, 1, \ldots, \infty$. Thus, we shall say that a submodule $N \subseteq M$ is the closure in M if $\overline{N} = \cap_{t=0}^{\infty}(N + H_t(M))$, and closed in terms of the h-topology if $\overline{N} = N$. By closed module M, we mean those modules that do not have any element of infinite height and have a limit in M for every Cauchy sequence. Moreover, the socle of M, or the sum of its simple submodules, is represented by $Soc(M)$. For any $t \geq 0$, $Soc^t(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{t+1}(M)/Soc^t(M) = Soc(M/Soc^t(M))$.

We also add some fundamental definitions from [14, 15] (see [12] too). The module M is named elementary if $H^{t-1}(0) \leq H_1(M)$ but $H^t(0) \nsubseteq H_1(M)$, and $H^t(0) = M$, for every $t \geq 0$. The module M is said to be bounded, if for any uniform element $u \in M$, there exists an integer t such that $H_M(u) \leq t$. Moreover, M is h-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$. Therefore, if module M has no h-divisible submodules, we say that it is h-reduced. A submodule N of M is h-pure in M if $N \cap H_t(M) = H_t(N)$ for every integer $t \geq 0$. In particular, if $t = 1$, N is called h-neat in M. A submodule $N \subseteq M$ is said to be high if it is a complement of M^1, that is, $M = N \oplus M^1$. A submodule $N \subseteq M$ is termed a basic submodule of M, if N is a direct sum of uniserial modules, N is an h-pure submodule of M and M/N is h-divisible. A submodule $N \subseteq M$ is named essential in M if $N \cap K = 0$ for every non-zero submodule K of M and M is known as the essential extension of N.

It is intriguing to observe that almost all the findings for the TAG-modules are also applicable to the $QTAG$-modules [17]. The many significant results of [1] are clearly the inspiration for our current work. For some other interesting generalizations of the topic mentioned here, the reader can see in [5, 6, 7]. As such, it can be viewed as an extension of the work contained in [2, 13]. The notation and terminology used in this paper are standard and can be found in the books [8, 9]; for specific information, we refer the readers to [4, 22]. As usual, U_k denotes a direct sum of uniserial modules of the same exponent k for some $k \in \mathbb{N}$.

2. Main results

A subclass of torsion abelian groups that plays a prominent role in $QTAG$-module theory is the one consisting of all α-pureness, where α is an ordinal, defined as: A submodule N of M can be described as α-pure if there is an ordinal α that depends on N and $H_\gamma(M) \cap N = H_\gamma(N)$ for all ordinals γ. This definition going from [17] is very important and it is a generalization of the ordinary purity (i.e., h-purity) in the classical theory of $QTAG$-modules. In addition, a submodule N of M is termed isotype when it is α-pure for each ordinal α. It is convenient to consider the class of extended ordinal numbers, which is obtain by adjoining the element ∞ to the ordinal numbers as the last element (i.e., $\alpha < \infty$ for every ordinal α). If we let $H_\infty(M) = \cap_{\alpha<\infty} H_\alpha(M)$, then N is t-isotype in M provided that $H_\alpha(M) \cap N = H_\alpha(N)$ for every $t \in \mathbb{N}$ and $\alpha < \infty$. Similarly, N is λ-isotype in M if $H_\gamma(M) \cap N = H_\gamma(N)$ for every ordinal $\gamma \leq \lambda$.

Our purpose here is to introduce a new concept of the above-mentioned machinery. Namely, we state the following definition of our main term.

Definition 2.1. Let $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ be an increasing sequence of ordinals and symbol ∞; that is, for each t, either λ_t is an ordinal or $\lambda_t < \infty$. With each such sequence λ we associate the submodule N of the $QTAG$-module M such that $H_\gamma(M) \cap N = H_\gamma(N)$ for every ordinal $\gamma \leq \lambda_t$. This submodule is known as the λ_t-isotype submodule.

According to the discussion above, the consequences are immediate.
$(2a)$ If $\lambda = (0, 0, \dots)$, $\lambda = (1, 1, \dots)$, $\lambda = (\omega, \omega, \dots)$, $\lambda = (\infty, \infty, \dots)$, then λ_t-isotype submodules of M are precisely submodules, h-neat submodules, h-pure submodules, isotype submodules, respectively.
$(2b)$ If $\lambda = (\lambda_1, \lambda_2, \dots)$, $\sigma = (\sigma_1, \sigma_2, \dots)$ and $\lambda \leq \sigma$ (i.e., $\lambda_t \leq \sigma_t$ for each $t \in \mathbb{N}$) then every σ_t-isotype submodule of M is λ_t-isotype in M.
$(2c)$ A submodule N of M is λ_t-isotype in M if and only if $Soc^t(N)$ is λ_t-isotype in $Soc^t(M)$ for every $t \in \mathbb{N}$.
$(2d)$ If $u \in M$ is any uniform element and $e(u) = t$, where $t \in \mathbb{N}$ and $t_1 \leq t$ for some integer t_1, then $\langle u \rangle$ is t_1-isotype in $M \Leftrightarrow H_M(H_{t_1-1}(uR)) = t_1 - 1$. Moreover, $\langle u \rangle$ is h-pure (isotype) in $M \Leftrightarrow H_M(H_{t-1}(uR)) = t - 1$.

The following affirmation extends the corresponding one from [12].

Lemma 2.2. *Let t be a natural number and M a $QTAG$-module with an h-neat submodule N such that $Soc^t(M)$ is h-divisible, then N is isotype in M.*

Proof. Note that

$$Soc^t(N) \cap H_1(Soc^t(M)) = H_1(Soc^t(N)) = Soc^t(N),$$

for some t. Thus, we write $N = Soc^t(N) + N_1$ and $M = Soc^t(M) + M_1$ where $N_1 \subset M_1$. Now suppose that $\overline{M_1}$ is the closure of M_1 and $\overline{N_1}$ is the closure of N_1 in M_1; since $(\overline{M_1/N_1}) = 0$, the result follows from [12]. We are done. □

We continue with another straightforward observation.

Lemma 2.3. *Suppose that M is a QTAG-module and $t_1 \in \mathbb{N}$. If every t_1-isotype submodule of M is an h-pure submodule of M, then either $M = M_1 \oplus M_2$, where M_1 is h-divisible and $H_{t_1-1}(M_2) = 0$, or $H_{t_1-1}(M) = U_k \oplus U_{k+1}$ for some $k \in \mathbb{N}$.*

Proof. Let $M = M_1 \oplus M_2$, where M_1 is h-divisible and M_2 is h-reduced. If M_1 is non-zero, then we write $M_2 = \langle x\rangle \oplus M_3$, where $e(x) = t$ and $t \geq t_1$ for some $t > 0$. Now, let $y \in M_1$ be an element of exponent $t+1$. Then by point $(2d)$, there exists a t_1-isotype submodule $\langle x+y\rangle$ of M, which is not h-pure in M. This is contradictory, and so $H_{t_1-1}(M_2) = 0$.

For the second part, we can apply the same concept. Suppose that M is h-reduced and let $M = \langle x\rangle \oplus \langle y\rangle \oplus M_4$, where $e(x) = t$, $e(y) = t_2$ and $t_2 - 2 \geq t \geq t_1$. And again by point $(2d)$, the submodule $\langle x + y'\rangle$ is t_1-isotype in M such that $d(yR/y'R) = 1$. Clearly, $\langle x + y'\rangle$ is not h-pure in M where $d(yR/y'R) = 1$, this is a contradiction. Certainly, if M_2 is a basic submodule of M, then $M = M_2 = S_1 \oplus \dots S_{t_1-1} \oplus S_{t_2} \oplus S_{t_2+1}$, where $t_2 \geq t_1$, and hence $H_{t_1-1}(M) = U_k \oplus U_{k+1}$ where $k = t_2 - t_1 + 1$. □

The following improves (Lemma 2.2 of [12]) to the new framework.

Lemma 2.4. *Let M be a QTAG-module and $\alpha < \beta$ ordinals. If $Soc^t(H_\beta(M))$ is not essential in $Soc^t(H_\alpha(M))$ and either $Soc^t(H_{\beta+1}(M))$ is non-zero or $H_\beta(M)$ is not a closed module for some natural number t, then there is a submodule N of M such that $H_\gamma(M) \cap N = H_\gamma(N)$ for every ordinal $\gamma \leq \alpha + 1$ and $H_{\beta+1}(M) \cap N \neq H_{\beta+1}(N)$.*

Proof. By Lemma 2.2 of [12], there is a non-zero uniform element $x \in Soc^t(H_\alpha(M))$ such that $\langle x\rangle \cap Soc^t(H_\beta(M)) = 0$ for some t. Let $y \in H_\beta(M)$ such that either $0 \neq y' \in Soc^t(M)$ or $e(y) = \infty$ with $d(yR/y'R) = 1$. Write $K = \langle Soc(H_\beta(M)), y', x+y\rangle$. It is easy to see that $\langle x\rangle \cap K = 0$. Let N be an $\langle x\rangle$-high submodule of M containing K. Thus, [17]'s result is applicable to deduce that $H_\gamma(M) \cap N = H_\gamma(N)$ for every ordinal $\gamma \leq \alpha + 1$. Since $Soc(H_\beta(M)) \subset N$, we can infer that $H_{\beta+1}(M) \cap N \neq H_{\beta+1}(N)$, we have our result. □

We turn now to our very satisfactory result.

Theorem 2.5. *If $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ is a sequence of ordinals and M is a QTAG-module. Then the next two points are equivalent.*
(i) Every λ_t-isotype submodule of M is h-pure in M.
(ii) For every $t \in \mathbb{N}$, if $\lambda_t < \omega$ then either $Soc^t(M) = M_1 \oplus M_2$ where M_1 is h-divisible and $H_{\lambda_t-1}(M_2) = 0$, or M is a closed module and $Soc^t(M)$ is elementary or M is a closed module and $H_{\lambda_t-1}(Soc^t(M)) = U_k \oplus U_{k+1}$ for some $k \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$. For each $t \in \mathbb{N}$, every λ_t-isotype submodule of $Soc^t(M)$ is λ_t-isotype in M and hence h-pure in $Soc^t(M)$. Furthermore, by utilizing Lemma

2.3, we get that $Soc^t(M) = M_1 \oplus M_2$; note that this automatically forces M_1 to be h-divisible, and $H_{\lambda_t - 1}(M_2) = 0$. Now, assume that M is not a closed module and $\lambda_t = 0$ for some $t \in \mathbb{N}$. Let $x \in M$ be any uniform element such that $e(x) = \infty$, then $x \notin \langle x', Soc^t(M)\rangle$ with $d(xR/x'R) = 1$, so there is a submodule N of M maximal with respect to the properties $x \notin N, \langle x', Soc^t(M)\rangle \subset N$ is λ_t-isotype in M such that $d(xR/x'R) = 1$. Since $N \cap H_t(M) = H_t(N)$ for some $t \in \mathbb{N}$, there exists $y \in N$ such that $d(xR/yR) = 1$. Thus, $x - y \in Soc^t(M) \subset N$, a contradiction. Finally, if M is not closed, $\lambda_t < \omega$ for some $t \in \mathbb{N}$ and $H_{\lambda_t - 1}(Soc^t(M)) = U_k \oplus U_{k+1} \neq 0$ then $H_{\lambda_t + k}(Soc^t(M))$ is not essential in $H_{\lambda_t - 1}(Soc^t(M))$ and $H_{\lambda_t + k}(M)$ is not closed in virtue of Lemma 2.4, which is absurd, so we get the contradiction.
$(ii) \Rightarrow (i)$. Let $N \subseteq M$ be a λ_t-isotype submodule and $t \in \mathbb{N}$. Letting $\gamma = \lambda_t$. In case that $\gamma \geq \omega$, we are able to observe that $N \cap H_t(M) = H_t(N)$ for some $t \in \mathbb{N}$. For the second case where $\gamma = 0$, it is simple to notice that M is closed and $Soc^t(M)$ is elementary for some t. Write $M = Soc^t(M) \oplus M_3$ and $N = Soc^t(N) \oplus N_1$. Then, for any natural number n, it follows that

$$H_n(N) = N_1 = N \cap M_3 = N \cap H_n(M),$$

i.e., N is h-pure in M. Let $0 < \gamma < \omega$. Supposing $H_{\gamma-1}(Soc^t(M)) = U_k \oplus U_{k+1}$ and M is closed, it follows directly from point $(2a)$ that

$$\begin{aligned} H_1(H_{\gamma-1}(Soc^t(N))) &= Soc^t(N) \cap H_1(H_{\gamma-1}(Soc^t(M))), \\ &= H_{\gamma-1}(Soc^t(N)) \cap H_1(H_{\gamma-1}(Soc^t(M))), \end{aligned}$$

i.e., $H_{\gamma-1}(Soc^t(N))$ is h-neat in $H_{\gamma-1}(Soc^t(M))$. Henceforth, in view of [15], we see that

$$H_{\gamma-1}(Soc^t(N)) \cap H_t(H_{\gamma-1}(Soc^t(M))) = H_t(H_{\gamma-1}(Soc^t(N)))$$

and $Soc^t(N) \cap H_t(Soc^t(M)) = H_t(Soc^t(N))$. Consequently, $N \cap H_t(M) = H_t(N)$ for some t. Now, suppose that $Soc^t(M) = M_1 + M_2$, where M_1 is h-divisible and $H_{\gamma-1}(M_2) = 0$. Then,

$$\begin{aligned} H_1(H_{\gamma-1}(N)) &= N \cap H_1(H_{\gamma-1}(M)), \\ &= H_{\gamma-1}(N) \cap H_1(H_{\gamma-1}(M)). \end{aligned}$$

Since $H_{\gamma-1}(Soc^t(M))$ is h-divisible, it therefore follows from Lemma 2.2 that

$$H_{\gamma-1}(N) \cap H_t(H_{\gamma-1}(M)) = H_t(H_{\gamma-1}(N)).$$

Consequently, $N \cap H_t(M) = H_t(N)$, as expected. □

Regarding the above theorem, we obtain the following result.

Theorem 2.6. *If $\lambda = (\lambda_t)_{t \in \mathbb{N}}$ is a sequence of ordinals and M is a QTAG-module. Then, every λ_t-isotype submodule of M is a direct summand of M if and only if all of the next points are fulfilled.*
(i) $M = M_1 \oplus M_2 \oplus M_3$, where M_1 is closed, M_3 is h-reduced and $H_1(M_2) = M_2$;
(ii) if $\lambda_t < \omega$ then either $H_{\lambda_t - 1}(Soc^t(M_1)) = 0$ or M is closed and $Soc^t(M)$ is elementary or M is closed and $H_{\lambda_t - 1}(Soc^t(M)) = U_k \oplus U_{k+1}$ for some $k \in \mathbb{N}$;
(iii) if $\omega \leq \lambda_t$ then $Soc^t(M_1)$ is bounded.

Proof. Concerning the necessity, if every λ_t-isotype submodule of M is a direct summand of M. Then, every isotype submodule of M is a direct summand of M, and it follows that every λ_t-isotype submodule of M is h-pure in M. We furthermore appeal to Theorem 2.1 of [12] and Theorem 2.5 to obtain the desired claim.

To treat the sufficiency, assuming that the conditions are fulfilled. By the usage of Theorem 2.5 listed above, we see that every λ_t-isotype submodule of M is h-pure in M, and consequently, every h-pure submodule of M is a direct summand of M, as required. □

The next assertion demonstrates a slight extension of Theorem 2.6.

Corollary 2.7. *If $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ is a sequence of ordinals and M is a $QTAG$-module. Then, every λ_t-isotype submodule of M is an absolute direct summand of M if and only if M satisfies one of the next two points.*
(i) M is a closed module and for every $t \in \mathbb{N}$,
(a) if $\lambda_t = 0$ then $Soc^t(M)$ is elementary,
(b) if $\lambda_t > 0$ then either $Soc^t(M)$ is h-divisible or $Soc^t(M) = U_k$ for some $k \in \mathbb{N}$.
(ii) $\lambda_t \neq 0$ for every $t \in \mathbb{N}$ and either $M = H_1(M)$ or $M = M_1 + \overline{M}$ where M_1 is h-reduced and $H_1(\overline{M}) = \overline{M}$.

Proof. Since, as observed before, every λ_t-isotype submodule of M is an absolute direct summand of M if and only if every λ_t-isotype submodule of M is a direct summand of M, and hence every direct summand of M is an absolute direct summand of M. Thus, appealing to Theorem 2.6, we get the desired claim. □

We now briefly discuss one special case in which an h-neatness result occurs.

Theorem 2.8. *If $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ is a sequence of ordinals and M is a $QTAG$-module. Then the next two points are equivalent.*
(i) Every λ_t-isotype submodule of M is h-neat in M.
(ii) If $\lambda_t = 0$ for some $t \in \mathbb{N}$ then $Soc^t(M)$ is elementary and M is a closed module.

Proof. $(i) \Rightarrow (ii)$. For each $t \in \mathbb{N}$, every λ_t-isotype submodule of $Soc^t(M)$ is λ_t-isotype in M and therefore h-neat in $Soc^t(M)$. In fact, if $\lambda_t = 0$ then obviously $Soc^t(M)$ is elementary. Suppose, instead, that the module M is not closed and $\lambda_t = 0$ for some $t \in \mathbb{N}$. To that end, let x be the uniform element of the module M. In fact, if $e(x) = \infty$, then there is a submodule N of M maximal with respect to the properties $x \notin N$, $\langle x', Soc^t(M)\rangle \subset N$ is λ_t-isotype in M such that $d(xR/x'R) = 1$; but obviously it is not h-neat in M, as needed.
$(ii) \Rightarrow (i)$. If $\lambda_t > 0$ for each $t \in \mathbb{N}$, one observes that every λ_t-isotype submodule of M is h-neat in M. If $\lambda_t = 0$ and M is a closed module, one sees that $Soc^t(M)$ is elementary. Furthermore, by what we have shown previously, N being λ_t-isotype submodule of M ensures that $Soc^t(N)$ is λ_t-isotype in $Soc^t(M)$ for each $t \in \mathbb{N}$ and hence $Soc^t(N) \cap H_1(Soc^t(M)) = H_1(Soc^t(N))$. Consequently, $N \cap H_1(M) = H_1(N)$ and the claim is sustained. □

The following technical matter is one of the keys for further investigations.

Lemma 2.9. *Suppose that α is an ordinal. For any QTAG-module M there exists a $H_\alpha(M)$-high submodule N such that $x \in H_\alpha(M)$. If $H_\beta(Soc^t(M)) \neq 0$ for some $t \in \mathbb{N}$ and for each ordinal $\beta < \alpha$, then there is a submodule K of M with $H_\beta(M) \cap K = H_\beta(K)$ for each ordinal $\beta \leq \alpha$ and $H_\alpha(K) = \langle x \rangle$.*

Proof. For all ordinals α we can distinguish the following basic cases:
Case 1: If α is not a limit ordinal, then there is an element $y \in H_{\alpha-1}(M)$ with the property that $y' = x$ and $d(yR/y'R) = 1$. Now, if $y \notin H_\alpha(M)$, we see that $x_\beta = y$ for every ordinal $\beta < \alpha$ and $K_\alpha = \langle y \rangle$. But, on the other hand, if $y \in H_\alpha(M)$, there exists an element $z \in Soc(H_{\alpha-1}(N))$ such that $w \in H_{\alpha-1}(M) \backslash H_\alpha(M)$ and $w' = x$ where $w = y + z$ and $d(wR/w'R) = 1$. Hence $x_\beta = w$ for every ordinal $\beta < \alpha$ and $K_\alpha = \langle w \rangle$. Finally, one infers that $K_\alpha \cap H_\alpha(M) = \langle x \rangle$, as expected.
Case 2: Let α be a limit ordinal. For each ordinal $\beta < \alpha$ there is an element $a \in H_\beta(M) \backslash H_\alpha(M)$ with the property that $x = a'$ and $d(aR/a'R) = 1$. We will use a transfinite induction on K_β with $\beta \leq \alpha$ such that $K_0 = \langle x \rangle$. Clearly, $K_0 \cap H_\alpha(M) = \langle x \rangle$ and $Soc(M \cap K_0) \subset \langle x \rangle$. Furthermore, $K_1 = \langle K_0, b \rangle$, where $b \in H_1(M) \backslash H_\alpha(M)$ and $b' = x$ where $d(bR/b'R) = 1$. Apparently, $K_1 \cap H_\alpha(M) = \langle x \rangle$ and $Soc(H_1(M) \cap K_1) \subset \langle x \rangle$. Appealing to [12], one can write that $K_{\beta-1} \cap H_\alpha(M) = \langle x \rangle$ and $Soc(H_{\beta-1}(M) \cap K_{\beta-1}) \subset \langle x \rangle$ for some submodule $K_{\beta-1} \supset K_\beta$. If there is an element $a \in K_{\beta-1} \cap H_\beta(M)$ such that $a' = x$ where $d(aR/a'R) = 1$, and $x_\beta = a$ with $K_\beta = K_{\beta-1}$. Let $K_\beta = \langle K_{\beta-1}, x_\beta \rangle$, where $x_\beta \in H_\beta(M) \backslash H_\alpha(M)$ and $x'_\beta = x$ such that $d(x_\beta R/x'_\beta R) = 1$. Now, we need to show that $K_\beta \cap H_\alpha(M) = \langle x \rangle$. Let $c \in K_{\beta-1}$ be any element such that $c + t_1 x_\beta \in H_\alpha(M)$ where t_1 is an integer. Then $c' \in K_{\beta-1} \cap H_\alpha(M) = \langle x \rangle$ and $c' = t_2 x$ where $d(cR/c'R) = 1$ and t_2 is an integer. Thus, one may write that $t_3 x' + t_2 t_4 x = x$ where t_3 and t_4 are integers and $d(xR/x'R) = 1$. So, $x = u'$, where $u = t_3 x + t_4 c \in K_{\beta-1} \cap H_\beta(M)$ and $d(uR/u'R) = 1$. This is a contradiction. Hence $t_2 = v'$, $p' = 0$ and $p = c - qx \in Soc(H_{\beta-1}(M) \cap K_{\beta-1}) \subset \langle x \rangle$ where $d(vR/v'R) = d(pR/p'R) = 1$. Therefore $c \in \langle x \rangle$ and $c + t_1 x_\beta \in \langle x_\beta \rangle \cap H_\alpha(M) = \langle x \rangle$. We next will show that $Soc(H_\beta(M) \cap K_\beta) \subset \langle x \rangle$. Let $c + t_1 x_\beta \in Soc(H_\beta(M) \cap K_\beta)$, where $c \in K_{\beta-1}$ and t_1 is an integer. In fact, $c' = -t_1 x$ such that $d(cR/c'R) = 1$ and so $x = q'$ where $q = t_3 x - t_4 c$ such that $d(qR/q'R) = 1$. Consequently, $q \in K_{\beta-1} \cap H_\beta(M)$, which is a contradiction. Hence $c + t_1 x \in Soc(H_{\beta-1}(M) \cap K_{\beta-1}) \subset \langle x \rangle$, and therefore $c \in \langle x \rangle$. Finally, if β is a limit ordinal and $c + t_1 x_\beta \in \langle x \rangle$, then $K_\beta = \cup_{\gamma \in \beta} K_\gamma$.

Let K be a submodule of M maximal with respect to the conditions: $K \cap H_\alpha(M) = \langle x \rangle$, $K_\alpha \subset K$. We next assert that $H_\beta(M) \cap K = H_\beta(K)$ for every $\beta \leq \alpha$, it suffices to show that if this equality holds for $\beta - 1$ then it holds for β. To this aim, given $a \in K \cap H_\beta(M)$ with $a = s'$, where $s \in H_{\beta-1}(M)$ and $d(sR/s'R) = 1$. If $s \in K$ is chosen such that $s \in K \cap H_{\beta-1}(M) = H_{\beta-1}(K)$, then $a \in H_\beta(K)$. But, on the other hand, if $s \notin K$, there exists an element $c \in K$ such that $c + t_1 s \in H_\alpha(M) \backslash \langle x \rangle$, where t_1 is an integer, and $c \in H_{\beta-1}(M)$. Since $s' t_1 + c' \in K \cap H_\alpha(M) = \langle x \rangle$ with $d(sR/s'R) = d(cR/c'R) = 1$, it is easily follows that $t_1 a + c' = t_5 x = t_5 x'_{\beta-1}$ and $t_1 a = H_1((t_5 x_{\beta-1} - c)R)$ where $d(cR/c'R) = d(x_{\beta-1}R/x'_{\beta-1}R) = 1$ and t_5 is an integer. Therefore, $t_5 x_{\beta-1} - c \in K \cap H_{\beta-1}(M) = H_{\beta-1}(K)$, whence $t_1 a \in H_\beta(K)$ and thus $a \in H_\beta(K)$. □

We turn next to the following observation.

Lemma 2.10. *Let M be a $QTAG$-module and α an ordinal. Then the next two points are equivalent.*
(i) Every α-isotype submodule of M is isotype in M.
(ii) Either $M = M_1 \oplus M_2$, where $H_1(M_1) = M_1$ and $H_\beta(M_2) = 0$ for some ordinal $\beta < \alpha$, or $H_\alpha(M)$ is elementary or $H_{\alpha-1}(M) = U_k \oplus U_{k+1}$ for some $k \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$. First observe that if $\alpha = 0$, then M is elementary by [12]. Next, choose α is a limit ordinal. Then, we write $\gamma = \alpha$, or $\gamma = \alpha - 1$. Thus $H_\gamma(M) = M_1 \oplus M_3$, where M_3 is h-reduced and $H_1(M_1) = M_1$. If both M_1 and M_3 are non-zero, therefore $M_3 = \langle x\rangle + M_4$ such that $e(x) = t_1$, for some $t_1 \in \mathbb{N}$. Indeed, the submodule $H_{\gamma+t_1}(M)$ is not essential in $H_\gamma(M)$. It therefore follows that $H_{\gamma+t_1+1}(M) \neq 0$. But this contradicts Lemma 2.4. If $H_\gamma(M)$ is h-reduced and $H_\gamma(M) = \langle x\rangle \oplus \langle y\rangle \oplus M_4$, where $e(x) = t_1$, $e(y) = t_2$ and $t_2 - t_1 \geq 2$. Note that the submodule $H_{\gamma+t_1}(M)$ is not essential in $H_\gamma(M)$, so that $H_{\gamma+t_1+1}(M) \neq 0$. But this would contradict Lemma 2.4. Consequently, either $H_1(H_\gamma(M)) = H_\gamma(M)$ or $H_\gamma(M) = U_k + U_{k+1}$ for some $k \in \mathbb{N}$. Now, if $\gamma = \alpha - 1$, we are done. Since $H_1(H_\gamma(M)) = H_\gamma(M)$, it follows that $M = H_\gamma(M) \oplus M_2$ and $H_\gamma(M_2) = 0$. Hence, we suppose that $\gamma = \alpha$. Let $H_1(H_\alpha(M)) = H_\alpha(M)$; we write $M = H_\alpha(M) \oplus M_2$. Observe that $H_\beta(M_2) \neq 0$ for every ordinal $\beta < \alpha$ and $0 \neq x \in Soc(H_\alpha(M))$. Thus, with Lemma 2.9 at hand, there exists an α-isotype submodule N of M with the property that $H_\alpha(N) = \langle x\rangle$. Thus, $H_{\alpha+1}(N) = 0 \neq \langle x\rangle = N \cap H_{\alpha+1}(M)$, a contradiction. Hence $H_\beta(M_2) = 0$, for some ordinal $\beta < \alpha$. Let $H_\alpha(M) = U_k \oplus U_{k+1}$ and suppose that $H_\alpha(M)$ is not elementary. If K is a $H_\alpha(M)$-high submodule of M, then $H_\beta(K) \neq 0$ for every ordinal $\beta < \alpha$. Let $x \in Soc(H_{\alpha+1}(M))$ be any non-zero uniform element. Then, there is an α-isotype submodule N of M such that $H_\alpha(N) = \langle x\rangle$, in conjunction with Lemma 2.9. Thus, $H_{\alpha+1}(N) \neq N \cap H_{\alpha+1}(M)$, a contradiction. Hence $H_\alpha(M)$ is elementary.
$(ii) \Rightarrow (i)$. Let K be an α-isotype submodule of M and $H_{\alpha-1}(M) = U_k \oplus U_{k+1}$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} H_1(H_{\alpha-1}(K)) &= H_\alpha(K), \\ &= K \cap H_\alpha(M), \\ &= H_{\alpha-1}(N) \cap H_1(H_{\alpha-1}(M)), \end{aligned}$$

hence $H_{\alpha-1}(K)$ is h-neat in $H_{\alpha-1}(M)$, and therefore $H_{\alpha-1}(N)$ is h-pure in $H_{\alpha-1}(M)$ by [15]. Consequently,

$$\begin{aligned} H_r(H_{\alpha-1}(K)) &= H_{\alpha-1}(K) \cap H_r(H_{\alpha-1}(M)), \\ &= K \cap H_r(H_{\alpha-1}(M)), \end{aligned}$$

for some $r \in \mathbb{N}$. Moreover, if $r \geq k+1$, then $H_r(H_{\alpha-1}(K)) = 0$. If $M = M_1 \oplus M_2$, where M_1 is h-divisible and $H_\beta(M_2) = 0$ for some $\beta < \alpha$, then

$$H_\beta(K) = K \cap H_\beta(M) = K \cap H_\alpha(M) = H_\alpha(K).$$

If $H_\alpha(M)$ is elementary, then

$$H_{\alpha+1}(K) = K \cap H_{\alpha+1}(M) = 0.$$

Therefore, we conclude that K is isotype in M. □

We proceed by proving now our important goal mentioned in [1] (see [12] too), namely:

Theorem 2.11. *If $\lambda = (\lambda_t)_{t\in\mathbb{N}}$ is a sequence of ordinals and M is a QTAG-module. Then the next two points are equivalent.*
(i) Every λ_t-isotype submodule of M is isotype in M.
(ii) For every $t \in \mathbb{N}$, either $Soc^t(M) = M_1 \oplus M_2$, where $H_1(M_1) = M_1$ and $H_\beta(M_2) = 0$ for some ordinal $\beta < \lambda_t$ or $H_{\lambda_t}(M)$ is closed and $H_{\lambda_t}(Soc^t(M))$ is elementary or $H_{\lambda_t}(M)$ is closed and $H_{\lambda_t-1}(Soc^t(M)) = U_k + U_{k+1}$ for some $k \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$. For each $t \in \mathbb{N}$, every λ_t-isotype submodule of $Soc^t(M)$ is isotype in $Soc^t(M)$ and hence $Soc^t(M) = M_1 \oplus M_2$ in virtue of Lemma 2.10. Suppose that $H_{\lambda_t}(Soc^t(M))$ is not a closed module. If $H_{\lambda_t-1}(Soc^t(M)) = U_k \oplus U_{k+1} \neq 0$, then $H_{\lambda_t+k}(Soc^t(M))$ is not essential in $H_{\lambda_t-1}(Soc^t(M))$ and $H_{\lambda_t+k}(M)$ is not a closed module. We will now demonstrate something more, namely that $H_{\lambda_t}(Soc^t(M))$ is even elementary. In fact, $H_{\lambda_t+1}(Soc^t(M))$ is not essential in $H_{\lambda_t}(Soc^t(M))$ and $H_{\lambda_t+1}(M)$ is not a closed module. In these both cases, we get a contradiction because of Lemma 2.4.

Suppose now that $H_{\lambda_t}(M)$ is not closed, $H_{\lambda_t}(Soc^t(M)) = 0$ and $H_\beta(Soc^t(M)) \neq 0$ for each ordinal $\beta < \lambda_t$. Let $x \in H_{\lambda_t}(M)$ with $e(x) = \infty$ and K be a $H_{\lambda_t}(M)$-high submodule of M containing $Soc^t(M)$. Therefore, $H_\beta(Soc^t(K)) \neq 0$ for each ordinal $\beta < \lambda_t$. By Lemma 2.9, there is a submodule L of M with the property that $H_\beta(L) = L \cap H_\beta(M)$ for every ordinal $\beta \leq \lambda_t$ and $H_{\lambda_t}(L) = \langle x' \rangle$ where $d(xR/x'R) = 1$; note that N be a submodule of M maximal with respect to the conditions: $L \subset N$, $x \notin N$. Thus, in view of [12], $\overline{(M/N)} = 0$. Then, we prove that $H_\beta(M) \cap N = H_\beta(N)$ for each ordinal $\beta \leq \lambda_t$. It is enough to prove that if this equality holds for $\beta - 1 \leq \lambda_t$ then it holds also for β. Let $y \in N \cap H_\beta(M)$, there exists an element $z \in H_{\beta-1}(M)$ such that $y = z'$ where $d(zR/z'R) = 1$, because $y \in H_{\beta-1}(N)$. Now, we shall consider two cases about z. First, if $z \in N$, then $y \in H_\beta(N)$. For the second case where $z \notin N$, it is obvious that $x \in \langle z, N \rangle$ and $x = t_1 z + a$ where, $a \in N$ and t_1 is an integer. Therefore, $a \in N \cap H_{\beta-1}(M) = H_{\beta-1}(N)$. Furthermore, $x' = t_1 y + a' \in H_{\lambda_t}(L) \subset H_\beta(L)$ such that $d(xR/x'R) = d(aR/a'R) = 1$. Then, there exists an element $b \in H_{\beta-1}(L)$ such that $t_1 y + a' = b'$ where $d(aR/a'R) = d(bR/b'R) = 1$. Hence $t_1 y = H_1((b-a)R)$, where $b - a \in H_{\beta-1}(N)$ and therefore $t_1 y \in H_\beta(N)$. Since y', $t_1 y \in H_\beta(N)$, then $y \in H_\beta(N)$ where $d(yR/y'R) = 1$. Thus, N is λ_t-isotype in M. Finally, if $x' \in N \cap H_{\lambda_t+1}(M) \backslash H_{\lambda_t+1}(N)$ and $x' = c'$ where $c \in H_{\lambda_t}(N)$ and $d(xR/x'R) = d(cR/c'R) = 1$, then $x - c \in Soc^t(M) \cap H_{\lambda_t}(M) = 0$ and $x \in N$, a contradiction. Henceforth, N is not isotype in M.
$(ii) \Rightarrow (i)$. If N is a λ_t-isotype submodule of M, then $\overline{N}$ is λ_t-isotype in $\overline{M}$, and therefore, point $(2c)$ would imply that $Soc^t(N)$ is λ_t-isotype in $Soc^t(M)$. Thus, by Lemma 2.10, $Soc^t(N)$ is isotype in $Soc^t(M)$, and so $\overline{N}$ is isotype in $\overline{M}$ in conjunction with point $(2c)$. If $H_{\lambda_t}(M)$ is a closed module for some $t \in \mathbb{N}$,

$$H_\beta(N) = H_\beta(\overline{N}) = \overline{N} \cap H_\beta(\overline{M}) = N \cap H_\beta(\overline{M}) = N \cap H_\beta(M),$$

for some ordinal $\beta \geq \lambda_t$. Suppose now that $Soc^t(M) = M_1 \oplus M_2$ where M_1 is h-divisible and $H_\beta(M_2) = 0$ for some ordinal $\beta < \lambda_t$. Then $H_\beta(Soc^t(M))$ and $H_\beta(Soc^t(N))$ are h-divisible. Setting $H_\beta(N) = H_\beta(Soc^t(N)) \oplus S$. Since $H_\beta(Soc^t(M)) \cap S = 0$, we get $H_\beta(M) = H_\beta(Soc^t(M)) \oplus L$, where $S \subset L$. We next assert that $H_\gamma(S) = S \cap H_\gamma(L)$ for each ordinal γ, it is sufficient to show that if this equality holds for γ then it holds also for $\gamma+1$. Let $c \in S \cap H_{\gamma+1}(L)$, there exists an element $y \in H_\beta(N)$ such that $c = y'$ where $d(yR/y'R) = 1$. Clearly, $y = a + s$, where $a \in H_\beta(Soc^t(N))$ and $s \in S$. Since $c = a' + s'$, we have $a' \in S \cap H_\beta(Soc^t(N)) = 0$ where $d(aR/a'R) = d(sR/s'R) = 1$. Therefore, $c = s'$ with $d(sR/s'R) = 1$, and consequently, $u - s \in Soc^t(L) = 0$, where $u \in S \cap H_\gamma(L) = H_\gamma(S)$. This, in turn, implies that $c \in H_{\gamma+1}(S)$. Finally,

$$\begin{aligned} H_\gamma(H_\beta(N)) &= H_\beta(Soc^t(N)) \oplus H_\gamma(S), \\ &= H_\beta(Soc^t(N)) \oplus (S \cap H_\gamma(L)), \\ &= H_\beta(N) \cap (H_\beta(Soc^t(M)) \oplus H_\gamma(L)), \\ &= H_\beta(N) \cap H_\gamma(H_\beta(M)), \\ &= N \cap H_\gamma(H_\beta(M)), \end{aligned}$$

for each ordinal γ, and we complete the proof. □

3. Conclusion and Future Work

The concept of the λ_t-isotype submodule in $QTAG$-modules has been introduced and initially investigated in this study. Some conditions under which a λ_t-isotype submodule admits an h-neat, an h-pure, an isotype submodule, a direct summand, an absolute direct summand, respectively, are provided. Following this, we investigated the connections of these concepts and found that every λ_t-isotype submodule is an isotype submodule. Our analysis in depth aims to enhance our understanding of these λ_t-isotype submodules in the context of $QTAG$-modules. According to our findings, our proposed study is promising and applicable to examine the decompositions of various types of submodules or other algebraic structures.

We close the work with certain challenging problems which are worthwhile for a further study.

Problem 3.1. Find conditions on a closed module such that it is a λ_t-isotype submodule (resp., an isotype) submodule.

Problem 3.2. Construct, if possible, a λ_t-isotype submodule which is not an isotype submodule.

Problem 3.3. If $t > 0$, and N_1 and N_2 are λ_t-isotype submodules such that $Soc^t(N_1)$ is isometric to $Soc^t(N_2)$, can we conclude that N_1 is isomorphic to N_2?

Problem 3.4. Describe the properties of those h-reduced $QTAG$-modules M for which there exist λ_t-isotype submodules N of M such that M/N are h-divisible.

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