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ORDERED SEMIGROUPS CONTAINING COVERED (m,n)-IDEALS

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Abstract. In this article, we extend the concept of covered (m,n)-ideals by defining covered (m,n)-ideals of ordered semigroups, where m,n are nonnegative integers. The results obtained generalize those of covered one-sided ideals and covered bi-ideals. We additionally investigate the conditions under which, in an (m,n)-regular ordered semigroup S, every covered (m,n)-ideal of an (m,n)-ideal I of S is also a covered (m,n)-ideal of S.

1. Introduction

In the world of algebra, there are many interesting algebraic structures, such as rings, semirings, semigroups and ordered semigroups. The concept of an ordered semigroup is an extension of the semigroup notion, achieved by adding a partial order compatible with the semigroup operation [1]. In exploring the structural properties of ordered semigroups, there are several important tools, one of which is ideal theory. The various kinds of ideals have been thoroughly studied in ordered semigroups, including ideals (two-sided ideals), one-sided ideals (left and right ideals), bi-ideals, quasi-ideals, interior ideals, etc.

The notion of (m,n)-ideals in semigroups (without order) was first defined by Lajos in 1961 [13]. In 2012, Sanborisoot and Changphas extended this idea to ordered semigroups in [15], where they provided some characterizations of (m,n)-regular ordered semigroups based on (m,n)-ideals. After that, in 2015, Changphas examined (m,n)-ideals of ordered semigroups in [3], offering a criterion to determine whether a given ordered semigroup will contain a proper (m,n)-ideal or not. Bussaban and Changphas explored (m,n)-ideals and (m,n)-regular ordered semigroups in [2], where they established the necessary and sufficient conditions for any (m,n)-ideal of an (m,n)-ideal I of an ordered semigroup S to also be an (m,n)-ideal of S.

On the other hand, another concept that mathematicians widely study is that of covered ideals. This concept was introduced by Fabrici [7], who initially explored semigroups containing covered one-sided ideals [8] in 1981 and two-sided ideals [9] in 1984. Later, in 2016, Changphas and Summaprab generalized Fabrici's results

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by investigating ordered semigroups containing covered ideals [4]. In the following year, they examined ordered semigroups containing covered one-sided ideals in [5] and demonstrated that the set of all covered left ideals of an ordered semigroup S forms a sublattice within the lattice of all left ideals of S. Khan et al. extended the investigation of the structure of ordered semigroups to ordered ternary semigroups by studying covered lateral ideals of ordered ternary semigroups in [12]. In 2022, Jantanan et al. focused on studying ordered semigroups containing covered bi-ideals in [10]. They provided a condition under which any covered bi-ideal B_1 of a bi-ideal B of an ordered semigroup S is also a covered bi-ideal of S. A year later, they studied covered left ideals of ordered ternary semigroups in [11], expanding upon their earlier work in [12]. They provided two conditions regarding the union and intersection of any two covered left ideals that guarantee the result is again a covered left ideal. As the concept of covered right ideals can be described in a dual manner, this completes the study of covered one-sided ideals of partially ordered ternary semigroups. In 2023, Chinram et al. also examined covered left ideals and extended the concept from semigroups to ternary semigroups (without order) in [6].

Over the years, the concepts of (m, n)-ideals and covered ideals have captured the attention of many mathematicians. In this article, we introduce the notion of covered (m, n)-ideals in ordered semigroups. This paper will explore the structure of ordered semigroups that contain covered (m, n)-ideals. We investigate the conditions that determine when the union and intersection of two covered (m, n)-ideals of an ordered semigroup are covered (m, n)-ideals as well. It can be seen that some results on covered one-sided ideals and bi-ideals discussed in [5] and [10] are special cases of our main findings.

2. Preliminaries

This section will recall some fundamental terminology of ordered semigroups and covered (m, n)-ideals.

A structure $(S, \cdot, \leq)$ is called an *ordered semigroup* if it is a semigroup with an associative binary operation together with a partial order $\leq$ which is compatible with the semigroup operation. This means that, if $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. In this paper, we will denote an ordered semigroup $(S, \cdot, \leq)$ simply by S.

Given nonempty subsets A and B of S, we define the product of A and B as $AB = \{ab \mid a \in A, b \in B\}$. For a fixed positive integer $n \geq 2$, the n-fold product of A is the set $A^n = \underbrace{A \cdot A \cdot A \cdots A}_{n \text{ terms}}$.

In particular, if $A = \{a\}$, we use aB in place of $\{a\}B$; likewise, we write Ab when $B = \{b\}$.

Moreover, the set of all elements x in S for which there exists an $a \in A$ such that $x \leq a$ is denoted by $(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}$.

It is noted that the following well-known properties are valid:

(1) $A \subseteq (A]$ and $((A]] = (A]$,

(2) if $A \subseteq B$, then $(A] \subseteq (B]$,

(3) $(A](B] \subseteq (AB]$,

(4) $(A \cup B] = (A] \cup (B]$ and $(A \cap B] \subseteq (A] \cap (B]$.

A nonempty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. In this paper, we consider m, n as nonnegative integers. A subsemigroup A of S is defined as an (m,n)-*ideal* of S if it satisfies the following conditions:

(1) $A^m S A^n \subseteq A$,

(2) $A = (A]$, which means that for any $x \in A$ and $y \in S$, if $y \le x$, then $y \in A$.

The symbol $\subset$ represents a proper subset of sets. An (m,n)-ideal A of S is called a *proper* (m,n)-*ideal* if $A \subset S$. A *maximal* (m,n)-*ideal* of S is a proper ideal I of S such that there is no other (m,n)-ideal lying between I and S. If S has no proper (m,n)-ideal, then S is called (m,n)-*simple*.

An element a of S is said to be (m,n)-*ordered regular* if there exists $x \in S$ such that $a \le a^m x a^n$. If every element of S is (m,n)-ordered regular, then we call S an (m,n)-*regular ordered semigroup*. Moreover, an *idempotent* element is an element a of S such that $a \le a^2$.

The (m,n)-ideal of S generated by a nonempty subset A of S is given by the following form

$$(A)_{(m,n)} = (A \cup A^2 \cup A^3 \cup \cdots \cup A^{m+n} \cup A^m S A^n].$$

If $A = \{a\}$, then we denote $(\{a\})_{(m,n)}$ as $(a)_{(m,n)}$. Furthermore, we have

$$(a)_{(m,n)} = (\{a, a^2, a^3, \cdots, a^{m+n}\} \cup a^m S a^n],$$

which is known as the *principal* (m,n)-*ideal* of S generated by a.

3. Main results

We begin this section by introducing the concept of covered (m,n)-ideals in ordered semigroups.

Definition 3.1. Let S be an ordered semigroup. A proper (m,n)-ideal I of S is called a *covered* (m,n)-*ideal* of S if $I \subseteq ((S-I)^m S (S-I)^n]$.

Note that the notion of covered (m,n)-ideals in ordered semigroups serves as a generalization of covered one-sided ideals and covered bi-ideals. The reader may refer to [5] and [10] for further details.

Example 3.2. Consider $S = \{a,b,c,d,e\}$, where the multiplication and the partial order on S are defined as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\le = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,c),(d,d),(e,e)\}.$$

It is evident that S is an ordered semigroup; the reader is encouraged to see [10]. By means of direct calculation, the proper $(1,2)$-ideals of S are $\{a\}$, $\{a,b\}$, $\{a,c\}$,

$\{a,d\}, \{a,e\}$, $\{a,b,d\}, \{a,b,e\}, \{a,c,d\}$, and $\{a,c,e\}$. Furthermore, we find that the proper $(2,2)$-ideals of S are the same: $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,d\}$, $\{a,b,e\}, \{a,c,d\}$, and $\{a,c,e\}$.

In addition, it can be seen that the covered $(1,2)$-ideals of S are $\{a\}, \{a,b\}, \{a,c\}$, $\{a,d\}$, and $\{a,e\}$, which yield the same results for the covered $(2,2)$-ideals of S.

Next, we examine some properties of covered (m,n)-ideals in ordered semigroups.

Theorem 3.3. *Let S be an ordered semigroup containing two different proper (m,n)-ideals I and J such that $I \cup J = S$. Then I and J are not covered (m,n)-ideals of S.*

Proof. Assume that S contains two different proper (m,n)-ideals I and J such that $I \cup J = S$. Since $I \cup J = S$, we obtain $S - I \subseteq J$ and $S - J \subseteq I$. Suppose, on the contrary, that I is a covered (m,n)-ideal of S. Then

$$I \subseteq ((S-I)^m S (S-I)^n] \subseteq (J^m S J^n] \subseteq (J] = J.$$

This implies that $S = J$, which leads to a contradiction. Similarly, it can be noted that J is not a covered (m,n)-ideal of S either. Thus, the assertion is confirmed. □

Corollary 3.4. *Let S be an ordered semigroup containing two different maximal proper (m,n)-ideals I and J such that $I \cup J$ is also an (m,n)-ideal of S. Then neither I nor J can be a covered (m,n)-ideal of S.*

Proof. Assume that S contains two different maximal proper (m,n)-ideals I and J such that $I \cup J$ is also an (m,n)-ideal of S. By the maximality of I and J, we see that $I \subset I \cup J$, and hence $I \cup J = S$. Therefore, by Theorem 3.3, neither I nor J qualifies as a covered (m,n)-ideal of S. □

Theorem 3.5. *Let I and J be a covered (m,n)-ideal and an (m,n)-ideal of an ordered semigroup S, respectively. If $I \cap J$ is a nonempty set, then $I \cap J$ is a covered (m,n)-ideal of S.*

Proof. Assume that I is a covered (m,n)-ideal and J is an (m,n)-ideal of S such that $I \cap J \neq \emptyset$. First of all, we will show that $I \cap J$ is an (m,n)-ideal of S. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, we get

$$(I \cap J)^m S (I \cap J)^n \subseteq I^m S I^n \subseteq I \text{ and}$$
$$(I \cap J)^m S (I \cap J)^n \subseteq J^m S J^n \subseteq J.$$

Then $(I \cap J)^m S (I \cap J)^n \subseteq I \cap J$. Furthermore, since

$$(I \cap J] \subseteq (I] = I \text{ and } (I \cap J] \subseteq (J] = J,$$

we obtain $(I \cap J] \subseteq I \cap J$. Consequently, we have that $I \cap J = (I \cap J]$ because the condition $I \cap J \subseteq (I \cap J]$ always holds.

Since I is a covered (m,n)-ideal of S, we obtain $I \subseteq ((S-I)^m S (S-I)^n]$. Note that $I \cap J$ is a proper (m,n)-ideal of S. Hence,

$$I \cap J \subseteq I \subseteq ((S-I)^m S (S-I)^n] \subseteq ((S-(I \cap J))^m S (S-(I \cap J))^n].$$

This demonstrates that $I \cap J$ is a covered (m,n)-ideal of S. □

From Theorem 3.5, we can derive the following corollary.

Corollary 3.6. *If both I and J are covered (m,n)-ideals of an ordered semigroup S such that $I \cap J$ is a nonempty set, then $I \cap J$ is a covered (m,n)-ideal of S.*

The union of any two covered (m,n)-ideals of an ordered semigroup S is not necessarily a covered (m,n)-ideal of S: We see from Example 3.2 that $I := \{a,d\}$ and $J := \{a,e\}$ are covered $(1,2)$-ideals of S. Since $I \cup J = \{a,d,e\}$ but $(I \cup J)^2 = \{a,b,c,d\} \nsubseteq I \cup J$, we see that $I \cup J$ is not a subsemigroup of S. This implies that $I \cup J$ is not a covered $(1,2)$-ideal of S. We need an additional condition that ensures the union of covered (m,n)-ideals of S is again a covered (m,n)-ideal of S.

Theorem 3.7. *Let S be an ordered semigroup containing two different covered (m,n)-ideals I and J. If $I \cup J \neq S$ and one of the ideals is contained within the other, then $I \cup J$ is a covered (m,n)-ideal of S.*

Proof. Let S be an ordered semigroup containing two different covered (m,n)-ideals I and J and $I \cup J \neq S$. Without any loss of generality, we will assume $I \subset J$, so $I \cup J = J$. Since J is a covered (m,n)-ideal of S, the union $I \cup J$ is guaranteed to be a covered (m,n)-ideal of S. $\square$

In the next theorem, we will examine the structure of ordered semigroups that contain covered (m,n)-ideals.

Theorem 3.8. *Let S be an ordered semigroup. If the intersection of any two proper (m,n)-ideals is nonempty and S is not (m,n)-simple, then S contains a covered (m,n)-ideal.*

Proof. Suppose that the intersection of any two proper (m,n)-ideals is nonempty and S is not (m,n)-simple. Hence, S contains a proper (m,n)-ideal I. Firstly, we will verify that $J := ((S-I)^m S (S-I)^n]$ is an (m,n)-ideal of S. Since

$$\begin{aligned} J^2 &= ((S-I)^m S (S-I)^n]((S-I)^m S (S-I)^n] \\ &\subseteq ((S-I)^m SSSS (S-I)^n] \\ &\subseteq ((S-I)^m S (S-I)^n] = J, \end{aligned}$$

J is a subsemigroup of S. Next, we consider

$$\begin{aligned} J^m S J^n &= J J^{m-1} S J^{n-1} J \\ &\subseteq JSSSJ \\ &\subseteq ((S-I)^m S (S-I)^n] S ((S-I)^m S (S-I)^n] \\ &\subseteq ((S-I)^m S (S-I)^n] = J. \end{aligned}$$

Thus, we obtain that $J^m S J^n \subseteq J$. Moreover, as $J = ((S-I)^m S (S-I)^n]$, it follows that

$$(J] = (((S-I)^m S (S-I)^n]] = ((S-I)^m S (S-I)^n] = J.$$

Therefore, J is an (m,n)-ideal of S. In the next step, we will show that S contains a covered (m,n)-ideal. Consider the case where $J = S$. Then

$$I \subseteq S = J = ((S-I)^m S (S-I)^n].$$

This shows that I is a covered (m,n)-ideal of S. On the other hand, if $J \neq S$, then J is a proper (m,n)-ideal of S. Put $K := I \cap J$. By assumption, we see that $K \neq \emptyset$. This implies that $K \subseteq I$ is a proper (m,n)-ideal of S, and also $S - I \subseteq S - K$. Finally, we obtain

$$K \subseteq J = ((S-I)^m S(S-I)^n] \subseteq ((S-K)^m S(S-K)^n].$$

Therefore, K is a covered (m,n)-ideal of S, as desired. □

Theorem 3.9. *Let S be an (m,n)-regular ordered semigroup and I a proper (m,n)-ideal of S. If for any $a \in I$, $(a)_{(m,n)} \subseteq (b)_{(m,n)}$ for some $b \in S - I$, then I is a covered (m,n)-ideal of S.*

Proof. Let I be a proper (m,n)-ideal of S, and assume that for any $a \in I$, we have $(a)_{(m,n)} \subseteq (b)_{(m,n)}$ for some $b \in S - I$. Let $a \in I$. By assumption, there exists $b \in S - I$ such that $(a)_{(m,n)} \subseteq (b)_{(m,n)}$. Given that S is an (m,n)-regular ordered semigroup, there is an element $x \in S$ such that

$$b \leq b^m x b^n \in ((S-I)^m S(S-I)^n].$$

As demonstrated in Theorem 3.8, $((S-I)^m S(S-I)^n]$ is an (m,n)-ideal of S. This implies that, for any positive integer k, the power b^k is always a member of $((S-I)^m S(S-I)^n]$. As a consequence,

$$\begin{aligned} b^m S b^n &\subseteq ((S-I)^m S(S-I)^n] S ((S-I)^m S(S-I)^n] \\ &\subseteq ((S-I)^m S(S-I)^n]. \end{aligned}$$

This shows that,

$$\begin{aligned} (b)_{(m,n)} &= (\{b, b^2, b^3, \cdots, b^{m+n}\} \cup b^m S b^n] \\ &\subseteq ((S-I)^m S(S-I)^n]. \end{aligned}$$

Therefore, we can deduce that $a \in (a)_{(m,n)} \subseteq (b)_{(m,n)} \subseteq ((S-I)^m S(S-I)^n]$ for all $a \in I$, suggesting that

$$I \subseteq ((S-I)^m S(S-I)^n].$$

This establishes that I is indeed a covered (m,n)-ideal of S. □

Theorem 3.10. *Let S be an (m,n)-regular ordered semigroup. If I is an (m,n)-ideal of S such that every element of I is idempotent, then any covered (m,n)-ideal of I is also a covered (m,n)-ideal of S.*

Proof. Consider I to be an (m,n)-ideal of S with the property that every element of I is idempotent. Then I is a subsemigroup of S such that $I^m S I^n \subseteq I$ and $(I] = I$. First of all, we will verify that I itself is (m,n)-regular. Let $t \in I \subseteq S$. By assumption, t is an idempotent element of I and we obtain $t \leq t^2$. Since S is an (m,n)-regular ordered semigroup, there is an element $x \in S$ such that $t \leq t^m x t^n$. Furthermore, because

$$\begin{aligned} t^m x t^n \leq t^{2m} x t^{2n} &= t^m (t^m x t^n) t^n \\ &\in t^m (I^m S I^n) t^n \\ &\subseteq t^m I t^n, \end{aligned}$$

it can be concluded that, $t \in (t^m I t^n]$. This means that I is an (m,n)-regular ordered subsemigroup of S. Next, let I_1 be taken as any covered (m,n)-ideal I. We will now show that I_1 is an (m,n)-ideal of S as well. By definition, it is clear that I_1 is a proper subsemigroup of S. Let $r_1, r_2 \in I_1 \subseteq I$. By assumption, we have $r_1 \leq r_1^2$ and $r_2 \leq r_2^2$. Moreover, for any $s \in S$,

$$r_1^m s r_2^n \in I_1^m S I_1^n \subseteq I^m S I^n \subseteq I.$$

Let $h := r_1^m s r_2^n$. Since $h \in I$ and I itself satisfies the (m,n)-regularity condition, it follows that $h \leq h^m k h^n$ for some $k \in I$. Thus we obtain

$$\begin{aligned} h \leq h^m k h^n &= h h^{m-1} k h^{n-1} h \\ &= (r_1^m s r_2^n) h^{m-1} k h^{n-1} (r_1^m s r_2^n) \\ &\leq (r_1^{2m} s r_2^{2n}) h^{m-1} k h^{n-1} (r_1^{2m} s r_2^{2n}) \\ &\in (I_1^{2m} S I_1^{2n}) I^{m-1} I I^{n-1} (I_1^{2m} S I_1^{2n}) \\ &\subseteq (I_1^{2m} S I_1^{2n}) S (I_1^{2m} S I_1^{2n}) \\ &\subseteq I_1^{2m} S I_1^{2n} \\ &= I_1^m I_1^m S I_1^n I_1^n \\ &\subseteq I_1^m (I^m S I^n) I_1^n \\ &\subseteq I_1^m I I_1^n \subseteq I_1 \end{aligned}$$

because I is an (m,n)-ideal of S and I_1 is an (m,n)-ideal of I. Then we have $h \in (I_1] = I_1$, which leads to $I_1^m S I_1^n \subseteq I_1$. To show that $(I] = I$ in S, let $y \in S$ and $y \in (I_1]$. Then $y \leq x$ for some $x \in I_1$. Since I is an (m,n)-ideal of S, $y \in (I_1] \subseteq (I] = I$. Due to the fact that I_1 is an (m,n)-ideal of I and $y \in I$ with $y \leq x$ for some $x \in I_1$, it follows that $y \in I_1$. Hence, I_1 is an (m,n)-ideal of S. Note also that $I_1 \subseteq ((I - I_1)^m I (I - I_1)^n]$ and $I_1 \subset I \subseteq S$ as I_1 is a covered (m,n)-ideal of I. Furthermore, since $\emptyset \neq I - I_1 \subseteq S - I_1$, we finally obtain

$$I_1 \subseteq ((I - I_1)^m I (I - I_1)^n] \subseteq ((S - I_1)^m S (S - I_1)^n].$$

Therefore, I_1 is a covered (m,n)-ideal of S. □

Example 3.11. Let $S = \{a,b,c,d,e\}$ with the multiplication and the partial order on S defined by

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq = \{(a,a),(a,b),(b,b),(c,a),(c,b),(c,c),(d,a),(d,b),(d,d),(e,e)\}.$$

One can confirm that S is an ordered semigroup; the reader may refer to [10]. It can be verified that S is an (m,n)-regular ordered semigroup for any nonnegative integers m and n. We have $I = \{a,b,c,d\}$ is a proper (m,n)-ideal of S and every element of I is idempotent. Moreover, it can be seen that $I_1 = \{a,c,d\}$ is a proper (m, n)-ideal of I. One can further check that I_1 is a covered (m, n)-ideal of both I and S.

4. Conclusions

This paper presents the concept of covered (m, n)-ideals of ordered semigroups. We generalize results from semigroups (without order) to the case of ordered semigroups. We examine some properties of covered (m, n)-ideals in ordered semigroups, including conditions that determine when the union and intersection of two covered (m, n)-ideals of an ordered semigroup are themselves covered (m, n)-ideals of S. Furthermore, we explore the structure of ordered semigroups that contain covered (m, n)-ideals. Some results from earlier works, such as covered one-sided ideals and covered bi-ideals in [5] and [10], respectively, can be considered as special cases of the results presented here. In addition, fuzzy ideals have been of interest; see, for example, [14]. Thus, in future work, we may study another kind of ideals in ordered semigroups such as fuzzy covered ideals.

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