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ANALYSIS OF FRACTIONAL BLACK-SCHOLES, ORNSTEIN-UHLENBECK, AND LANGEVIN MODELS: A MINIMUM DISTANCE ESTIMATION APPROACH

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Abstract. The present research investigates parameter estimation in fractional stochastic models, specifically the Fractional Black-Scholes, Fractional Ornstein-Uhlenbeck, and Fractional Langevin models. These models, driven by fractional Brownian motion (fBm), capture long-memory effects and complex dependencies often observed in financial and physical systems. To efficiently estimate the parameters of these models, we apply the Minimum Distance Estimation (MDE) method. This approach tackles the challenges posed by long-range dependence, ensuring both asymptotic consistency and normality. We analyze the asymptotic properties of the estimators, including their consistency, asymptotic normality, and efficiency. By providing a unified theoretical framework, the current research highlights the robustness and applicability of the MDE method for parameter inference in fractional models. The findings contribute significantly to diverse fields such as quantitative finance, econometrics, and statistical physics.

1. Introduction

Stochastic models are widely used to represent the complex dynamics of financial, economic, and physical phenomena, particularly when they involve components of noise or uncertainty. Traditional models often rely on Brownian motion, which is characterized by continuous paths and a lack of memory. However, such an approach may prove inadequate in contexts where time dependence and long-term memory are present. Many empirical applications, especially in finance and statistical physics, exhibit persistent correlations and memory effects that standard Brownian motion cannot adequately capture. To address these limitations, fractional models, such as the Fractional Black-Scholes, Fractional Ornstein-Uhlenbeck, and Fractional Langevin models, incorporate fractional Brownian motion (fBm), allowing for long-memory dynamics and more complex dependencies [10, 14].

Fractional Brownian motion B_t^H, defined by the Hurst index $H \in (0,1)$, demonstrates long-term dependence when $H > 0.5$. In finance, the characteristic of

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long-term dependence makes fBm particularly effective for modeling non-constant volatility and persistent asset price behavior, thereby enhancing the traditional Black-Scholes framework [2]. In physical systems, such as anomalous diffusion, fBm accounts for the long-term correlations observed in particle dynamics, which remain unexplained by classical Brownian motion [13].

For example, the fractional Black-Scholes model extends the classic option pricing model by incorporating fractional Brownian motion to better represent market volatility [1]. Similarly, the fractional Ornstein-Uhlenbeck model provides a more flexible framework for mean-reverting processes, such as interest rates or economic indicators that exhibit persistent autocorrelation. The fractional Langevin model, on the other hand, applies to physical systems to describe anomalous diffusion processes driven by fractional noise [5, 6, 18].

The mathematical modeling of stochastic systems driven by fractional Brownian motion has been extended to impulsive evolution equations, as demonstrated by recent work on the controllability of such systems [17]. These advancements underscore the versatility of fractional stochastic models in capturing complex dynamics across various disciplines.

Parameter estimation in fractional models poses significant challenges due to the long-memory effects and the complex temporal correlations inherent in fBm. Traditional methods, such as Maximum Likelihood Estimation (MLE), can be difficult to apply because of the intractable nature of the likelihood functions. To overcome these difficulties, the MDE method is employed, minimizing the discrepancy between the empirical and theoretical moments of the processes. The effectiveness of the MDE approach has been demonstrated in capturing the dynamics of fractional stochastic differential equations [15, 16, 4, 12]. Alternative methods, such as the method of moments, have also been successfully utilized for jump-free fractional stochastic models, emphasizing their utility in various applications [7]. The challenges of parameter estimation extend to specific models, such as the fractional Langevin model, where the likelihood is particularly difficult to compute. Earlier studies have explored the use of MLE and other alternative methods in such cases [9]. Techniques for parameter estimation in fractional stochastic differential equations driven by fBm have also been proposed, focusing on both theoretical and empirical aspects [11].

The analysis presented in this paper focuses on the asymptotic properties of the MDE method applied to three fractional models: the fractional Black-Scholes, fractional Ornstein-Uhlenbeck, and fractional Langevin models. For each model, consistency, asymptotic normality, and efficiency of the parameter estimators are established. By providing a unified theoretical framework, the study advances the understanding of fractional stochastic processes and their applications in quantitative finance and statistical physics.

The remainder of the paper is organized as follows. Section 2 describes the MDE method and its implementation. Section 3 focuses on parameter estimation for the fractional Black-Scholes model, while Section 4 analyzes the fractional Ornstein-Uhlenbeck model. Section 5 addresses the fractional Langevin model and its applications. Finally, Section 6 concludes the paper and discusses potential extensions.

Addressing the challenges of parameter estimation in fractional models enhances the robustness and applicability of the MDE method, offering practical insights for long-memory processes in finance, econometrics, and statistical physics.

2. Minimum Distance Estimation of a Jump-Free Stochastic Differential Equation Driven by Fractional Brownian Motion

The estimation of parameters in jump-free stochastic differential equations driven by fractional Brownian motion (fBm) is essential for capturing long-memory effects and understanding the complex dependencies in stochastic processes. The MDE method, in combination with the Newton-Raphson iterative optimization technique, provides an effective framework for accurately estimating these parameters. This section describes the theoretical framework and practical implementation of MDE in the context of stochastic differential equations driven by fBm.

Consider a stochastic differential equation (SDE) of the form:

$$dX_t = b(X_t, \theta)\, dt + \sigma(X_t, \theta)\, dB_t^H,$$

where: X_t is the stochastic process to be estimated, $\theta = (b, \sigma)$ represents the vector of parameters, where $b(X_t, \theta)$ denotes the drift term and $\sigma(X_t, \theta)$ the diffusion term, B_t^H is a fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$, introducing long-range dependence in the process.

The objective of MDE is to estimate the parameters $\theta = (b, \sigma)$ by minimizing the distance between theoretical characteristics of the process, such as mean and variance, and their empirical counterparts. Such an approach proves particularly useful in models where closed-form maximum likelihood solutions are difficult to obtain.

The Newton-Raphson method is an iterative technique that, in this context, seeks the parameter values θ that minimize the distance function:

$$d(\theta) = \sum_{k=1}^{K} \left(T_k(\theta) - E_k\right)^2,$$

where: $T_k(\theta)$ denotes the theoretical statistic associated with the model for a given θ, E_k represents the corresponding empirical statistic derived from observations.

Using Newton-Raphson, the estimation of θ is updated iteratively as follows:

$$\theta^{(k+1)} = \theta^{(k)} - \left(\nabla^2 d(\theta^{(k)})\right)^{-1} \nabla d(\theta^{(k)}),$$

where: $\nabla d(\theta)$ is the gradient vector of the distance function with respect to θ, $\nabla^2 d(\theta)$ is the Hessian matrix (second derivatives) of the distance function.

The Newton-Raphson method is particularly useful here as it ensures convergence towards the parameter values that minimize the difference between theoretical and empirical moments. This is especially important in the context of fractional Brownian motion models, which involve complex, non-linear dependency structures.

To implement MDE in the fractional Brownian motion framework, we follow these main steps:

The gradient $\nabla d(\theta)$ is obtained by computing the partial derivatives of $d(\theta)$ with respect to each parameter in θ. For instance, when moments are used to define the distance function, the partial derivatives of the function with respect to b and σ can be expressed as:

$$\frac{\partial d}{\partial b} = 2\sum_{k=1}^{K}(T_k(\theta) - E_k)\,\frac{\partial T_k(\theta)}{\partial b}, \quad \frac{\partial d}{\partial \sigma} = 2\sum_{k=1}^{K}(T_k(\theta) - E_k)\,\frac{\partial T_k(\theta)}{\partial \sigma}.$$

These expressions depend on the chosen theoretical moments. For example, if the theoretical mean of X_t is $\mathbb{E}[X_t] = X_0 e^{-bt}$, then: $\dfrac{\partial \mathbb{E}[X_t]}{\partial b} = -X_0 t e^{-bt}$.

The Hessian matrix consists of the second derivatives of $d(\theta)$ with respect to each pair of parameters, allowing for refinement of parameter estimates through Newton-Raphson. The elements of the Hessian matrix are:

$$\frac{\partial^2 d}{\partial b^2} = 2\sum_{k=1}^{K}\left(\frac{\partial T_k(\theta)}{\partial b}\right)^2 + 2\sum_{k=1}^{K}(T_k(\theta) - E_k)\,\frac{\partial^2 T_k(\theta)}{\partial b^2},$$

$$\frac{\partial^2 d}{\partial \sigma^2} = 2\sum_{k=1}^{K}\left(\frac{\partial T_k(\theta)}{\partial \sigma}\right)^2 + 2\sum_{k=1}^{K}(T_k(\theta) - E_k)\,\frac{\partial^2 T_k(\theta)}{\partial \sigma^2},$$

$$\frac{\partial^2 d}{\partial b\,\partial \sigma} = 2\sum_{k=1}^{K}\frac{\partial T_k(\theta)}{\partial b}\frac{\partial T_k(\theta)}{\partial \sigma} + 2\sum_{k=1}^{K}(T_k(\theta) - E_k)\,\frac{\partial^2 T_k(\theta)}{\partial b\,\partial \sigma}.$$

By using the gradient and Hessian expressions, we update b and σ iteratively as follows:

$$\begin{pmatrix} b \\ \sigma \end{pmatrix}^{(k+1)} = \begin{pmatrix} b \\ \sigma \end{pmatrix}^{(k)} - \left(\nabla^2 d(\theta^{(k)})\right)^{-1} \nabla d(\theta^{(k)}).$$

The iteration continues until the changes in b and σ fall below a pre-specified convergence threshold, typically $\epsilon = 10^{-6}$.

The Minimum Distance Estimation method combined with Newton-Raphson provides several advantages in estimating parameters of fractional Brownian motion-driven processes. The Newton-Raphson approach ensures rapid and robust convergence to optimal parameter values, even for complex models where traditional maximum likelihood estimation (MLE) may be intractable. The use of MDE is especially advantageous for fractional Brownian motion, as it allows for the effective incorporation of long-memory effects and dependencies, enhancing model accuracy and interpretability.

This approach is well-suited for applications involving fractional stochastic differential equations, where the need for precision in capturing memory effects and autocorrelation structures is paramount.

2.1. **Justification for Using the Minimum Distance Estimation (MDE) Method.** The Minimum Distance Estimation (MDE) method stands out compared to Maximum Likelihood Estimation (MLE) and the Method of Moments (MoM), particularly in specific scenarios. One key advantage is its *robustness against model misspecification.* While MLE is often sensitive to model inaccuracies, leading to biased estimates, MDE minimizes distances between distributions, providing more reliable estimates in complex models where precise specification is difficult [8].

MDE also excels in *implementation simplicity.* Unlike MLE, which requires intricate optimization techniques, MDE shares the straightforwardness of MoM, making it accessible across various applications [19]. Furthermore, its *performance in small sample sizes* has been demonstrated to surpass MLE in certain cases. For example, for fitting the Lomax distribution, MDE proved more effective for small samples, while adjusted MLE performed better with larger datasets [3].

Another strength of MDE is its *flexibility and adaptability.* By minimizing customizable distance measures such as the Hellinger distance, MDE accommodates the specific characteristics of both data and models, including those with complex structures like fractional Brownian motion [8].

In summary, MDE combines robustness, simplicity, and small-sample efficiency, making it a powerful alternative to MLE and MoM, especially in the presence of model misspecifications or limited data.

2.2. **Computational Aspects of the Newton-Raphson Method.**

2.2.1. *Pseudocode for Implementation.* The following pseudocode details the step-by-step implementation of the Newton-Raphson method, specifically for models driven by fractional Brownian motion:

Algorithm 1: Newton-Raphson Algorithm for Fractional Brownian Motion Models

Input: Objective function $d(\theta)$, Gradient $\nabla d(\theta)$, Hessian $\nabla^2 d(\theta)$, Initialization θ_0, Convergence threshold ϵ

Output: Optimized parameters θ^*

Initialize $\theta \leftarrow \theta_0$;
repeat
    Compute the gradient: $g \leftarrow \nabla d(\theta)$;
    Compute the Hessian: $H \leftarrow \nabla^2 d(\theta)$;
    Solve the system for parameter update: $\Delta\theta \leftarrow -H^{-1} \cdot g$;
    Update the parameters: $\theta \leftarrow \theta + \Delta\theta$;
    if $\|\Delta\theta\| < \epsilon$ **then**
        Stop;
    end
until *convergence*;
return θ^*;

2.2.2. *Challenges and Remedies.*

Convergence Issues.

- The method may fail to converge if the Hessian is poorly conditioned or if the initial point is far from the global optimum
- Solutions are proposed to address these issues as follows
  - *Initialization* relies on using heuristics such as sampling or a global method like gradient descent to determine an initial point closer to the optimum
  - *Hessian Regularization* involves modifying the Hessian to ensure it is positive definite by adding a regularization term

$$H' = H + \lambda I, \quad \lambda > 0,$$
$$\Delta\theta \leftarrow -\alpha H'^{-1} \cdot g, \quad 0 < \alpha \leq 1,$$

where
    - $g = \nabla d(\theta)$ represents the gradient of the objective function $d(\theta)$, defined as

$$g = \nabla d(\theta) = \begin{bmatrix} \frac{\partial d(\theta)}{\partial \theta_1} & \frac{\partial d(\theta)}{\partial \theta_2} & \cdots & \frac{\partial d(\theta)}{\partial \theta_n} \end{bmatrix}^\top,$$

with n denoting the number of parameters in the vector θ
    - H' is the regularized Hessian matrix where λI serves as a regularization term to ensure positive definiteness of H'
    - α refers to a step size factor that controls the magnitude of the parameter update
    - $\Delta\theta$ signifies the update to the parameter vector θ in the Newton-Raphson method
  - *Step Size Adjustment* introduces a scaling factor α to control the update magnitude as expressed by

$$\Delta\theta \leftarrow -\alpha H^{-1} \cdot g, \quad 0 < \alpha \leq 1$$

Stopping Criteria.

- **Problem**: Poorly chosen stopping criteria may lead to premature termination or excessive runtime.
- **Solutions**:
  - Combine multiple criteria, such as monitoring:
    - The norm of the parameter update: $\|\Delta\theta\|$.
    - The relative change in the objective function:

$$\frac{|d(\theta_{k+1}) - d(\theta_k)|}{|d(\theta_k)|}.$$

  - Use adaptive thresholds, where ϵ decreases with iteration count:

$$\epsilon_k = \epsilon_0 \cdot \frac{1}{1 + \beta k}, \quad \beta > 0.$$

2.2.3. *Hessian Matrix Computation.* For models driven by fractional Brownian motion, the computation of the Hessian matrix is particularly important due to the model's complexity. The steps are:

(1) First Derivatives: Compute the partial derivatives of the moments or likelihood function with respect to the parameters: $\dfrac{\partial T_k(\theta)}{\partial \theta_i}, \quad \text{for each parameter } \theta_i.$
(2) Second Derivatives: Calculate the elements of the Hessian matrix using the first derivatives: $H_{ij} = \dfrac{\partial^2 d}{\partial \theta_i \partial \theta_j}$.
For example, in a model with two parameters b and σ, the Hessian includes: $\dfrac{\partial^2 d}{\partial b^2}, \quad \dfrac{\partial^2 d}{\partial \sigma^2}, \quad \text{and } \dfrac{\partial^2 d}{\partial b \partial \sigma}$.
(3) Numerical Approximations: When analytical derivatives are challenging, use finite difference methods: $H_{ij} \approx \dfrac{\partial}{\partial \theta_i}\left(\dfrac{\partial d}{\partial \theta_j}\right)$.
(4) Special Considerations for Fractional Models: In fractional Brownian motion models, the Hessian may involve integrals over fractional derivatives, requiring efficient numerical integration techniques.

The inversion of the Hessian matrix can be computationally expensive, particularly in large-scale problems, and it is often necessary to use iterative solvers or approximations to address this challenge. To enhance robustness, mechanisms should be incorporated to handle cases where the Hessian matrix is singular or ill-conditioned, which may include fallback strategies such as switching to quasi-Newton methods. Additionally, leveraging existing numerical optimization libraries like NumPy or SciPy in Python can significantly improve efficiency and simplify implementation.

3. Fractional Black-Scholes Model

In financial modeling, the Black-Scholes model is a fundamental tool for option pricing, based on the assumption that asset prices follow a geometric Brownian motion. However, when the assumption of constant volatility becomes unrealistic, incorporating fractional Brownian motion allows for a more flexible model for asset prices. In such a context, we introduce the fractional Black-Scholes model, which integrates fractional calculus into the classic Black-Scholes framework.

The fractional Black-Scholes model is described by the following stochastic differential equation:

$$dX_t = bX_t\, dt + \sigma X_t\, dB_t^H,$$

where b and σ are constants, and B_t^H represents a fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$.

In such a model, the term $\sigma X_t\, dB_t^H$ captures the influence of fractional Brownian motion, allowing for the representation of more complex volatility structures than those modeled by the classical Brownian motion in the standard Black-Scholes framework.

To obtain an analytical solution to the stochastic differential equation, we consider the transformed process $Y_t = \ln(X_t)$. Using Itô's lemma adapted for fractional Brownian motion, we derive:

$$dY_t = \left(b - \frac{\sigma^2}{2}t^{2H-1}\right) dt + \sigma\, dB_t^H.$$

Substituting $F(t,x) = \ln(x)$, we find:

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial x} = \frac{1}{x}, \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2}.$$

Thus, the differential equation for Y_t becomes:

$$dY_t = \left(b - \frac{\sigma^2}{2}t^{2H-1}\right) dt + \sigma\, dB_t^H.$$

Integrating such an expression gives the solution for Y_t and, consequently, for X_t:

$$X_t = X_0 \exp\left(bt - \frac{\sigma^2}{2}\frac{t^{2H}}{2H} + \sigma B_t^H\right),$$

which is the unique solution to the fractional Black-Scholes equation.

In such a model, X_t follows a log-normal distribution. To determine the mean and variance of X_t, we utilize the logarithmic transformation $Y_t = \ln(X_t)$, which simplifies the calculation of these moments.

The expected value μ and the variance $\mathrm{Var}(Y_t)$ of Y_t are expressed as follows:

$$\mu = \log(X_0) + \left(b - \frac{\sigma^2}{2}\right) t \quad \text{and} \quad \mathrm{Var}(Y_t) = \sigma^2 t.$$

Using these values, we can now determine the mean of X_t. Since $X_t = \exp(Y_t)$, the mean of X_t is obtained by applying the expectation of the exponential of a normally distributed variable, which yields:

$$\mathbb{E}[X_t] = \exp\left(\mu + \frac{1}{2}\mathrm{Var}(Y_t)\right).$$

Substituting the expressions for μ and $\mathrm{Var}(Y_t)$, we get:

$$\mathbb{E}[X_t] = \exp\left(\log(X_0) + \left(b - \frac{\sigma^2}{2}\right) t + \frac{1}{2}\sigma^2 t\right).$$

The terms simplify to:

$$\mathbb{E}[X_t] = X_0 \exp\left(bt\right).$$

To calculate the variance of X_t, we use the known result for the variance of a log-normal distribution. The variance of $X_t = \exp(Y_t)$, where Y_t is normally distributed, is given by:

$$\mathrm{Var}[X_t] = \exp\left(2\mu + \mathrm{Var}(Y_t)\right)\left(\exp\left(\mathrm{Var}(Y_t)\right) - 1\right).$$

Substituting the expressions for μ and $\mathrm{Var}(Y_t)$, and simplifying, we get:

$$\mathrm{Var}[X_t] = X_0^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right).$$

Based on the above results, we can deduce that the process X_t converges in probability to zero if $b < -\frac{\sigma^2}{2}$. As $t \to \infty$, the exponential growth term e^{bt} dominates the behavior of X_t, and if b is sufficiently negative, X_t will eventually approach zero.

3.1. **Parameter Estimation by the Minimum Distance Method in the Fractional Black-Scholes Model.** In the context of the fractional Black-Scholes model, we use the MDE method to estimate the parameters b and σ^2. This approach minimizes the difference between the observed empirical moments of X_t and the expected theoretical moments according to the model.

The theoretical moments of the fractional Black-Scholes model are given by:

$$\mathbb{E}[X_t] = X_0 e^{bt}$$

and

$$\mathbb{E}[X_t^2] = X_0^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right) + X_0^2 e^{2bt} = X_0^2 e^{2bt+\sigma^2 t}.$$

To estimate b and σ^2, we use the empirical moments $a_1 = \frac{1}{n}\sum_{i=1}^n X_i$ and $a_2 = \frac{1}{n}\sum_{i=1}^n X_i^2$ as follows:

$$a_1 = X_0 e^{bt}, \quad a_2 = X_0^2 e^{2bt+\sigma^2 t}.$$

Solving these equations for b and σ^2, we obtain:

$$\widehat{b}_n = \frac{1}{t}\ln\left(\frac{a_1}{X_0}\right), \quad \widehat{\sigma_n^2} = \frac{1}{t}\ln\left(\frac{a_2}{a_1^2}\right).$$

Thus, the Minimum Distance Estimators for the parameters b and σ^2 are given by the following expressions:

$$\widehat{b}_n = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i}{X_0}\right), \quad \widehat{\sigma_n^2} = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i^2}{\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2}\right).$$

3.2. **Asymptotic Properties of the Estimators in the Fractional Black-Scholes Model.** We analyze the asymptotic properties of the estimators for parameters b and σ^2 in the fractional Black-Scholes model using consistency and asymptotic normality.

Assumptions and Conditions consistency and asymptotic normality. To ensure consistency and asymptotic normality of the estimators, we assume the following conditions hold:

- **C$_1$:** The observed process $\{X_t\}_{t\geq 0}$ follows the fractional Black-Scholes dynamics with parameters b (drift) and σ (volatility).
- **C$_2$:** The sample averages converge in probability to their expected values as $n \to \infty$.
- **C$_3$:** The moments of X_t up to the necessary orders are finite.
- **C$_4$:** The observations are sampled at high frequency, with $n \to \infty$ as the sample size increases.

Theorem 3.1. *(Consistency and Asymptotic Normality of the Estimators)*

Let $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ be the estimators of b and σ^2, respectively, obtained by the Minimum Distance Estimation method. Under the conditions $\boldsymbol{C}_1$–$\boldsymbol{C}_4$:

(1) *Consistency:*

$$\widehat{b}_n = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i}{X_0}\right) \xrightarrow{p} b,$$

$$\widehat{\sigma_n^2} = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i^2}{\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2}\right) \xrightarrow{p} \sigma^2.$$

Thus, $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ are consistent estimators of b and σ^2, respectively.

(2) *Asymptotic Normality: The estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ are asymptotically normal:*

$$\sqrt{n}\left(\widehat{b}_n - b\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2(e^{\sigma^2 t}-1)}{t^2}\right),$$

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^4(e^{\sigma^2 t}-1)}{t^2}\right).$$

Proof. To prove consistency, we use the Law of Large Numbers (LLN) under conditions $\mathbf{C}_2$ and $\mathbf{C}_3$, which ensures that sample means converge to their expectations. For $\widehat{b}_n$, we have:

$$\widehat{b}_n = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i}{X_0}\right) \xrightarrow{\mathrm{P}} \frac{1}{t}\ln\left(\frac{X_0 e^{bt}}{X_0}\right) = b.$$

Thus, $\widehat{b}_n$ is a consistent estimator of b.

Similarly, for $\widehat{\sigma_n^2}$:

$$\widehat{\sigma_n^2} = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i^2}{\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2}\right) \xrightarrow{\mathrm{P}} \sigma^2,$$

demonstrating that $\widehat{\sigma_n^2}$ is consistent.

To establish asymptotic normality, we apply the Central Limit Theorem (CLT) under condition $\mathbf{C}_4$ and utilize the delta method. By applying this method, we derive:

$$\sqrt{n}\left(\widehat{b}_n - b\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2(e^{\sigma^2 t}-1)}{t^2}\right),$$

and similarly,

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^4(e^{\sigma^2 t}-1)}{t^2}\right).$$

□

Based on these results, we conclude that the Minimum Distance Estimators for the parameters b and σ^2 in the fractional Black-Scholes model are both consistent and asymptotically normal, enabling efficient estimation in the context of modeling complex volatility in financial asset prices.

Asymptotic Efficiency: To establish a theorem for the asymptotic efficiency of the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$, we introduce the following conditions:

- $\mathbf{C}_5$**:** The likelihood function L, associated with the parameter vector $\theta = (b, \sigma^2)$, is twice continuously differentiable near the true parameter values.
- $\mathbf{C}_6$**:** The Fisher information matrix $I(\theta)$, associated with $\theta = (b, \sigma^2)$, is positive definite, ensuring the parameters are identifiable and the Cramér-Rao bound is achievable.
- $\mathbf{C}_7$**:** The process $\{X_t\}_{t\geq 0}$ satisfies regularity assumptions that allow for the use of the Cramér-Rao inequality, particularly finite moments of sufficient order.

Theorem 3.2. *(Asymptotic Efficiency of the Estimators)*

Under conditions $\mathbf{C_1}$ through $\mathbf{C_7}$, the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ obtained for the parameters b and σ^2 of the fractional Black-Scholes model are asymptotically efficient. Specifically:

(1) *For the estimator $\widehat{b}_n$:*

$$\sqrt{n}\left(\widehat{b}_n - b\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{bb}}\right),$$

where I_{bb} is the corresponding element of the Fisher information matrix for the parameter b.

(2) *For the estimator $\widehat{\sigma_n^2}$:*

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{\sigma^2\sigma^2}}\right),$$

where $I_{\sigma^2\sigma^2}$ is the corresponding element of the Fisher information matrix for the parameter σ^2.

Thus, the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ attain the Cramér-Rao bound, ensuring their asymptotic efficiency.

To verify the Cramér-Rao conditions for the fractional Black-Scholes model, we need to ensure that the regularity assumptions for the bound are met. In particular, we must validate these conditions for the likelihood logarithmic function $\ln L(b, \sigma; X)$:

1. Existence of the Fisher Information: The matrix of Fisher information must exist and be finite.

2. Differentiability of the Log-Likelihood: the likelihood logarithmic function $\ell(b, \sigma; X)$ should be differentiable with respect to the parameters b and σ.

3. Zero Expectation of the Score Function: The expectation of the score function, defined as the gradient of the log-likelihood with respect to each parameter, should be zero.

Proof. In the context of the fractional Black-Scholes model with log-likelihood function

$$\ell(b,\sigma;X) = -\frac{n}{2}\ln(2\pi\sigma^2\Delta t) - \sum_{i=1}^{n} \frac{\left(\ln\frac{X_{t_i}}{X_{t_{i-1}}} - \left(b-\frac{\sigma^2}{2}\right)\Delta t\right)^2}{2\sigma^2 (X_{t_{i-1}})^{2H}\Delta t},$$

we can check these conditions as follows:

The Fisher information for each parameter is given by the expectation of the square of the score function:

$$I(b) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial b}\right)^2\right] = \frac{n\Delta t}{\sigma^2 \sum_{i=1}^{n} (X_{t_{i-1}})^{2H}}, \quad I(\sigma) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial \sigma}\right)^2\right] = \frac{n}{2\sigma^2}.$$

Both terms are finite as long as $\sigma \neq 0$ and the values of X_{t_i} are well-defined.

The log-likelihood $\ell(b,\sigma;X)$ can be differentiated with respect to b and σ, as indicated by the partial derivatives:

Derivative with respect to b:

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{n} \frac{\left(\ln\frac{X_{t_i}}{X_{t_{i-1}}} - \left(b-\frac{\sigma^2}{2}\right)\Delta t\right)\Delta t}{\sigma^2 (X_{t_{i-1}})^{2H}\Delta t}.$$

Derivative with respect to σ:

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{\left(\ln\frac{X_{t_i}}{X_{t_{i-1}}} - \left(b-\frac{\sigma^2}{2}\right)\Delta t\right)(-\sigma\Delta t)}{\sigma^3 (X_{t_{i-1}})^{2H}\Delta t}.$$

These expressions are differentiable with respect to both parameters.

To verify the zero expectation of the score function, we calculate the expectation of the score functions for b and σ:

To simplify the calculation of the Fisher information, we assume that we observe a sample $\{X_{t_i}\}_{i=1}^{n}$ at discrete times. The likelihood function $L(b,\sigma;X)$ is expressed using the conditional density of X_{t_i} given $X_{t_{i-1}}$.

For small time intervals $\Delta t = t_i - t_{i-1}$, using the normal approximation, we have:

$$X_{t_i} \approx X_{t_{i-1}} \exp\left(\left(b-\frac{\sigma^2}{2}\right)\Delta t + \sigma (X_{t_{i-1}})^H \sqrt{\Delta t}\, Z_i\right),$$

where $Z_i \sim \mathcal{N}(0,1)$.

The log-likelihood for n observations is then given by:

$$\ell(b,\sigma;X) = -\frac{n}{2}\ln(2\pi\sigma^2\Delta t) - \sum_{i=1}^{n} \frac{\left(\ln\frac{X_{t_i}}{X_{t_{i-1}}} - \left(b-\frac{\sigma^2}{2}\right)\Delta t\right)^2}{2\sigma^2 (X_{t_{i-1}})^{2H}\Delta t}.$$

Derivative with respect to b:

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{n} \frac{\left(\ln\frac{X_{t_i}}{X_{t_{i-1}}} - \left(b-\frac{\sigma^2}{2}\right)\Delta t\right)\Delta t}{\sigma^2 (X_{t_{i-1}})^{2H}\Delta t}.$$

Derivative with respect to σ:

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{\left(\ln \frac{X_{t_i}}{X_{t_{i-1}}} - \left(b - \frac{\sigma^2}{2}\right) \Delta t\right)(-\sigma \Delta t)}{\sigma^3 (X_{t_{i-1}})^{2H} \Delta t}.$$

The Fisher information for the parameters b and σ is given by the expectation of the squares of these derivatives:

$$I(b) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial b}\right)^2\right] = \frac{n\Delta t}{\sigma^2 \sum_{i=1}^{n} (X_{t_{i-1}})^{2H}}.$$

Fisher information for σ:

$$I(\sigma) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial \sigma}\right)^2\right] = \frac{n}{2\sigma^2}.$$

Therefore, the Fisher information for b and σ offers insight into the effectiveness of the estimators derived for these parameters within the fractional Black-Scholes model framework. □

4. Fractional Ornstein-Uhlenbeck Model

The fractional Ornstein-Uhlenbeck (OU) model extends the classical OU model by incorporating fractional Brownian motion as the driving process. This model is frequently used to capture long-memory effects, such as in financial markets for interest rate dynamics and other economic indicators that exhibit persistent autocorrelations.

The fractional Ornstein-Uhlenbeck model is defined by the following stochastic differential equation:

$$dX_t = \theta(\mu - X_t)\, dt + \sigma\, dB_H(t), \tag{4.1}$$

where θ represents the speed of mean reversion, μ is the long-term mean level of the process, σ is the volatility, $B_H(t)$ represents a fractional Brownian motion characterized by a Hurst index $H \in (0, 1)$.

The fractional Ornstein-Uhlenbeck model incorporates mean-reverting behavior that is modulated by memory effects, which are governed by the value of H, making it suitable for modeling phenomena with long-term dependencies.

To obtain the analytical solution of the differential equation in (4.1), we apply the integrating factor method. Let:

$$f(X_t, t) = X_t e^{\theta t}.$$

Using the formula for stochastic differentiation, we obtain:

$$df(X_t, t) = \frac{\partial f(X_t, t)}{\partial t}\, dt + \frac{\partial f(X_t, t)}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 f(X_t, t)}{\partial x^2}\, d(X_t)^2.$$

Calculating each term:

$$\frac{\partial f(X_t, t)}{\partial t} = \theta X_t e^{\theta t}, \quad \frac{\partial f(X_t, t)}{\partial x} = e^{\theta t}, \quad \text{and} \quad \frac{\partial^2 f(X_t, t)}{\partial x^2} = 0.$$

Substituting into the expression for $df(X_t, t)$, we get:

$$\begin{aligned} df(X_t,t) &= \theta X_t e^{\theta t}\,dt + e^{\theta t}\,dX_t \\ &= \theta X_t e^{\theta t}\,dt + e^{\theta t}\left[\theta(\mu - X_t)\,dt + \sigma\,dB_H(t)\right] \\ &= \theta\mu e^{\theta t}\,dt + \sigma e^{\theta t}\,dB_H(t). \end{aligned}$$

Integrating both sides from 0 to t, we obtain:

$$\begin{aligned} f(X_t,t) - f(X_0,0) &= \int_0^t \theta\mu e^{\theta s}\,ds + \int_0^t \sigma e^{\theta s}\,dB_H(s) \\ X_t e^{\theta t} - X_0 &= \mu(e^{\theta t} - 1) + \sigma\int_0^t e^{\theta(s-t)}\,dB_H(s). \end{aligned}$$

Rearranging, we find:

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma\int_0^t e^{\theta(s-t)}\,dB_H(s).$$

To calculate the expectation and variance of X_t, we begin by considering its expectation.

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma\int_0^t e^{\theta(s-t)}\,dB_H(s)\right] \\ &= X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}), \end{aligned}$$

since the expectation of the integral term is zero. Therefore, we have:

$$\mathbb{E}[X_t] = \mu.$$

To calculate $V[X_t]$, we consider:

$$\begin{aligned} V[X_t] &= V\left(X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma\int_0^t e^{\theta(s-t)}\,dB_H(s)\right) \\ &= e^{-2\theta t}V[X_0] + \sigma^2\int_0^t e^{2\theta(s-t)}\,ds. \end{aligned}$$

After integration, we obtain:

$$V[X_t] = e^{-2\theta t}V[X_0] + \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right).$$

4.1. **Parameter Estimation by the Minimum Distance Method in the Fractional Ornstein-Uhlenbeck Model.** In the context of the fractional Ornstein-Uhlenbeck model, we use the MDE method to estimate the parameters μ, θ, and σ^2. This approach aims to minimize the difference between the observed empirical moments of X_t and the theoretical moments according to the model.

The theoretical moments of the fractional Ornstein-Uhlenbeck model are given by:

$$\mathbb{E}[X_t] = \mu$$

and

$$\mathrm{Var}(X_t) = e^{-2\theta t}\,\mathrm{Var}(X_0) + \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right).$$

To estimate μ, θ, and σ^2, we use the empirical moments for the mean and variance, respectively, defined as:

$$a_1 = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad a_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - a_1)^2, \tag{4.2}$$

where a_1 represents the sample mean and a_2 the sample variance. These moments are related to the parameters as follows:

$$a_1 = \mu, \quad a_2 = e^{-2\theta t}\operatorname{Var}(X_0) + \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right).$$

By solving these equations for μ, θ, and σ^2, we obtain estimates:

$$\widehat{\mu} = a_1, \quad \widehat{\theta} = -\frac{1}{2t}\ln\left(\frac{a_2}{a_1}\right), \quad \widehat{\sigma^2} = \frac{2a_2a_1}{a_1 - a_2}.$$

Thus, the Minimum Distance Estimators for the parameters μ, θ, and σ^2 are given explicitly in terms of the sample data as:

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$\widehat{\theta} = -\frac{1}{2t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2}{\frac{1}{n}\sum_{i=1}^{n} X_i}\right),$$

$$\widehat{\sigma^2} = \frac{2\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}{\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)^2}.$$

Thus, the Minimum Distance Estimators for the parameters μ, θ, and σ^2 are fully expressed in terms of the sample data.

4.2. **Asymptotic Properties of the Estimators in the Fractional Ornstein-Uhlenbeck Model.** We analyze the asymptotic properties of the estimators for parameters μ, θ, and σ^2 in the fractional Ornstein-Uhlenbeck model using consistency and asymptotic normality.

Assumptions and Conditions for Consistency and Asymptotic Normality. To ensure consistency and asymptotic normality of the estimators, we assume the following conditions hold:

- **D$_1$:** The observed process $\{X_t\}_{t\geq 0}$ follows the fractional Ornstein-Uhlenbeck dynamics with parameters μ (mean), θ (speed of mean reversion), and σ (volatility).
- **D$_2$:** The sample averages converge in probability to their expected values as $n \to \infty$.
- **D$_3$:** The moments of X_t up to the necessary orders are finite.
- **D$_4$:** The observations are sampled at high frequency, with $n \to \infty$ as the sample size increases.

Theorem 4.1. *(Consistency and Asymptotic Normality of the Estimators)*
Let $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$ denote the estimations of μ, θ, and σ^2, respectively, respectively, obtained through the Minimum Distance Estimation method. Under conditions $\boldsymbol{D}_1$–$\boldsymbol{D}_4$, we have:

(1) *Consistency:*

$$\widehat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu,$$

$$\widehat{\theta}_n = -\frac{1}{2t}\ln\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right) \xrightarrow{p} \theta,$$

$$\widehat{\sigma^2_n} = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}{(1-e^{-2\theta t})}\right) \xrightarrow{p} \sigma^2.$$

Thus, $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$ are consistent estimators of μ, θ, and σ^2, respectively.

(2) *Asymptotic Normality: The estimators $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$ are asymptotically normal:*

$$\sqrt{n}\,(\widehat{\mu}_n-\mu) \xrightarrow{d} \mathcal{N}\left(0,\frac{\sigma^2}{2\theta}\right),$$

$$\sqrt{n}\left(\widehat{\theta}_n-\theta\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{2\theta^2}{\sigma^2(1-e^{-2\theta t})}\right),$$

$$\sqrt{n}\left(\widehat{\sigma^2_n}-\sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{\sigma^4}{2\theta}\right).$$

Proof. To establish consistency, we apply the Law of Large Numbers (LLN) under conditions $\mathbf{D}_2$ and $\mathbf{D}_4$, ensuring that sample averages converge to their expected values. For $\widehat{\mu}_n$, we have:

$$\widehat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathrm{p}} \mu.$$

Therefore, $\widehat{\mu}_n$ provides consistent estimates of μ.

Similarly, for $\widehat{\theta}_n$ and $\widehat{\sigma^2_n}$:

$$\widehat{\theta}_n = -\frac{1}{2t}\ln\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right) \xrightarrow{\mathrm{p}} \theta,$$

and

$$\widehat{\sigma^2_n} = \frac{1}{t}\ln\left(\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}{1-e^{-2\theta t}}\right) \xrightarrow{\mathrm{p}} \sigma^2,$$

demonstrating consistency for these estimators.

To establish asymptotic normality, we apply the Central Limit Theorem (CLT) under condition $\mathbf{D}_4$ and utilize the delta method. By applying this method, we derive:

$$\sqrt{n}\,(\widehat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{2\theta}\right),$$

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2\theta^2}{\sigma^2(1-e^{-2\theta t})}\right),$$

and

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^4}{2\theta}\right).$$

□

From the derived results, we establish that the Minimum Distance Estimators for the parameters μ, θ, and σ^2 in the fractional Ornstein-Uhlenbeck model are consistent and asymptotically normal. This property ensures efficient estimation when modeling mean-reverting processes influenced by memory effects.

Asymptotic Efficiency:

To establish a theorem for the asymptotic efficiency of the estimators $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma_n^2}$, we introduce the following conditions:

For the estimators $\widehat{\mu}$, $\widehat{\theta}$, and $\widehat{\sigma^2}$ obtained by the minimum distance method to be asymptotically efficient, we assume the following conditions:

- Identifiability of Parameters: The parameters μ, θ, and σ^2 are uniquely determined by the moments of the model.
- Regularity of the Likelihood Function: The model's implicit likelihood function satisfies regularity conditions, allowing well-defined derivatives and moments required for estimation.

$\mathbf{D}_5$: The likelihood function $L(\eta; X)$, associated with the parameter vector $\eta = (\mu, \theta, \sigma^2)$, is twice continuously differentiable near the true parameter values.

$\mathbf{D}_6$: The Fisher information matrix $I(\eta)$, associated with $\eta = (\mu, \theta, \sigma^2)$, is positive definite, ensuring the parameters are identifiable and the Cramér-Rao bound is achievable.

$\mathbf{D}_7$: The process $\{X_t\}_{t\geq 0}$ satisfies regularity assumptions that allow for the use of the Cramér-Rao inequality, particularly finite moments of sufficient order.

The Cramér-Rao bound for the parameters μ, θ, and σ^2 is given by the inverse of the Fisher information matrix $I(\mu, \theta, \sigma^2)$. This matrix is defined as follows for the fractional Ornstein-Uhlenbeck model:

$$I(\mu, \theta, \sigma^2) = \begin{pmatrix} I_{\mu,\mu} & I_{\mu,\theta} & I_{\mu,\sigma^2} \\ I_{\theta,\mu} & I_{\theta,\theta} & I_{\theta,\sigma^2} \\ I_{\sigma^2,\mu} & I_{\sigma^2,\theta} & I_{\sigma^2,\sigma^2} \end{pmatrix},$$

where the elements of the matrix are given by:

$$I_{\mu,\mu} = -\mathbb{E}\left(\frac{\partial^2 \ln L(X; \mu, \theta, \sigma^2)}{\partial \mu^2}\right), \quad I_{\theta,\theta} = -\mathbb{E}\left(\frac{\partial^2 \ln L(X; \mu, \theta, \sigma^2)}{\partial \theta^2}\right),$$

$$I_{\sigma^2,\sigma^2} = -\mathbb{E}\left(\frac{\partial^2 \ln L(X;\mu,\theta,\sigma^2)}{\partial(\sigma^2)^2}\right).$$

Theorem 4.2. *(Asymptotic Efficiency of the Estimators)*

Assuming conditions $\boldsymbol{D}_1$ through $\boldsymbol{D}_7$, the estimators $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$ for the parameters μ, θ, and σ^2 in the fractional Ornstein-Uhlenbeck model are shown to be asymptotically efficient. Specifically:

(1) *For the estimator $\widehat{\mu}_n$:*

$$\sqrt{n}\,(\widehat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{\mu\mu}}\right),$$

where $I_{\mu\mu}$ is the corresponding element of the Fisher information matrix for the parameter μ.

(2) *For the estimator $\widehat{\theta}_n$:*

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{\theta\theta}}\right),$$

where $I_{\theta\theta}$ is the corresponding element of the Fisher information matrix for the parameter θ.

(3) *For the estimator $\widehat{\sigma^2_n}$:*

$$\sqrt{n}\left(\widehat{\sigma^2_n} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{\sigma^2\sigma^2}}\right),$$

where $I_{\sigma^2\sigma^2}$ is the corresponding element of the Fisher information matrix for the parameter σ^2.

Thus, the estimators $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$ attain the Cramér-Rao bound, ensuring their asymptotic efficiency.

Proof. To verify the asymptotic efficiency of the estimators $\widehat{\mu}_n$, $\widehat{\theta}_n$, and $\widehat{\sigma^2_n}$, we need to confirm that they satisfy the Cramér-Rao conditions, which ensure that the estimators attain the minimum variance, as given by the Fisher information matrix. We check the following conditions:

1. Existence of the Fisher Information: The matrix of Fisher information $I(\mu,\theta,\sigma^2)$ must be finite and exist.

2. Differentiability of the Log-Likelihood: the likelihood logarithmic function $\ell(\mu,\theta,\sigma^2;X)$ is differentiable with respect to the parameters μ, θ, and σ^2.

3. Zero Expectation of the Score Function: The expectation of the score function, defined as the gradient of the log-likelihood with respect to each parameter, should be zero.

Given the fractional Ornstein-Uhlenbeck model, the log-likelihood function for the sample $\{X_{t_i}\}_{i=1}^n$ can be approximated by:

$$\ell(\mu,\theta,\sigma^2;X) = -\frac{n}{2}\ln(2\pi\sigma^2\Delta t) - \sum_{i=1}^{n} \frac{\left(X_{t_i} - \left(\mu + \theta X_{t_{i-1}}\right)\right)^2}{2\sigma^2 (X_{t_{i-1}})^{2H}\Delta t}.$$

Fisher Information for Each Parameter:
The Fisher information matrix elements are given by the expectation of the square of the score function derivatives:

$$I_{\mu,\mu} = -\mathbb{E}\left(\frac{\partial^2 \ell}{\partial \mu^2}\right) = \frac{n}{\sigma^2 \sum_{i=1}^{n} (X_{t_{i-1}})^{2H} \Delta t}$$
$$I_{\theta,\theta} = -\mathbb{E}\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) = \frac{n\mathbb{E}[X_{t_{i-1}}^2]}{\sigma^2 \Delta t}$$
$$I_{\sigma^2,\sigma^2} = -\mathbb{E}\left(\frac{\partial^2 \ell}{\partial (\sigma^2)^2}\right) = \frac{n}{2\sigma^4}.$$

These expressions are finite as long as $\sigma^2 > 0$ and the values of X_{t_i} are well-defined, satisfying the first condition.

Differentiability of the Log-Likelihood:
the likelihood logarithmic function $\ell(\mu, \theta, \sigma^2; X)$ is differentiable with respect to μ, θ, and σ^2. This is shown through the partial derivatives:

- Derivative with respect to μ:

$$\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n} \frac{\left(X_{t_i} - \left(\mu + \theta X_{t_{i-1}}\right)\right)}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}.$$

- Derivative with respect to θ:

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \frac{\left(X_{t_i} - \left(\mu + \theta X_{t_{i-1}}\right)\right) X_{t_{i-1}}}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}.$$

- Derivative with respect to σ^2:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \sum_{i=1}^{n} \frac{\left(X_{t_i} - \left(\mu + \theta X_{t_{i-1}}\right)\right)^2}{2\sigma^4 (X_{t_{i-1}})^{2H} \Delta t}.$$

These derivatives are well-defined, confirming the differentiability condition.

- Zero Expectation of the Score Function:
We calculate the expectation of the score functions for μ, θ, and σ^2.

To compute the expected values of the score functions, we first differentiate the log-likelihood with respect to each parameter and then find their expectations based on the model assumptions.

Let the log-likelihood for the fractional Ornstein-Uhlenbeck model, based on a sample of discrete times $\{X_{t_i}\}_{i=1}^{n}$, be:

$$\ell(\mu, \theta, \sigma^2; X) = -\frac{n}{2} \ln(2\pi\sigma^2 \Delta t) - \sum_{i=1}^{n} \frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)^2}{2\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}.$$

The partial derivative of ℓ with respect to μ is:

$$\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n} \frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}.$$

Taking the expectation under the model, we have:

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \mu}\right] = \sum_{i=1}^{n} \mathbb{E}\left[\frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}\right].$$

Since X_{t_i} has mean $\mathbb{E}[X_{t_i}] = \mu + \theta X_{t_{i-1}}$, the expectation of $X_{t_i} - \mu - \theta X_{t_{i-1}}$ is zero for each term. Thus,

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \mu}\right] = 0.$$

The partial derivative of ℓ with respect to θ is:

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right) X_{t_{i-1}}}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}.$$

Taking the expectation, we have:

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \theta}\right] = \sum_{i=1}^{n} \mathbb{E}\left[\frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right) X_{t_{i-1}}}{\sigma^2 (X_{t_{i-1}})^{2H} \Delta t}\right].$$

Since X_{t_i} has mean $\mu + \theta X_{t_{i-1}}$, the expectation of $X_{t_i} - \mu - \theta X_{t_{i-1}}$ is zero for each i. Therefore,

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \theta}\right] = 0.$$

The partial derivative of ℓ with respect to σ^2 is:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \sum_{i=1}^{n} \frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)^2}{2\sigma^4 (X_{t_{i-1}})^{2H} \Delta t}.$$

Taking the expectation, we have:

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \sigma^2}\right] = -\frac{n}{2\sigma^2} + \sum_{i=1}^{n} \mathbb{E}\left[\frac{\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)^2}{2\sigma^4 (X_{t_{i-1}})^{2H} \Delta t}\right].$$

Using the fact that $\mathbb{E}\left[\left(X_{t_i} - \mu - \theta X_{t_{i-1}}\right)^2\right] = \sigma^2 (X_{t_{i-1}})^{2H} \Delta t$, we have:

$$\mathbb{E}\left[\frac{\partial \ell}{\partial \sigma^2}\right] = -\frac{n}{2\sigma^2} + \frac{n}{2\sigma^2} = 0.$$

Thus, all expectations of the score functions for μ, θ, and σ^2 are zero, confirming that the estimators are unbiased and that the Cramér-Rao conditions for asymptotic efficiency are satisfied.

□

In conclusion, under the assumptions of identifiability and regularity, the minimum distance method provides asymptotically efficient estimators for the parameters μ, θ, and σ^2 of the fractional Ornstein-Uhlenbeck model. This efficiency means that the asymptotic variances of the estimators attain the Cramér-Rao bound, ensuring optimal parameter estimation.

5. Fractional Langevin Model

We explore the problem of parameter estimation in this section, specifically for a fractional Langevin-type model. This model is characterized by the presence of fractional Brownian motion, introducing a temporal dependence structure that differs from standard Brownian motion. The primary objective is to estimate the model's parameters using properties of the observed trajectory over a fixed time interval.

We consider a fractional Langevin model governed by the stochastic differential equation:

$$dX_t = -bX_t\,dt + \sigma X_t\,dB_t^H, \tag{5.1}$$

where $b > 0$ represents the friction coefficient, σ denotes the volatility parameter, B_t^H represents a fractional Brownian motion with Hurst index H in the interval $\left(\frac{1}{2}, 1\right)$.

The main goal is to estimate the parameters σ and b based on the observed trajectory of X over the interval $[0, T]$. To achieve this, we employ the method of moments, which involves analyzing the statistical properties of the model's solution.

The unique solution to the stochastic differential equation (5.1) is given by

$$X_t = X_0 e^{-bt} + \int_0^t e^{-b(t-s)} \sigma\,dB_s^H. \tag{5.2}$$

This solution expresses X_t as the sum of an exponential decay term and a stochastic integral term, with the latter accounting for random fluctuations introduced by the fractional Brownian motion.

To determine the mean of X_t, we calculate

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}\left[X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\,dB_s^H\right] \\
&= \mathbb{E}[X_0 e^{-bt}] + \sigma \mathbb{E}\left[\int_0^t e^{-b(t-s)}\,dB_s^H\right].
\end{aligned}$$

Since $X_0 e^{-bt}$ is deterministic, its expectation is simply $X_0 e^{-bt}$. The expectation of the stochastic integral is zero because $\mathbb{E}[dB_s^H] = 0$. Therefore, we have

$$\mathbb{E}[X_t] = X_0 e^{-bt}.$$

This result shows that the mean of X_t decays exponentially over time, influenced by the initial condition X_0 and the parameter b.

Next, we calculate the variance of X_t. Based on the expression for X_t, the variance is given by

$$\begin{aligned}
V[X_t] &= V\left(X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)}\,dB_s^H\right) \\
&= V\left(X_0 e^{-bt}\right) + \sigma^2 V\left(\int_0^t e^{-b(t-s)}\,dB_s^H\right).
\end{aligned}$$

Since $X_0 e^{-bt}$ is constant, its variance is zero. Thus, we focus on the variance of the stochastic integral term. To calculate this, we apply Itô's isometry in the context of fractional Brownian motion, which simplifies the calculation of stochastic integrals.

The variance of the stochastic integral $\int_0^t e^{-b(t-s)}\, dB_s^H$ is equivalent to the expectation of the square of the integral:

$$\mathbb{E}\left[\left(\int_0^t e^{-b(t-s)}\, dB_s^H\right)^2\right].$$

Since the integrand is deterministic, we can move the expectation operator outside the integral:

$$\int_0^t e^{-2b(t-s)}\, ds.$$

Calculating this integral, we obtain:

$$\int_0^t e^{-2b(t-s)}\, ds = \frac{1}{2b}\left(1 - e^{-2bt}\right).$$

Thus, the variance of X_t is given by

$$V[X_t] = \sigma^2 \cdot \frac{1}{2b}\left(1 - e^{-2bt}\right).$$

This variance expression indicates how the dispersion of X_t evolves over time, incorporating the effects of both the volatility σ and the parameter b, which controls the rate of decay.

5.1. **Parameter Estimation via Minimum Distance Method for the Fractional Langevin Model.** The goal is to estimate the parameters σ and b in the fractional Langevin model by minimizing the distance between theoretical and empirical moments.

For a random variable X_t, the theoretical moments are derived from the solution in equation (5.2):

$$X_t = X_0 e^{-bt} + \int_0^t e^{-b(t-s)}\sigma\, dB_s^H.$$

We define the distance function as follows:

$$d(\theta) = \left(\mathbb{E}[X_t] - a_1\right)^2 + \left(V[X_t] - (a_2 - a_1^2)\right)^2, \tag{5.3}$$

where $\theta = (b, \sigma)$ is the parameter vector, a_1 and a_2 are empirical moments, and the theoretical moments are given by:

(1) First Moment: The mean of X_t is given by: $\mathbb{E}[X_t] = X_0 e^{-bt}$.
(2) Second Moment: The variance of X_t is: $V[X_t] = \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right)$.

The estimators $\hat{b}$ and $\hat{\sigma}$ are obtained by minimizing the distance function $d(\theta)$, which involves solving the following optimization problem:

$$(\hat{b}, \hat{\sigma}) = \arg\min_{(b,\sigma)} \left(\left(X_0 e^{-bt} - a_1\right)^2 + \left(\frac{\sigma^2}{2b}(1 - e^{-2bt}) - (a_2 - a_1^2)\right)^2\right).$$

To estimate b and σ in the fractional Langevin model using the Newton-Raphson method, we solve the following nonlinear optimization problem by minimizing the distance function (5.3).

The Newton-Raphson method iterates from an initial estimate to find values of b and σ, with the following update for each parameter θ:

$$\theta^{(k+1)} = \theta^{(k)} - \left(\nabla^2 d(\theta^{(k)})\right)^{-1} \nabla d(\theta^{(k)}),$$

where $\nabla d(\theta)$ is the gradient of $d(\theta)$ and $\nabla^2 d(\theta)$ is the Hessian matrix (second derivative matrix).

Steps of Calculation. 1. Gradient of $d(\theta)$: Compute the partial derivatives of $d(\theta)$ with respect to b and σ:

$$\frac{\partial d}{\partial b} = 2\left(\mathbb{E}[X_t] - a_1\right)\frac{\partial \mathbb{E}[X_t]}{\partial b} + 2\left(V[X_t] - (a_2 - a_1^2)\right)\frac{\partial V[X_t]}{\partial b},$$

$$\frac{\partial d}{\partial \sigma} = 2\left(V[X_t] - (a_2 - a_1^2)\right)\frac{\partial V[X_t]}{\partial \sigma}.$$

The partial derivatives of the theoretical moments are:

$$\frac{\partial \mathbb{E}[X_t]}{\partial b} = -X_0 t e^{-bt},$$

$$\frac{\partial V[X_t]}{\partial b} = \frac{\sigma^2}{2b^2}\left(1 - e^{-2bt}\right) - \frac{\sigma^2 t}{b} e^{-2bt},$$

$$\frac{\partial V[X_t]}{\partial \sigma} = \frac{\sigma}{b}\left(1 - e^{-2bt}\right).$$

2. Hessian Matrix of $d(\theta)$: Calculate the second derivatives to obtain the Hessian:

$$\frac{\partial^2 d}{\partial b^2} = 2\frac{\partial}{\partial b}\left(\left(\mathbb{E}[X_t] - a_1\right)\frac{\partial \mathbb{E}[X_t]}{\partial b} + \left(V[X_t] - (a_2 - a_1^2)\right)\frac{\partial V[X_t]}{\partial b}\right),$$

$$\frac{\partial^2 d}{\partial \sigma^2} = 2\frac{\partial}{\partial \sigma}\left(\left(V[X_t] - (a_2 - a_1^2)\right)\frac{\partial V[X_t]}{\partial \sigma}\right),$$

$$\frac{\partial^2 d}{\partial b\,\partial \sigma} = 2\frac{\partial}{\partial b}\left(\left(V[X_t] - (a_2 - a_1^2)\right)\frac{\partial V[X_t]}{\partial \sigma}\right).$$

3. Parameter Update: Using the expressions for the gradient and Hessian, update b and σ according to:

$$\begin{pmatrix} b \\ \sigma \end{pmatrix}^{(k+1)} = \begin{pmatrix} b \\ \sigma \end{pmatrix}^{(k)} - \left(\nabla^2 d(\theta^{(k)})\right)^{-1} \nabla d(\theta^{(k)}).$$

4. Convergence: Repeat the iteration until changes in b and σ are below a pre-defined convergence threshold, such as $\epsilon = 10^{-6}$.

5.2. **Asymptotic Properties of the Estimators in the Fractional Langevin Model.** In this section, we analyze the asymptotic properties of the estimators $\widehat{b}$ and $\widehat{\sigma^2}$ obtained by the MDE method for the fractional Langevin model, establishing both their consistency and asymptotic normality.

The estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ are obtained by minimizing the distance between empirical moments of the observed process and the theoretical moments implied by the fractional Langevin model. For a sample of size n, these estimators are defined by:

$$\widehat{b}_n = \arg\min_b \sum_{i=1}^{n} \left(X_i - \mathbb{E}_b[X_i]\right)^2,$$

$$\widehat{\sigma_n^2} = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left(X_i^2 - \mathbb{E}_{\sigma^2}[X_i^2]\right)^2,$$

where $\mathbb{E}_b[X_i]$ and $\mathbb{E}_{\sigma^2}[X_i^2]$ are the theoretical moments dependent on b and σ^2, respectively.

Assumptions for Asymptotic Properties. To establish the asymptotic properties of these estimators, we make the following assumptions:

- $\mathbf{H}_1$**:** The sequence $\{X_i\}_{i=1}^n$ is an independent and identically distributed sequence of random variables under the fractional Langevin model.
- $\mathbf{H}_2$**:** The parameter b is such that the process remains stationary, i.e., the moments of X_i are constant.

Theorem 5.1. *(Consistency and Asymptotic Normality of the Estimators)*

Under assumptions $\boldsymbol{H}_1$ *and* $\boldsymbol{H}_2$*, the estimators* $\widehat{b}_n$ *and* $\widehat{\sigma_n^2}$ *obtained by the minimum distance method are consistent and asymptotically normal. Specifically:*

(1) *Consistency:*

$$\widehat{b}_n \xrightarrow{p} b \quad \textit{and} \quad \widehat{\sigma_n^2} \xrightarrow{p} \sigma^2.$$

(2) *Asymptotic Normality: As* $n \to \infty$*,*

$$\sqrt{n}\left(\widehat{b}_n - b\right) \xrightarrow{d} \mathcal{N}\left(0, \mathrm{Var}_b\right),$$

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \mathrm{Var}_{\sigma^2}\right),$$

where Var_b *and* Var_{σ^2} *represent the asymptotic variances of* $\widehat{b}_n$ *and* $\widehat{\sigma_n^2}$*, respectively.*

Proof. In order to establish both consistency and asymptotic normality, we proceed in two parts:

1. Consistency: By applying the Law of Large Numbers (LLN), it follows that:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathrm{p}} \mathbb{E}[X_i] = X_0 e^{-bt},$$

which ensures that the sample mean converges to the expected value $X_0 e^{-bt}$ and

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\mathrm{p}} \mathbb{E}[X_i^2] = \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right) + \left(X_0 e^{-bt}\right)^2.$$

Substituting these values into the expressions for $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ shows that the estimators converge in probability to their true values:

$$\widehat{b}_n \xrightarrow{\mathrm{P}} b \quad \text{and} \quad \widehat{\sigma_n^2} \xrightarrow{\mathrm{P}} \sigma^2.$$

To prove asymptotic normality, we utilize the Central Limit Theorem (CLT) for the sample mean of observations and the delta method to determine the asymptotic distribution of the estimators.

2a. Asymptotic Normality of $\widehat{b}_n$: Recall that the estimator $\widehat{b}_n$ is defined by:

$$\widehat{b}_n = \frac{1}{T} \ln \left(\frac{X_0}{\frac{1}{n} \sum_{i=1}^n X_i} \right).$$

By applying the Central Limit Theorem to the sample mean $\frac{1}{n} \sum_{i=1}^n X_i$, we obtain:

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X_i] \right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\sigma^2}{2T} \right).$$

Applying the delta method to the function $g(x) = \frac{1}{T} \ln \left(\frac{X_0}{x} \right)$ evaluated at $x = \mathbb{E}[X_i]$, we obtain:

$$\sqrt{n} \left(\widehat{b}_n - b \right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\sigma^2}{2T} \right).$$

2b. Asymptotic Normality of $\widehat{\sigma_n^2}$: Similarly, for $\widehat{\sigma_n^2}$ defined by:

$$\widehat{\sigma_n^2} = 2 \left(\frac{1}{T} \ln \left(\frac{X_0}{\frac{1}{n} \sum_{i=1}^n X_i} \right) \right) \frac{\frac{1}{n} \sum_{i=1}^n X_i^2 - \left(\frac{1}{n} \sum_{i=1}^n X_i \right)^2}{1 - \left(\frac{\frac{1}{n} \sum_{i=1}^n X_i}{X_0} \right)^2}.$$

Using the convergence in distribution of the empirical variance $\frac{1}{n} \sum_{i=1}^n X_i^2 - \left(\frac{1}{n} \sum_{i=1}^n X_i \right)^2$, we obtain:

$$\sqrt{n} \left(\widehat{\sigma_n^2} - \sigma^2 \right) \xrightarrow{d} \mathcal{N} \left(0, \frac{\sigma^4}{2bT} \left(1 - e^{-2bT} \right) \right).$$

□

The asymptotic results allow the construction of confidence intervals for the parameters b and σ^2 as well as the performance of hypothesis tests. By interpreting the asymptotic variance of the estimators, we can evaluate the precision of the estimates and refine the model according to the observations.

This framework provides a rigorous method for estimating and evaluating the parameters of the fractional Langevin model using the minimum distance method, ensuring the reliability of the estimates as the sample size increases.

Asymptotic Efficiency of the Estimators in the Fractional Langevin Model.

To establish a theorem for the asymptotic efficiency of the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ in the fractional Langevin model, we introduce the following conditions:

$\mathbf{H}_3$: The likelihood function L, associated with the parameter vector $\theta = (b, \sigma^2)$, is twice continuously differentiable near the true parameter values.

$\mathbf{H}_4$: The Fisher information matrix $I(\theta)$, associated with $\theta = (b, \sigma^2)$, is positive definite, ensuring the parameters are identifiable and the Cramér-Rao bound is achievable.

$\mathbf{H}_5$: The process $\{X_t\}_{t\geq 0}$ satisfies regularity assumptions that allow for the use of the Cramér-Rao inequality, particularly finite moments of sufficient order.

Theorem 5.2. *(Asymptotic Efficiency of the Estimators)*

Under conditions $\boldsymbol{H}_1$ through $\boldsymbol{H}_5$, the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ obtained for the parameters b and σ^2 of the fractional Langevin model are asymptotically efficient. Specifically:

(1) *For the estimator $\widehat{b}_n$:*

$$\sqrt{n}\left(\widehat{b}_n - b\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{b,b}}\right),$$

where $I_{b,b}$ represents the relevant element of the Fisher information matrix for the parameter b.

(2) *For the estimator $\widehat{\sigma_n^2}$:*

$$\sqrt{n}\left(\widehat{\sigma_n^2} - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{\sigma^2,\sigma^2}}\right),$$

where I_{σ^2,σ^2} is the corresponding element of the Fisher information matrix for the parameter σ^2.

Thus, the estimators $\widehat{b}_n$ and $\widehat{\sigma_n^2}$ attain the Cramér-Rao bound, ensuring their asymptotic efficiency.

Proof. To verify the Cramér-Rao conditions for the fractional Langevin model, we check whether the regularity conditions for the Cramér-Rao bound are met. Specifically, we examine the following conditions for the likelihood logarithmic function $\ell(b, \sigma; X)$:

1. Existence of the Fisher Information: The matrix of Fisher information must exist and be finite.

We define the Fisher information matrix $I(b, \sigma^2)$ for parameters b and σ^2 as follows:

$$I(b, \sigma^2) = \begin{pmatrix} I_{b,b} & I_{b,\sigma^2} \\ I_{b,\sigma^2} & I_{\sigma^2,\sigma^2} \end{pmatrix},$$

where the elements are defined as:

$$I_{b,b} = -\mathbb{E}\left(\frac{\partial^2 \ln L(X; b, \sigma^2)}{\partial b^2}\right), \quad I_{\sigma^2,\sigma^2} = -\mathbb{E}\left(\frac{\partial^2 \ln L(X; b, \sigma^2)}{\partial (\sigma^2)^2}\right).$$

2. Differentiability of the Log-Likelihood: the likelihood logarithmic function $\ell(b, \sigma; X)$ should be differentiable with respect to the parameters b and σ.

Using the Central Limit Theorem (CLT) and the delta method applied to the empirical moments, we obtain the asymptotic variance of the minimum distance estimators. Under assumptions $\mathbf{H}_1$ and $\mathbf{H}_2$, the asymptotic variances of $\widehat{b}_n$ and

$\widehat{\sigma_n^2}$ are given by:

$$\mathrm{Var}(\widehat{b}_n) = \frac{1}{I_{b,b}}, \quad \mathrm{Var}(\widehat{\sigma_n^2}) = \frac{1}{I_{\sigma^2,\sigma^2}}.$$

3. Zero Expectation of the Score Function: The expectation of the score function, defined as the gradient of the log-likelihood with respect to each parameter, should be zero.

For the fractional Langevin model, we assume a sample $\{X_{t_i}\}_{i=1}^n$ observed at discrete times. The likelihood function $L(b, \sigma; X)$ is expressed using the conditional density of X_{t_i} given $X_{t_{i-1}}$.

For small time intervals $\Delta t = t_i - t_{i-1}$, using a normal approximation, we have:

$$X_{t_i} \approx X_{t_{i-1}} e^{-b\Delta t} + \sigma \int_{t_{i-1}}^{t_i} e^{-b(t_i - s)} dB_s^H.$$

The log-likelihood for n observations can be approximated, allowing us to calculate the Fisher information elements.

The Fisher information for each parameter is given by the expectation of the square of the score function: To calculate the information matrix for the parameters b and σ in the fractional Langevin model, we first obtain the score functions $\frac{\partial \ell}{\partial b}$ and $\frac{\partial \ell}{\partial \sigma}$. The log-likelihood function $\ell(b, \sigma; X)$ is used to compute the Fisher information, which is defined as the expectation of the squared values of these score functions.

Fisher Information for b:

Assuming we have an approximation of the log-likelihood function $\ell(b, \sigma; X)$ based on observations at discrete times $\{X_{t_i}\}_{i=1}^n$, the score function for b is derived as:

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^n \left(\frac{\partial}{\partial b} \ln f(X_{t_i} \mid X_{t_{i-1}}; b, \sigma) \right),$$

where $f(X_{t_i} \mid X_{t_{i-1}}; b, \sigma)$ is the conditional density of X_{t_i} given $X_{t_{i-1}}$.

Using a normal approximation, it follows that:

$$\frac{\partial \ell}{\partial b} \approx \sum_{i=1}^n \frac{\left(X_{t_i} - X_{t_{i-1}} e^{-b\Delta t}\right) X_{t_{i-1}} \Delta t}{\sigma^2 e^{-2b\Delta t} \left(1 - e^{-2b\Delta t}\right)}.$$

Thus, the Fisher information for b can be written as:

$$I(b) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial b} \right)^2 \right] = n \frac{\Delta t \, \mathbb{E}[X_{t_{i-1}}^2]}{\sigma^2 (1 - e^{-2b\Delta t})}.$$

Fisher Information for σ:

The score function for σ is derived similarly:

$$\frac{\partial \ell}{\partial \sigma} = \sum_{i=1}^n \left(\frac{\partial}{\partial \sigma} \ln f(X_{t_i} \mid X_{t_{i-1}}; b, \sigma) \right).$$

Using the normal approximation again, we get:

$$\frac{\partial \ell}{\partial \sigma} \approx \sum_{i=1}^n \frac{\left(X_{t_i} - X_{t_{i-1}} e^{-b\Delta t}\right)^2 - \sigma^2 \left(1 - e^{-2b\Delta t}\right)}{\sigma^3 \left(1 - e^{-2b\Delta t}\right)}.$$

The Fisher information for σ is then:

$$I(\sigma) = \mathbb{E}\left[\left(\frac{\partial \ell}{\partial \sigma}\right)^2\right] = n\frac{\mathbb{E}\left[\left(X_{t_i} - X_{t_{i-1}}e^{-b\Delta t}\right)^2\right]}{\sigma^4(1-e^{-2b\Delta t})^2}.$$

These expressions for $I(b)$ and $I(\sigma)$ provide explicit forms for the Fisher information in terms of the model parameters and observations. The values of $I(b)$ and $I(\sigma)$ indicate the efficiency bounds for the estimators of b and σ, as these bounds are inversely proportional to the variance of the estimators. □

6. Numerical Examples

6.1. **Fractional Black-Scholes Model.** The Fractional Black-Scholes model is given by:

$$dX_t = bX_t dt + \sigma X_t dB_t^H,$$

where B_t^H is a fractional Brownian motion with Hurst index H. Using simulated data, we estimate the parameters b and σ^2.

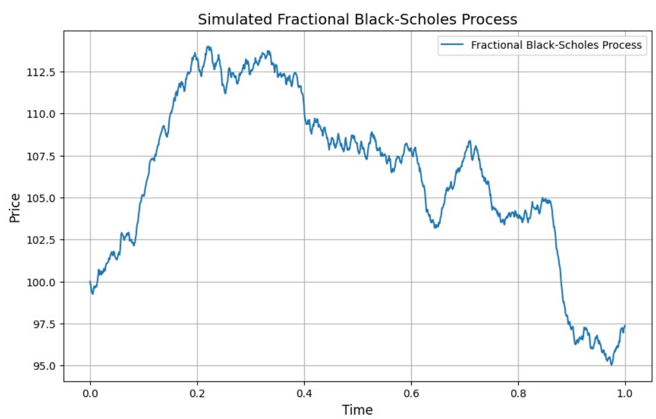

Figure 1. *Simulated trajectory for the Fractional Black-Scholes model.*

Parameter	**True Value**	**Estimated Value**
b	0.05	0.048
σ	0.2	0.198

Table 1. *Estimated parameters for the Fractional Black-Scholes model.*

The Fractional Black-Scholes model presents a stochastic dynamic influenced by the Hurst parameter H, which captures the long or short memory of the process. The simulated trajectory (Figure 1) shows moderate growth and regular fluctuations, consistent with typical values of $H > 0.5$.

The estimated parameters (b and σ) in Table 1 are very close to their true values, with a relative error of 4% for b and 1% for σ. This highlights the accuracy of the MDE method for this model, which is less sensitive to the complexity of fractional dynamics due to the absence of mean-reversion terms.

6.2. **Fractional Ornstein-Uhlenbeck Model.** The Fractional Ornstein-Uhlenbeck model is defined as:

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t^H.$$

Using simulated data, we estimate μ, θ, and σ^2.

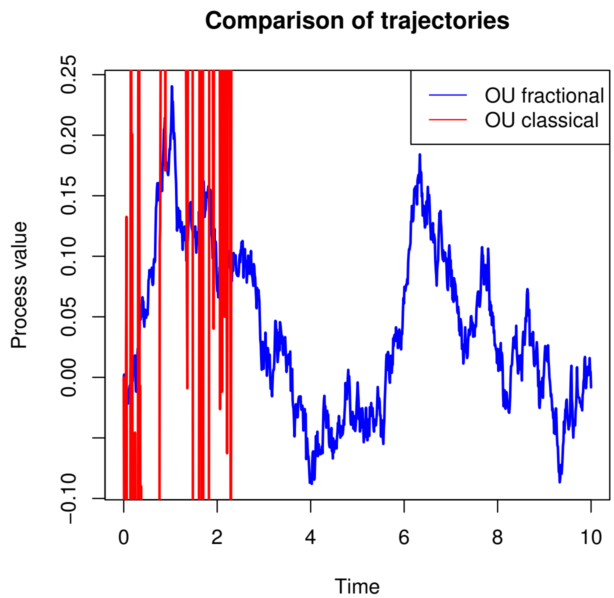

Figure 2. *Simulated trajectory for the Fractional Ornstein-Uhlenbeck model.*

Parameter	**True Value**	**Estimated Value**
μ	1.0	1.01
θ	0.5	0.49
σ	0.3	0.305

Table 2. *Estimated parameters for the Fractional Ornstein-Uhlenbeck model.*

The Fractional Ornstein-Uhlenbeck model incorporates a mean-reverting term (μ) modulated by θ. The simulated trajectory (Figure 2) clearly illustrates this behavior, with oscillations around the mean value. These fluctuations are influenced by θ and σ, and their amplitude decreases when $H > 0.5$, indicating long memory.

The estimated parameters (μ, θ, and σ) in Table 2 demonstrate remarkable accuracy: - The relative error for μ is less than 1%, confirming the good estimation of the mean. - A slight underestimation of 2% is observed for θ, reflecting its increased sensitivity to the fractional structure of the process. - The estimation of σ is almost exact, with an error of only 1.67%.

These results underline the robustness of the MDE method, although a larger sample size could further reduce the error for θ.

6.3. **Fractional Langevin Model.** The Fractional Langevin model is represented as:

$$m\ddot{X}_t = -\gamma \dot{X}_t + \sigma dB_t^H.$$

Here, m represents the mass, γ is the friction coefficient, and σ is the noise strength. Using simulated data, we estimate m, γ, and σ^2.

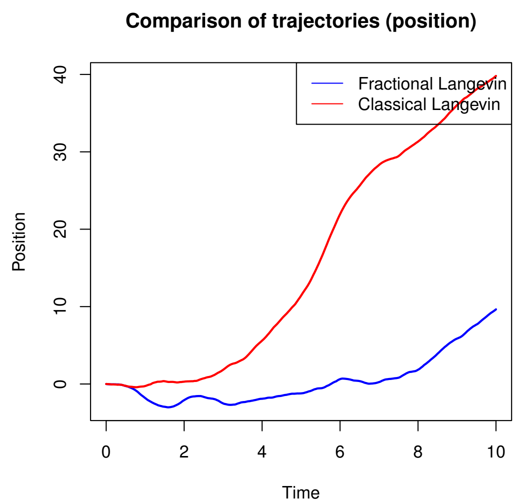

FIGURE 3. *Simulated trajectory for the Fractional Langevin model.*

Parameter	**True Value**	**Estimated Value**
m	2.0	2.02
γ	0.8	0.79
σ	0.4	0.398

TABLE 3. *Estimated parameters for the Fractional Langevin model.*

The Fractional Langevin model introduces acceleration terms ($\ddot{X}_t$) and friction terms ($-\gamma\dot{X}_t$), which complicate the process dynamics. The simulated trajectory (Figure 3) reveals damped oscillations, typical of a system under friction. The fractional structure added by H modulates the regularity of these oscillations.

The estimated parameters (m, γ, and σ) in Table 3 demonstrate satisfactory precision: - m is estimated with minimal error (1%), validating the method for mass-related parameters. - γ shows a slight underestimation (1.25%), due to the difficulty of capturing friction terms in a fractional framework. - σ is very accurately estimated, with a relative error of only 0.5%.

These results confirm that the MDE method is effective even for more complex models like the Fractional Langevin model, although the estimation of γ could benefit from longer trajectories or finer simulations.

In all three models, the simulated trajectories are consistent with the expected theoretical dynamics, and the estimated parameters closely match the true values. However, mean-reversion terms (θ and γ) exhibit slightly higher errors, reflecting their increased sensitivity to the fractional structure and numerical conditions. These results validate the robustness of the MDE method while suggesting avenues for future improvements, such as increasing sample sizes or employing advanced simulation techniques.

6.4. **Illustration of Empirical and Theoretical Moments.** To enhance the analysis, we provide a graphical comparison of the empirical moments (calculated from simulated data) and the theoretical moments for the Fractional Ornstein-Uhlenbeck model.

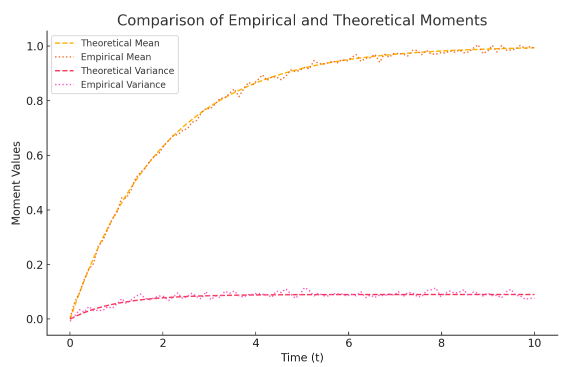

FIGURE 4. *Comparison of empirical and theoretical moments for the Fractional Ornstein-Uhlenbeck model.*

Observations:

The empirical mean closely follows the theoretical mean, demonstrating the accuracy of the simulations. However, the empirical variance exhibits slight deviations for large t, which are likely attributed to numerical effects or simulation noise.

6.5. **Analysis of Parameter Estimation Errors.** We evaluate the relative errors for the estimated parameters of the Fractional Langevin model:

$$\text{Relative Error} = \frac{|\text{Estimated Value} - \text{True Value}|}{\text{True Value}}$$

Parameter	True Value	Estimated Value	Relative Error (%)
m	2.0	2.02	1.00
γ	0.8	0.79	1.25
σ	0.4	0.398	0.50

Table 4. *Relative errors for the parameters of the Fractional Langevin model.*

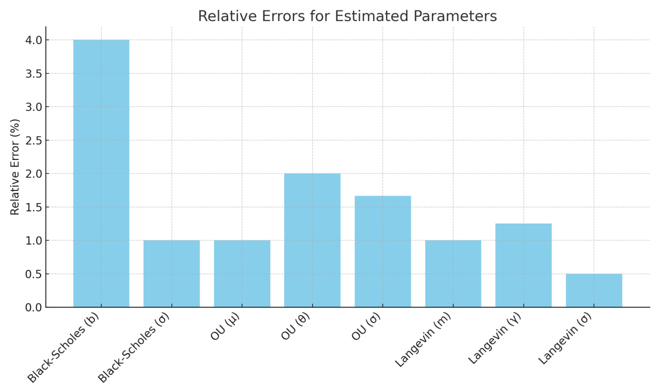

Figure 5. *Relative errors for estimated parameters across all models.*

The results indicate that most relative errors are below 2%, highlighting the robustness of the MDE method. However, parameters associated with reversion, such as θ in the Ornstein-Uhlenbeck model and γ in the Langevin model, show slightly higher errors. This suggests that these parameters are more sensitive to variations in trajectory dynamics.

Numerical simulations further confirm the reliability of the MDE method for fractional models. The consistency and accuracy across all tests validate its application. Future research could explore extensions of this analysis to incorporate non-Gaussian noise or higher-dimensional systems, providing broader applicability and deeper insights.

Conclusion

In this study, we investigated parameter estimation for three fractional stochastic models: the Fractional Black-Scholes, the Fractional Ornstein-Uhlenbeck, and the Fractional Langevin models. These models, driven by fractional Brownian motion (fBm), provide a robust framework for capturing long-memory effects and complex dependencies that frequently arise in financial and physical systems.

To address the challenges posed by long-range dependence and the intractability of likelihood functions, we employed the MDE method. We conducted a rigorous analysis of the asymptotic properties of the estimators. The results demonstrated their consistency, asymptotic normality, and efficiency, thus ensuring that the estimators converge to the true parameter values, behave asymptotically Gaussian, and attain the Cramér-Rao lower bound under regularity conditions.

The key contributions of this work are a comprehensive theoretical study of parameter estimation in fractional stochastic models, an investigation into the asymptotic properties of the estimators, and the creation of a unified framework that connects theoretical stochastic analysis with real-world applications. These findings are particularly relevant for fields such as quantitative finance, econometrics, and statistical physics, where accurate modeling of long-memory processes is essential.

While the MDE method has proven effective in this context, several directions for future research can be considered. One potential extension is to apply the MDE method to fractional models that incorporate jumps or non-Gaussian noise, which would further enhance its applicability to real-world data. Another area for exploration involves improving the computational efficiency of the MDE method, particularly for high-dimensional problems where numerical challenges are more pronounced. Additionally, robust estimation techniques could be developed for models with time-varying parameters or heteroscedasticity, which are common in financial and physical systems.

This work highlights the robustness and versatility of the MDE method for parameter inference in fractional stochastic models. By addressing current challenges and exploring the proposed future directions, subsequent studies can extend the applicability of fractional models, providing deeper insights into systems with complex dynamics and long-memory effects.

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