UNDER TOTALLY GEODETIC VARIETY OF $Geod\mathbb{C}P^n$

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Abstract. $\mathbb{C}P^n$ is a variety whose all geodesics are closed and periodic of the same period $\pi$. We can apply all studies made by M. SARIH in his thesis, see ([1]), in particular $\text{dim}(\text{Geod}\mathbb{C}P^n) = 4n - 2$. In this paper we will find some results concerning the open problem in [1], page 56.

1. Introduction

$S^{2n+1}$ is provided with the canonical riemannian structure induced by the scalar product of $\mathbb{R}^{2n+2}$. $S^1$ is the Lie group; it is the unit circle of $\mathbb{R}^2$.

$u : S^1 \times S^{2n+1} \rightarrow S^{2n+1}$ an action of $S^1$ on $S^{2n+1}$:

- $u$ is differentiable
- $\forall z \in S^1, \ u_z : x \mapsto u(z, x)$ is a diffeomorphism of $S^{2n+1}$
- $z \mapsto u_z$ is a homomorphism of $S^1$ on $\text{Diff}(S^{2n+1})$.

The action $u$ is free if: $\forall z \in S^1, \ u_z$ is without fixed point.

We define the relation: $x \sim y \Leftrightarrow \exists z \in S^1/ y = u(z, x)$; $\sim$ is an equivalence relation and $S^{2n+1}/S^1$ is the quotient set of $S^{2n+1}$ by $\sim$.

If $u$ is free then: [2, 3]

- $S^{2n+1}/S^1$ is a $C^\infty$-variety
- $S^1 \hookrightarrow S^{2n+1} \rightarrow S^{2n+1}/S^1$ is a principal bundle of fiber of type $S^1$

Theorem 1.1. If $S^1$ Operate isometrically and freely on $(S^{2n+1}, \text{can})$, the quotient riemannized by submersion is isometric to $(\mathbb{C}P^n, \text{can})$.

For the proof of this theorem, see [1], page 5.

2. Under Variety of $\mathbb{C}P^n$

Under the conditions of the theorem 1.1, we have $(S^{2n+1}, \text{can})$ is isomorphic to $(\mathbb{C}P^n, \text{can})$. The geodesics of $(\mathbb{C}P^n, \text{can})$ are simply closed and of the same length $\pi$.

Definition 2.1. Let $M$ and $N$ be two connected Riemannian manifolds. An application $f : M \rightarrow N$ of class $C^\infty$, is said to be totally geodesic, if for any geodesic $\gamma$ of $M$, $f \circ \gamma$ is a geodesic of $N$.

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Proposition 2.2. The injection $i : S^p \rightarrow S^q$ is totally geodesic.

Proof. Let $\gamma$ be a geodesic of $S^p$, $\gamma$ is a big circle, so $i(\gamma)$ is a big circle of $S^q$. 

Proposition 2.3. The injection $j : \mathbb{C}P^m \rightarrow \mathbb{C}P^n$ is totally geodesic.

Proof. We have $(S^{2n+1}, can)$ is isomorphic to $(\mathbb{C}P^n, can)$ and $(S^{2m+1}, can)$ is isomorphic to $(\mathbb{C}P^m, can)$. The injection $i : S^{2m+1} \rightarrow S^{2n+1}$ is totally geodesic, hence the result. 

Definition 2.4. We call $C_l$-manifold, a manifold $M$ such that there exists a metric $g$ on $M$ satisfying all geodesics of $M$ are periodical and of same period $l$.

Let $(M, g)$ be a $C_l$-manifold, $\zeta : \mathbb{R} \times UM \rightarrow UM (t, v) \mapsto \zeta^t(v)$ is the geodesic flow.

Proposition 2.5. $S^1 \hookrightarrow UM \rightarrow GeodM$ is a principal bundle of fiber of type $S^1$, and $\dim(\text{Geod}M) = 2n - 2$.

Remark 2.6. $\mathbb{C}P^m$ is a $C\pi$-manifold and $\dim(\text{Geod}\mathbb{C}P^m) = 4n - 2$.

If $M$ is a Riemannian manifold having closed geodesics of the same length $l$ and $N$ is a totally geodesic sub manifold whose all geodesics are also closed and of the same length $l$, then $\text{Geod}N$ is plunged into $\text{Geod}M$, as a submanifold totally geodesic.

Corollary 2.7. $\text{Geod}\mathbb{K}P^m$ is plunged as a totally geodesic submanifold into $\text{Geod}\mathbb{C}P^n$, with $m \leq n$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

3. SOME RESULTS CONCERNING THE OPEN PROBLEM

Open problem:
Is there any geodesic subvariety of $\text{Geod}\mathbb{C}P^n$ other than $\text{Geod}\mathbb{C}P^m$, with $m \leq n$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$?

As an important class of homogeneous spaces, the Grassmann manifolds $G_{\mathbb{C}}(2, n-1)$, have been extensively studied by group-theoretic methods. In this section, we show how their differential geometry. The geodesic in $G_{\mathbb{C}}(2, n-1)$, can be characterized by proprieties of their images in the Euclidean space. We have:

$$\text{Geod}S^2 \rightarrow_{f} \text{Geod}\mathbb{C}P^n \rightarrow_{\pi} G_{\mathbb{C}}(2, n-1)$$

is a bundle of fiber of type $S^2$. The following two lemmas are results of a more general theorem see [4] page 38.

Lemma 3.1. Let $f$ be a continuous application of $M$ in $\text{Geod}\mathbb{C}P^n$, it exists:

- A locally trivial bundle of the type fiber $S^1$; $E \rightarrow_{r} M$
- A continuous application $F$ of $E$ in $U\mathbb{C}P^n$ verifying:

$\begin{array}{c}
E \xrightarrow{F} U\mathbb{C}P^n \\
\downarrow{r} \quad \downarrow{p} \\
M \xrightarrow{f} \text{Geod}\mathbb{C}P^n
\end{array}$

Lemma 3.2. Let $h$ be a continuous application of $N$ in $G_{\mathbb{C}}(2, n-1)$, it exists:
• A locally trivial fiber of the type fiber $S^2; M \to_p N$
• A continuous application $H$ of $M$ in $\text{GeodCP}^n$ verifying:

$$
\begin{array}{ccc}
M & \xrightarrow{H} & \text{GeodCP}^n \\
p & \downarrow & \downarrow \pi \\
N & \xrightarrow{h} & G_C(2, n-1)
\end{array}
$$

(3.2)

Remark 3.3.
• If $M$ is a part of $\text{GeodCP}^n$, and $f$ is the canonical injection of $M$ into $\text{GeodCP}^n$, then $E$ is a part of $U\text{CP}^n$.
• If $N$ is a part of $G_C(2, n-1)$, and $h$ is the canonical injection of $N$ into $G_C(2, n-1)$, then $M$ is a part of $\text{GeodCP}^n$.

Lemma 3.4. If $f$ is an injective application, totally geodesic of $M$ in $\text{GeodCP}^n$, then $F$ transforms a horizontal geodesic into a horizontal geodesic.

Proof. $f$ transforms a geodesic into a geodesic. Let $\gamma$ be a horizontal geodesic of $E$, such that $\gamma(0) = x$ and $\gamma'(0) = u \in \text{Hor}(x)$, then $f \circ r \circ \gamma$ is a horizontal geodesic of $\text{GeodCP}^n$ of initial condition $f(x)$ and $(f \circ r \circ \gamma)'(0) \in \text{Hor}(f(x))$. So, $p \circ F \circ \gamma(t) = p(F(\gamma(t)))$, consequently $F \circ \gamma$ is a horizontal geodesic. □

Lemma 3.5. If $h$ is an injective application, totally geodesic of $N$ in $G_C(2, n-1)$, then $H$ is a $p$-injective application, totally geodesic of $M$ in $\text{GeodCP}^n$.

Proof. We have $h \circ p = \pi \circ H$. Let $x, y \in M$, then $H(x) = H(y) \implies h \circ p(x) = h \circ p(y) \implies p(x) = p(y)$, so $H$ is an $p$-injective application. Let $\gamma$ be a geodesic of $M$, $\forall \epsilon > 0$, $\gamma(\cdot - \epsilon, \epsilon]$ is a sub manifold of $M$, so $H(\gamma(\cdot - \epsilon, \epsilon])$ is a sub manifold of $\text{GeodCP}^n$. We have: $\pi(H(\gamma(\cdot - \epsilon, \epsilon])) = h(p(\gamma(\cdot - \epsilon, \epsilon)))$, i.e., $\forall \epsilon > 0$, $H(\gamma(\cdot - \epsilon, \epsilon])$ is a sub manifold of $\text{GeodCP}^n$, thereby $H$ is totally geodesic. □

Lemma 3.6. Let $M$ is a totally geodesic sub manifold of $N$. If $N$ is a totally geodesic sub manifold of $P$, then $M$ is a totally geodesic sub manifold of $P$.

Proof. We have:

$$
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
& & \xrightarrow{j} P
\end{array}
$$

$(3.3)$

$j \circ i$ is an injective application which turns a geodesic of $M$ into a geodesic of $P$. □

Lemma 3.7. $M$ is generated by a totally geodesic sub manifold of $\text{CP}^n$ if and only if $E$ is a unit bundle.

Proof. $\iff$ If $E = U\text{CP}^k$, then the following two fibrations:

$$
\begin{array}{ccc}
U\text{CP}^k & \xrightarrow{F} & U\text{CP}^n \\
r & \downarrow & \downarrow p \\
M & \xrightarrow{f} & \text{GeodCP}^n
\end{array}
$$

(3.4)
so, \( M \) is generated by a totally geodesic sub manifold of \( \mathbb{C}P^n \).

\( \implies \) If \( M \) is generated by a totally geodesic sub manifold of \( \mathbb{C}P^n \), then there exists \( k \) such that, \( E = U\mathbb{C}P^k \).

\( \square \)

**Lemma 3.8.** \( M \) is generated by a totally geodesic sub manifold of \( \mathbb{C}P^n \) if and only if \( N \) is a Grassmann manifolds.

**Proof.** \( \iff \) If \( N = G_C(2, m - 1) \), then the following two fibrations:

\[
\begin{align*}
G_C \mathbb{C}P^m & \xrightarrow{H} G_C \mathbb{C}P^n \\
G_C(2, m - 1) & \xrightarrow{h} G_C(2, n - 1)
\end{align*}
\]

so, \( M \) is generated by a totally geodesic sub manifold of \( \mathbb{C}P^n \).

\( \implies \) If \( M \) is generated by a totally geodesic sub manifold of \( \mathbb{C}P^n \), then there exists \( m \) such that, \( N = G_C(2, m - 1) \).

\( \square \)

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