

ELLIPTIC EQUATIONS INVOLVING MULTI-SINGULAR POTENTIALS AND CRITICAL NONLINEARITY

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ABSTRACT. Let Ω be a smooth open-bounded domain in $\mathbb{R}^N$ ($N \geq 5$). We deal with the following multi-singular critical elliptic problem

$$\Delta^2 v - \sum_{k=1}^m \frac{\mu_k}{|x - \omega_k|^4} v = G(x)|v|^{2^*-2}v, \quad v \in H_0^2(\Omega)$$

where $\mu_k \in \mathbb{R}$, $2^* := \frac{2N}{N-4}$, $\omega_k \in \Omega$ ($1 \leq k \leq m$) and $G(x)$ is a bounded positive function on $\bar{\Omega}$. By exploiting the impact of the coefficient of the critical nonlinearity, we, by making use of variational method, establish the existence of at least one solution to problem (1.1).

1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 5$). In the present paper, we consider the following problem:

$$\begin{cases} \Delta^2 v - \sum_{k=1}^m \frac{\mu_k}{|x - \omega_k|^4} v = G(x)|v|^{2^*-2}v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\mu_k \in \mathbb{R}$, $2^* := \frac{2N}{N-4}$, $\omega_k \in \Omega$ ($1 \leq k \leq m$) and $G(x)$ is a bounded positive function on $\bar{\Omega}$.

Singular potentials appear across various scientific fields providing models for phenomena in quantum mechanics, chemistry, cosmology, astrophysics, and differential geometry. In quantum chemistry, multi-singular potentials emerge when considering molecular systems consisting of m nuclei of unit charge situated at a finite number of points $\omega_k, k = 1, \dots, m$ and of m electrons, this class of systems are described by the Hartree-Fock model where Coulomb multi-singular potentials appear in correspondence to the interactions between the electrons and the fixed nuclei, we refer to [9, 22].

Problems involving the Laplacian operator with only one singular potential were first studied in [29]. More specifically, it is proven that if $\lambda \in [0, \frac{(N-2)^2}{4})$,

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then the following problem

$$\begin{cases} -\Delta v = \frac{\lambda}{|x|^2} v + |v|^{2^*-1}, & x \in \mathbb{R}^N, \\ v > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } v \in D^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.2)$$

possesses exactly a one-dimensional C^2 -manifold of positive solutions characterized by

$$Z_\lambda = \{w_\epsilon^\lambda(x) := \epsilon^{-\frac{N-2}{2}} w^{(\lambda)}(\epsilon^{-1}x), \epsilon > 0\}$$

where

$$w^{(\lambda)} := \frac{(N(N-2)\nu_\lambda^2)^{\frac{N-2}{4}}}{(|x|^{1-\nu_\lambda}(1+|x|^{2\nu_\lambda}))^{\frac{N-2}{2}}} \text{ and } \nu_\lambda := \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{1/2}.$$

Furthermore, every solution to (1.2) minimizes the related Rayleigh quotient

$$\inf_{v \in D^{1,2}(\mathbb{R}^N \setminus \{0\})} \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 - \lambda \frac{v^2}{|x|^2}) dx}{\left(\int_{\mathbb{R}^N} |v|^{2^*} dx\right)^{\frac{2}{2^*}}} = \frac{\int_{\mathbb{R}^N} (|\nabla w_\epsilon^\lambda|^2 - \lambda \frac{(w_\epsilon^\lambda)^2}{|x|^2}) dx}{\left(\int_{\mathbb{R}^N} |w_\epsilon^\lambda|^{2^*} dx\right)^{\frac{2}{2^*}}} = S \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{\frac{N-1}{N}}$$

where S corresponds to the best possible constant in the Sobolev inequality $S|v|_{L^{2^*}(\mathbb{R}^N)} \leq |v|_{D^{1,2}(\mathbb{R}^N)}^2$.

In recent years, much attention has been paid to the existence of nontrivial solutions to the following problem

$$\begin{cases} -\Delta v - \mu \frac{v}{|x-a|^2} = \lambda v + |v|^{2^*-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $a \in \Omega$, $\mu \in (0, \bar{\mu})$, $\bar{\mu} := \left(\frac{N-2}{2}\right)^2$ and $\lambda \in \mathbb{R}$.

Jannelli in [19] established that when $\mu \in (0, \bar{\mu} - 1]$, the problem (1.3) possesses a positive solution for any $\lambda \in (0, \lambda_1(\mu))$. The same author proved that if $\Omega = B_1(0)$ and $\mu \in (\bar{\mu} - 1, \bar{\mu})$, then there exists $\lambda^* \in (0, \lambda_1(\mu))$, such that (1.3) admits a positive solution if and only if $\lambda \in (\lambda^*, \lambda_1(\mu))$, where $\lambda_1(\mu)$ is the first eigenvalue of the positive operator $-\Delta - \frac{\mu}{|x|^2}$ with Dirichlet boundary condition.

Cao and Peng [7] also investigated the problem (1.3) when $N \geq 7$ and showed that for $\mu \in [0, \bar{\mu} - 4)$, the problem (1.3) possesses at least a pair of sign-changing solutions for any $\lambda \in (0, \lambda_1(\mu))$.

Cao and Han in [6] proved that if $\mu \in \left[0, \bar{\mu} - \left(\frac{N+2}{N}\right)^2\right)$, then (1.3) admits a nontrivial solution for any positive parameter λ . Ferrero and Gazzola in [15] established several results for problem (1.3). Further relevant papers are discussed in [2, 5, 11, 13, 14, 15, 16, 26], and the references therein.

Lately, biharmonic problems involving Rellich-type potential have been extensively treated, we refer to [21, 20, 18]. Ambrosio and Jannelli in [12] established some existence and nonexistence results to the following critical problem

$$\begin{cases} \mathcal{L}_\mu[v] := \Delta^2 v - \mu \frac{v}{|x|^4} = \lambda v + |v|^{2^*-2}v & \text{in } \Omega, \\ v = \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 5$) is a bounded open set with $0 \in \Omega$, $2^* := \frac{2n}{n-4}$, $\bar{\mu}$ is the best constant in Rellich inequality and $\mu \in (0, \bar{\mu})$. In order to accomplish existence

results, the authors explored some crucial properties describing the behavior of the radial solutions to the limit problem $\mathcal{L}_\mu[v] = |v|^{2^*-1}$ in the whole space $\mathbb{R}^n$.

Denote by $D^{2,2}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\Delta \cdot|^2 dx)^{1/2}$.

For any $\mu \in [0, \bar{\mu})$, $\omega \in \mathbb{R}^N$, define

$$R_\mu := \inf_{v \in D^{2,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\Delta v|^2 - \mu \frac{v^2}{|x-\omega|^4}) dx}{(\int_{\mathbb{R}^N} |v|^{2^*} dx)^{\frac{2}{2^*}}}.$$

From [15, 19], R_μ is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that if

$$R_\mu(\Omega) := \inf_{v \in H_0^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\Delta v|^2 - \mu \frac{v^2}{|x-\omega|^4}) dx}{(\int_{\Omega} |v|^{2^*} dx)^{\frac{2}{2^*}}},$$

then $R_\mu(\Omega) = R_\mu(\mathbb{R}^N) = R_\mu$.

The authors in [12] proved that for any $\mu \in [0, \bar{\mu})$, R_μ admits the unique positive minimizers

$$\{z_\epsilon^\mu(x - \omega) = \epsilon^{-\frac{N-4}{2}} U_\mu(\epsilon^{-1}(x - \omega)), \epsilon > 0\}, \quad (1.5)$$

where $U_\mu(x)$ is positive, radially decreasing, radially symmetric, and solves

$$\Delta^2 v - \mu \frac{v}{|x|^4} = v^{2^*-1}, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad v > 0,$$

that fulfills,

$$\int_{\mathbb{R}^N} \left(|\Delta z_\epsilon^\mu(x - \omega)|^2 - \mu \frac{|z_\epsilon^\mu(x - \omega)|^2}{|x - \omega|^4} \right) dx = \int_{\mathbb{R}^N} |z_\epsilon^\mu(x - \omega)|^{2^*} dx = R_\mu^{\frac{N}{4}}.$$

Moreover, there holds that $U_\mu(x) = O_1(|x|^{-\alpha(\mu)})$ at the origin and $U_\mu(x) = O_1(|x|^{-\beta(\mu)})$ at infinity, where $\alpha(\mu) := \phi(\mu)(N-4)/2$, $\beta(\mu) := (2-\phi(\mu))(N-4)/2$ and $\phi : [0, \bar{\mu}] \rightarrow [0, 1]$,

$$\phi(\mu) := 1 - \left(\sqrt{N^2 - 4N + 8 - 4\sqrt{(N-2)^2 + \mu}} \right) / (N-4).$$

In the present work, we consider a biharmonic equation involving multiple Rellich-type potentials and a critical Sobolev exponent. It is worth noting that the main challenge in studying problems like (1.1) is the lack of knowledge regarding the explicit form of minimizers to the best Rellich–Sobolev constant R_μ . However, as in [12] and [21], this difficulty can be overcome since the key tool to perform the required asymptotic expansions is the asymptotic behavior at the origin and infinity of Rellich–Sobolev extremals. On the other hand, the appearance of critical growth creates a lack of compactness, as both embeddings

$$H_0^2(\Omega) \hookrightarrow L^{2^*}(\Omega) \text{ and } H_0^2(\Omega) \hookrightarrow L^2(\Omega, |x - \omega_k|^{-4} dx),$$

are not compact. Consequently, the standard variational approach is not applicable directly. In order to ensure compactness conditions, we need to investigate the effect of the coefficient G and the energy range where the energy functional associated with (1.1) satisfies the $(P.S.)_c$ condition.

Without loss of generality, all through this study, we suppose that there exists an k_0 , $1 \leq k_0 < m$ ($m \geq 2$), such that

$$\mu_1 \leq \cdots \leq \mu_{k_0} < 0 < \mu_{k_0+1} \leq \mu_{k_0+2} \leq \cdots \leq \mu_m, \quad \sum_{k=k_0+1}^m \mu_k < \bar{\mu} := \left(\frac{N^2(N-4)^2}{16} \right).$$

(H₁) There is an $l \in \{k_0 + 1, \dots, m\}$, such that

$$\min \left\{ \frac{R_{\mu_{k_0+1}}^{\frac{N}{4}}}{G(\omega_{k_0+1})^{\frac{N-4}{4}}}, \frac{R_{\mu_{k_0+2}}^{\frac{N}{4}}}{G(\omega_{k_0+2})^{\frac{N-4}{4}}}, \dots, \frac{R_{\mu_m}^{\frac{N}{4}}}{G(\omega_m)^{\frac{N-4}{4}}} \right\} = \frac{R_{\mu_l}^{\frac{N}{4}}}{G(\omega_l)^{\frac{N-4}{4}}},$$

and

$$G(x) - G(\omega_l) = o(|x - \omega_l|^4)$$

in the neighborhood of ω_l .

(H₂) $0 < \mu_l < \mu_* := \frac{1}{16}(N^2 - 16)(N^2 - 8N)$ and $\sum_{k \neq l}^m \frac{\mu_k}{|\omega_l - \omega_k|^4} > 0$ where l is provided in (H₁).

(H₃) There exists an $x_0 \in \Omega \setminus \{\omega_1, \dots, \omega_m\}$ such that $G(x_0) = G_M = \max_{\bar{\Omega}} G(x)$ and

$$G(x) - G(x_0) = o(|x - x_0|^4)$$

as x tends to x_0 . Furthermore $\sum_{k=1}^m \frac{\mu_k}{|x_0 - \omega_k|^4} > 0$.

Our main result is the following theorem.

Theorem 1.1. *Suppose that (H₁)-(H₃) hold. Then problem (1.1) has at least one solution.*

2. EXISTENCE OF SOLUTIONS TO (1.1)

In this section, we will use the Mountain Pass lemma (see [1]) to establish the existence of at least one solution to (1.1). A function $v \in H_0^2(\Omega)$ is said to be a weak solution to problem (1.1) if v satisfies

$$\int_{\Omega} (\Delta v \Delta \phi - \sum_{k=1}^m \frac{\mu_k v \phi}{|x - \omega_k|^4} - G(x) |v|^{2^*-2} v \phi) dx = 0 \quad \forall \phi \in H_0^2(\Omega).$$

Clearly, nontrivial solutions to (1.1) are equivalent to the associated nonzero critical points of the energy functional

$$\mathcal{K}(v) := \frac{1}{2} \int_{\Omega} \left(|\Delta v|^2 - \sum_{k=1}^m \frac{\mu_k v^2}{|x - \omega_k|^4} \right) dx - \mathcal{L}(v), \quad (2.1)$$

where

$$\mathcal{L}(v) := \frac{1}{2^*} \int_{\Omega} G(x) |v|^{2^*} dx, \quad v \in H_0^2(\Omega).$$

Problem (1.1) is related to the following Rellich inequality (see [25]):

$$\int_{\Omega} \frac{v^2}{|x - \omega_k|^4} dx \leq \frac{1}{\bar{\mu}} \int_{\Omega} |\Delta v|^2 dx \quad \forall v \in H_0^2(\Omega),$$

and from $\sum_{k=k_0+1}^m \mu_k < \bar{\mu}$, we obtain that $\left(\int_{\Omega} (|\Delta v|^2 - \sum_{k=1}^m \frac{\mu_k v^2}{|x - \omega_k|^4}) dx \right)^{1/2}$ is an equivalent norm of $H_0^2(\Omega)$.

To establish existence of solution to (1.1), we need to show that $\mathcal{K}$ satisfies $(P.S.)_c$ condition for c in a suitable interval. We start with a lemma concerning the $(P.S.)_c$ condition of $\mathcal{K}$.

Lemma 2.1. *The functional $\mathcal{K}$ fulfills $(P.S.)_c$ condition with*

$$c < c^* := \frac{2}{N} \min \left\{ \frac{R_{\mu_l}^{\frac{N}{4}}}{G(\omega_l)^{\frac{N-4}{4}}}, \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}} \right\}.$$

Proof. Let $\{v_n\} \subset H_0^2(\Omega)$ satisfy

$$\mathcal{K}(v_n) \rightarrow c < c^*, \quad d\mathcal{K}(v_n) \rightarrow 0 \text{ strongly in } H^{-2}(\Omega).$$

Invoking Hardy inequality, we derive that $\|v_n\|_{H_0^2(\Omega)} \leq C$. Thus, up to a subsequence, we may suppose that

$$\begin{aligned} v_n &\rightharpoonup v \text{ weakly in } H_0^2(\Omega); \\ v_n &\rightharpoonup v \text{ weakly in } L^2(\Omega, |x - \omega_k|^{-4} dx) \text{ for } 1 \leq k \leq m; \\ v_n &\rightharpoonup v \text{ weakly in } L^{2^*}(\Omega); \\ v_n &\rightarrow v \text{ a.e. on } \Omega. \end{aligned}$$

Hence $v \in H_0^2(\Omega)$ is a weak solution to (1.1). Applying the concentration compactness principle [23], there exists a subsequence, still denoted by $\{v_n\}$, at most countable set $\mathcal{I}$, a set of distinct points $\{x_i\}_{i \in \mathcal{I}} \subset \Omega \setminus \{\omega_1, \omega_2, \dots, \omega_m\}$, nonnegative real numbers $\widetilde{\lambda}_{x_i}, \widetilde{\sigma}_{x_i}$, $i \in \mathcal{I}$ and $\widetilde{\lambda}_{\omega_k}, \widetilde{\sigma}_{\omega_k}, \widetilde{\gamma}_{\omega_k}$ ($1 \leq k \leq m$) such that

$$\begin{aligned} |\Delta v_n|^2 &\rightharpoonup d\widetilde{\lambda} \geq |\Delta v|^2 + \sum_{i \in \mathcal{I}} \widetilde{\lambda}_{x_i} \delta_{x_i} + \sum_{k=1}^m \widetilde{\lambda}_{\omega_k} \delta_{\omega_k}; \\ |v_n|^{2^*} &\rightharpoonup d\widetilde{\sigma} = |v|^{2^*} + \sum_{i \in \mathcal{I}} \widetilde{\sigma}_{x_i} \delta_{x_i} + \sum_{k=1}^m \widetilde{\sigma}_{\omega_k} \delta_{\omega_k}; \\ \frac{|v_n|^2}{|x - \omega_k|^4} &\rightharpoonup d\widetilde{\gamma} = \frac{|v|^2}{|x - \omega_k|^4} + \widetilde{\gamma}_{\omega_k} \delta_{\omega_k}. \end{aligned}$$

where δ_{x_i} is the Dirac mass at x_i and δ_{ω_k} is the Dirac mass at ω_k .

Claim. $\mathcal{I}$ is finite and for any $i \in \mathcal{I}$, either $\widetilde{\sigma}_{x_i} = 0$ or $G(x_i) \widetilde{\sigma}_{x_i} \geq \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}}$.

Indeed, assume $\varepsilon > 0$ small enough such that for each $1 \leq k \leq m$, we have $\omega_k \notin B_\varepsilon(x_i)$ ($i \in \mathcal{I}$).

Let ζ_ε^i be a smooth cut-off function centered at x_i such that $0 \leq \zeta_\varepsilon^i \leq 1$,

$$\zeta_\varepsilon^i(x) = 1 \text{ if } |x - x_i| \leq \frac{\varepsilon}{2}, \quad \zeta_\varepsilon^i(x) = 0 \text{ if } |x - x_i| \geq \varepsilon, \quad |\nabla \zeta_\varepsilon^i| \leq \frac{2}{\varepsilon} \text{ and } |\Delta \zeta_\varepsilon^i| \leq \frac{2}{\varepsilon^2}.$$

By our assumptions, we have

$$\begin{aligned} \langle d\mathcal{K}(v_n), v_n \zeta_\varepsilon^i \rangle &= \int_{\Omega} |\Delta v_n|^2 \zeta_\varepsilon^i dx + \int_{\Omega} v_n \Delta v_n \Delta \zeta_\varepsilon^i dx + 2 \int_{\Omega} \Delta v_n \nabla v_n \cdot \nabla \zeta_\varepsilon^i dx \\ &\quad - \sum_{k=1}^m \mu_k \int_{\Omega} \frac{|v_n|^2 \zeta_\varepsilon^i}{|x - \omega_k|^4} dx - \int_{\Omega} G(x) |v_n|^{2^*} \zeta_\varepsilon^i dx, \end{aligned} \quad (2.2)$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta v_n|^2 \zeta_\varepsilon^i dx = \int_{\Omega} \zeta_\varepsilon^i d\tilde{\lambda} \geq \int_{\Omega} |\Delta v|^2 \zeta_\varepsilon^i dx + \widetilde{\lambda}_{x_i}, \quad (2.3)$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} G(x) |v_n|^{2^*} \zeta_\varepsilon^i dx = \int_{\Omega} G(x) \zeta_\varepsilon^i d\tilde{\sigma} = \int_{\Omega} G(x) |v|^{2^*} \zeta_\varepsilon^i dx + G(x_i) \widetilde{\sigma}_{x_i}. \quad (2.4)$$

Invoking Hölder inequality, we get

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} v_n \Delta v_n \Delta \zeta_\varepsilon^i dx \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\left(\int_{\Omega} |\Delta v_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v_n|^2 |\Delta \zeta_\varepsilon^i|^2 dx \right)^{\frac{1}{2}} \right) \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |v|^2 |\Delta \zeta_\varepsilon^i|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\left(\int_{B_\varepsilon(x_i)} |\Delta \zeta_\varepsilon^i|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \left(\int_{B_\varepsilon(x_i)} |v|^{\frac{2N}{N-4}} dx \right)^{\frac{N-4}{2N}} \right) \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{B_\varepsilon(x_i)} |v|^{2^*} dx \right)^{\frac{1}{2^*}} = 0, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} \Delta v_n \nabla v_n \cdot \nabla \zeta_\varepsilon^i dx \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\left(\int_{\Omega} |\Delta v_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v_n|^2 |\nabla \zeta_\varepsilon^i|^2 dx \right)^{\frac{1}{2}} \right) \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |\nabla v|^2 |\nabla \zeta_\varepsilon^i|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\left(\int_{B_\varepsilon(x_i)} |\nabla \zeta_\varepsilon^i|^N dx \right)^{\frac{1}{N}} \left(\int_{B_\varepsilon(x_i)} |\nabla v|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \right) \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{B_\varepsilon(x_i)} |\nabla v|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} = 0. \end{aligned} \quad (2.6)$$

Moreover, we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sum_{k=1}^m \mu_k \int_{\Omega} \frac{|v_n|^2 \zeta_\varepsilon^i}{|x - \omega_k|^4} dx = 0, \quad (2.7)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\Delta v|^2 \zeta_{\varepsilon}^i dx = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} G(x) |v|^{2^*} \zeta_{\varepsilon}^i dx = 0. \quad (2.8)$$

Combining (2.3)-(2.8) with (2.2), we infer that

$$0 = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle d\mathcal{K}(v_n), v_n \zeta_{\varepsilon}^i \rangle \geq \widetilde{\lambda_{x_i}} - G(x_i) \widetilde{\sigma_{x_i}}. \quad (2.9)$$

Meanwhile, by Sobolev inequality, $R_0 \widetilde{\sigma_{x_i}}^{\frac{2}{2^*}} \leq \widetilde{\lambda_{x_i}}$ for any $i \in \mathcal{I}$, which together with (2.9), yields that $\widetilde{\sigma_{x_i}} = 0$ or $G(x_i) \widetilde{\sigma_{x_i}} \geq \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}}$. Thus $\mathcal{I}$ is finite.

Next, we examine the possibility of concentration at the points ω_k ($1 \leq k \leq m$).

Letting $\varepsilon > 0$ be small enough such that for any $i \in \mathcal{I}, x_i \notin B_{\varepsilon}(\omega_k)$ ($1 \leq k \leq m$), and ζ_{ε}^k be a smooth cut-off function centered at ω_k fulfilling $0 \leq \zeta_{\varepsilon}^k \leq 1$,

$$\zeta_{\varepsilon}^k(x) = 1 \text{ if } |x - \omega_k| \leq \frac{\varepsilon}{2}, \quad \zeta_{\varepsilon}^k(x) = 0 \text{ if } |x - \omega_k| \geq \varepsilon, \quad |\nabla \zeta_{\varepsilon}^k| \leq \frac{2}{\varepsilon} \text{ and } |\Delta \zeta_{\varepsilon}^k| \leq \frac{2}{\varepsilon^2}.$$

As a result we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} G(x) |v_n|^{2^*} \zeta_{\varepsilon}^k dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} G(x) \zeta_{\varepsilon}^k d\tilde{\sigma} \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} G(x) |v|^{2^*} \zeta_{\varepsilon}^k dx + G(\omega_k) \widetilde{\sigma_{\omega_k}} \right) \\ &= G(\omega_k) \widetilde{\sigma_{\omega_k}}, \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} \frac{|v_n|^2 \zeta_{\varepsilon}^k}{|x - \omega_k|^4} dx = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} \zeta_{\varepsilon}^k d\tilde{\gamma} = \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \frac{|v|^2 \zeta_{\varepsilon}^k}{|x - \omega_k|^4} dx + \widetilde{\gamma_{\omega_k}} \right) = \widetilde{\gamma_{\omega_k}},$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} v_n \Delta v_n \Delta \zeta_{\varepsilon}^k dx = 0, \quad \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} \Delta v_n \nabla v_n \cdot \nabla \zeta_{\varepsilon}^k dx = 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} \frac{|v_n|^2 \zeta_{\varepsilon}^k}{|x - \omega_i|^2} dx = 0 \text{ for } i \neq k.$$

Thus, we derive that

$$0 = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle d\mathcal{K}(v_n), v_n \zeta_{\varepsilon}^k \rangle \geq \widetilde{\lambda_{\omega_k}} - \mu_k \widetilde{\gamma_{\omega_k}} - G(\omega_k) \widetilde{\sigma_{\omega_k}}. \quad (2.10)$$

By the Rellich inequalities, we obtain $R_{\mu_k} \widetilde{\sigma_{\omega_k}}^{\frac{2}{2^*}} \leq \widetilde{\lambda_{\omega_k}} - \mu_k \widetilde{\gamma_{\omega_k}}$, which along with (2.10), implies

$$R_{\mu_k} \widetilde{\sigma_{\omega_k}}^{\frac{2}{2^*}} \leq G(\omega_k) \widetilde{\sigma_{\omega_k}},$$

so

$$\widetilde{\sigma_{\omega_k}} = 0 \text{ or } G(\omega_k) \widetilde{\sigma_{\omega_k}} \geq \frac{R_{\mu_k}^{\frac{N}{4}}}{G(\omega_k)^{\frac{N-4}{4}}} \text{ for all } 1 \leq k \leq m.$$

Based on the preceding arguments, we infer that

$$\begin{aligned} c &= \mathcal{K}(v_n) - \frac{1}{2} \langle d\mathcal{K}(v_n), v_n \rangle + o(1) \\ &= \frac{2}{N} \int_{\Omega} G(x) |v_n|^{2^*} dx + o(1) \\ &= \frac{2}{N} \left(\int_{\Omega} G(x) |v|^{2^*} dx + \sum_{i \in \mathcal{I}} G(x_i) \widetilde{\sigma}_{x_i} + \sum_{k=1}^m G(\omega_k) \widetilde{\sigma}_{\omega_k} \right). \end{aligned}$$

If there exists a $i \in \mathcal{I}$ such that $\widetilde{\sigma}_{x_i} \neq 0$, or there is an $k \in \{1, 2, \dots, m\}$, such that $\widetilde{\sigma}_{\omega_k} \neq 0$, then we derive that

$$c \geq \frac{2}{N} \min \left\{ \frac{R_{\mu_1}^{\frac{N}{4}}}{G(\omega_1)^{\frac{N-4}{4}}}, \frac{R_{\mu_2}^{\frac{N}{4}}}{G(\omega_2)^{\frac{N-4}{4}}}, \dots, \frac{R_{\mu_m}^{\frac{N}{4}}}{G(\omega_m)^{\frac{N-4}{4}}}, \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}} \right\}. \quad (2.11)$$

Noting that $\mu_k < 0$ for any $k \leq k_0$, hence $R_{\mu_k} \geq R_0$. Since $G(\omega_k) \leq G_M$ for $k = 1, 2, \dots, m$, we deduce that $\frac{R_{\mu_k}^{\frac{N}{4}}}{G(\omega_k)^{\frac{N-4}{4}}} \geq \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}}$ for every $k \leq k_0$. Invoking (2.11), we obtain that

$$c \geq \frac{2}{N} \min \left\{ \frac{R_{\mu_{k_0+1}}^{\frac{N}{4}}}{G(\omega_{k_0+1})^{\frac{N-4}{4}}}, \frac{R_{\mu_{k_0+2}}^{\frac{N}{4}}}{G(\omega_{k_0+2})^{\frac{N-4}{4}}}, \dots, \frac{R_{\mu_m}^{\frac{N}{4}}}{G(\omega_m)^{\frac{N-4}{4}}}, \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}} \right\} = c^*,$$

which contradicts the assumption $c < c^*$. Hence, up to a subsequence, we conclude that $v_n \rightarrow v$ strongly in $H_0^2(\Omega)$. $\square$

Set $v_{a,\epsilon}^\mu(x) := \eta(x-a)z_\epsilon^\mu(x-a)$ where $a \in \Omega$ with $\eta \in C_0^\infty(B_\rho(0))$ satisfies $0 \leq \eta \leq 1$, $\eta \equiv 1, \forall x \in B_{\frac{\rho}{2}}(0)$ and $z_\epsilon^\mu(x)$ is the same function as in (1.5). The next asymptotic properties hold.

Lemma 2.2. ([20], [21]) *Suppose that $N \geq 5$, $a, \omega \in \Omega$, $\mu \in [0, \mu^*)$ and $\epsilon \rightarrow 0$, we have*

$$\int_{\Omega} \left(|\Delta v_{a,\epsilon}^\mu|^2 - \mu \frac{|v_{a,\epsilon}^\mu|^2}{|x-a|^4} \right) dx = R_\mu^{\frac{N}{4}} + O(\epsilon^{2(\beta(\mu)-\delta)}), \quad (2.12)$$

$$\int_{\Omega} |v_{a,\epsilon}^\mu|^{2^*} dx = R_\mu^{\frac{N}{4}} + O(\epsilon^{2^*(\beta(\mu)-\delta)}), \quad (2.13)$$

$$\int_{\Omega} \frac{|v_{a,\epsilon}^\mu|^2}{|x-\omega|^4} dx = \int_{\mathbb{R}^N} \frac{|z_\epsilon^\mu|^2}{|x+(a-\omega)|^4} dx + O(\epsilon^{2(\beta(\mu)-\delta)}), \quad (2.14)$$

and

$$\int_{\mathbb{R}^N} \frac{|z_\epsilon^\mu|^2}{|x+(a-\omega)|^4} dx = \frac{\epsilon^4}{|a-\omega|^4} \int_{\mathbb{R}^N} |z_1^\mu|^2 dx + o(\epsilon^4). \quad (2.15)$$

Lemma 2.3. *Suppose that (H_1) – (H_3) hold, then there exists a nonnegative function $u \in H_0^2(\Omega) \setminus \{0\}$, such that $\sup_{s \geq 0} \mathcal{K}(su) < c^*$, where c^* is provided in Lemma 2.1.*

Proof. If $c^* = \frac{2}{N} \frac{R_{\mu_l}^{\frac{N}{4}}}{G(\omega_l)^{\frac{N-4}{4}}}$, by (H₁), and since $\beta(\mu_l) > \frac{N}{2}$, for any $\tau > 0$, we have

$$\int_{\Omega} |G(x) - G(\omega_l)| |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx \leq d\tau \int_{B_{\rho}(\omega_l)} |x - \omega_l|^4 |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx \leq d\tau \varepsilon^4,$$

which, along with (2.13), yields that

$$\begin{aligned} \int_{\Omega} G(x) |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx &= \int_{\Omega} G(\omega_l) |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx + \int_{\Omega} (G(x) - G(\omega_l)) |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx \\ &= G(\omega_l) R_{\mu_l}^{\frac{N}{4}} + O(\varepsilon^{2^*(\beta(\mu_l) - \delta)}) + o(\varepsilon^4). \end{aligned} \quad (2.16)$$

Thanks to the fact that for any $D_1, D_2 > 0$, we have

$$\sup_{s \geq 0} \left(\frac{s^2}{2} D_1 - \frac{s^{2^*}}{2^*} D_2 \right) = \frac{2}{N} D_1^{N/4} D_2^{-(N-4)/4},$$

and by (2.12), (2.14)-(2.16) and (H₂), we arrive at

$$\begin{aligned} \sup_{s \geq 0} \mathcal{K}(sv_{\omega_l, \varepsilon}^{\mu_l}) &= \frac{2}{N} \left(\int_{\Omega} \left(|\Delta v_{\omega_l, \varepsilon}^{\mu_l}|^2 - \mu_l \frac{|v_{\omega_l, \varepsilon}^{\mu_l}|^2}{|x - \omega_l|^4} - \sum_{\substack{k=1 \\ k \neq l}}^m \mu_k \frac{|v_{\omega_l, \varepsilon}^{\mu_l}|^2}{|x - \omega_k|^4} \right) dx \right)^{N/4} \\ &\quad \times \left(\int_{\Omega} G(x) |v_{\omega_l, \varepsilon}^{\mu_l}|^{2^*} dx \right)^{-(N-4)/4} \\ &= \frac{2}{N} \left(R_{\mu_l}^{\frac{N}{4}} - \varepsilon^4 \int_{\mathbb{R}^N} |z_1^{\mu_l}|^2 dx \sum_{\substack{k=1 \\ k \neq l}}^m \frac{\mu_k}{|\omega_l - \omega_k|^4} + o(\varepsilon^4) + O(\varepsilon^{2(\beta(\mu_l) - \delta)}) \right)^{N/4} \\ &\quad \times \left(G(\omega_l) R_{\mu_l}^{\frac{N}{4}} + O(\varepsilon^{2^*(\beta(\mu_l) - \delta)}) + o(\varepsilon^4) \right)^{-(N-4)/4} \\ &< \frac{2}{N} \frac{R_{\mu_l}^{\frac{N}{4}}}{G(\omega_l)^{\frac{N-4}{4}}}. \end{aligned} \quad (2.17)$$

If $c^* = \frac{2}{N} \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}}$, it follows from (2.12)-(2.13) that

$$\int_{\Omega} |\Delta v_{x_0, \varepsilon}^0|^2 dx = R_0^{\frac{N}{4}} + O(\varepsilon^{N-4}), \quad \int_{\Omega} |v_{x_0, \varepsilon}^0|^{2^*} dx = R_0^{\frac{N}{4}} + O(\varepsilon^N). \quad (2.18)$$

By (H₂), we have

$$\begin{aligned} \int_{\Omega} G(x) |v_{x_0, \varepsilon}^0|^{2^*} dx &= \int_{\Omega} G(x_0) |v_{x_0, \varepsilon}^0|^{2^*} dx + \int_{\Omega} (G(x) - G(x_0)) |v_{x_0, \varepsilon}^0|^{2^*} dx \\ &= G_M R_0^{\frac{N}{4}} + O(\varepsilon^N) + o(\varepsilon^4). \end{aligned} \quad (2.19)$$

Based on (2.14)-(2.15) in Lemma 2.2, for any $1 \leq k \leq m$, we have

$$\int_{\Omega} \frac{|v_{x_0, \varepsilon}^0|^2}{|x - \omega_k|^4} dx = \frac{\varepsilon^4}{|x_0 - \omega_k|^4} \int_{\mathbb{R}^N} |z_1^0|^2 dx + o(\varepsilon^4). \quad (2.20)$$

Making use of (H₃) and (2.18)-(2.20), we deduce that

$$\begin{aligned} \sup_{s \geq 0} \mathcal{K}(sv_{x_0, \varepsilon}^0) &= \frac{2}{N} \left(\int_{\Omega} \left(|\Delta v_{x_0, \varepsilon}^0|^2 - \sum_{k=1}^m \mu_k \frac{|v_{x_0, \varepsilon}^0|^2}{|x - \omega_k|^4} \right) dx \right)^{N/4} \\ &\quad \times \left(\int_{\Omega} G(x) |v_{x_0, \varepsilon}^0|^{2^*} dx \right)^{-(N-4)/4} \\ &= \frac{2}{N} \left(R_0^{\frac{N}{4}} - \varepsilon^4 \int_{\mathbb{R}^N} |z_1^0|^2 dx \sum_{k=1}^m \frac{\mu_k}{|x_0 - \omega_k|^4} + O(\varepsilon^{N-4}) + o(\varepsilon^4) \right)^{N/4} \\ &\quad \times \left(G_M R_0^{\frac{N}{4}} + O(\varepsilon^N) + o(\varepsilon^4) \right)^{-(N-4)/4} \\ &< \frac{2}{N} \frac{R_0^{\frac{N}{4}}}{G_M^{\frac{N-4}{4}}}. \end{aligned}$$

As a consequence, there is a nonnegative function $u \in H_0^2(\Omega) \setminus \{0\}$ such that $\sup_{s \geq 0} \mathcal{K}(su) < c^*$. $\square$ $\square$

Proof of Theorem 1.3.

Proof. Set $c = \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} \mathcal{K}(\gamma(s))$, where

$$\Gamma = \{ \gamma \in C([0, 1], H_0^2(\Omega)) : \gamma(0) = 0, \gamma(1) < 0 \}.$$

Applying Rellich inequality, we get

$$\mathcal{K}(v) \geq \frac{1}{2} \left(1 - \frac{1}{\bar{\mu}} \sum_{k=k_0+1}^m \mu_k \right) \int_{\Omega} |\Delta v|^2 dx - \frac{G_M}{2^* R_0^{\frac{N}{N-4}}} \left(\int_{\Omega} |\Delta v|^2 dx \right)^{\frac{N}{N-4}},$$

for any $v \in H_0^2(\Omega) \setminus \{0\}$. Therefore there exists a sufficiently small positive number ρ such that

$$b := \inf_{\|v\|=\rho} \mathcal{K}(v) > 0 = \mathcal{K}(0).$$

Since $\mathcal{K}(su) \rightarrow -\infty$ as $s \rightarrow \infty$, where $u \in H_0^2(\Omega) \setminus \{0\}$ is from Lemma 2.3. Hence, there exists $s_0 > 0$ such that $\|s_0 u\| > \rho$ and $\mathcal{K}(s_0 u) < 0$.

By the Mountain Pass lemma [1], there exists a sequence $\{v_n\} \subset H_0^2(\Omega)$ satisfying

$$\mathcal{K}(v_n) \rightarrow c, \quad d\mathcal{K}(v_n) \rightarrow 0.$$

By Lemma 2.3, we have

$$c \leq \sup_{s \in [0,1]} \mathcal{K}(ss_0 u) \leq \sup_{s \geq 0} \mathcal{K}(su) < c^*.$$

From Lemma 2.1, $\{v_n\}$ has a subsequence, still denoted by $\{v_n\}$, such that

$$v_n \rightarrow v \text{ strongly in } H_0^2(\Omega).$$

As a result v is a critical point for $\mathcal{K}$ and thus is a solution to (1.1). c is a critical value of the functional $\mathcal{K}$. $\square$

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