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IMPROVED BOUNDS FOR THE NUMERICAL RADIUS VIA MALIGRANDA INEQUALITY

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Abstract. This paper contributes to the study of numerical radius inequalities for a bounded linear operator on a complex Hilbert space. By employing the Maligranda inequality and the Cartesian decomposition of operators, we establish new inequalities that yield sharper estimates than previously existing results.

1. Introduction and preliminaries

Mathematical inequalities play a crucial role in the development of various fields in both pure and applied mathematics. These inequalities provide powerful methods for estimating solutions to practical problems in engineering and other scientific disciplines. In particular, the study of numerical radius inequalities plays a significant role in functional analysis due to its applications in operator theory. Researchers have devoted considerable effort to refine and generalize these inequalities to achieve greater precision and greater applicability (see [3, 6, 9, 10, 11, 14] for notable advances in this area). The paper improves the field by establishing better bounds on the numerical radius, providing greater accuracy than earlier findings.

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, associated with the inner product $\langle \cdot, \cdot \rangle$. For any operator $S \in \mathcal{B}(\mathcal{H})$, the set $W(S) = \{\langle Sx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ represents the numerical range of S, and the numerical radius $w(S)$ is defined as:

$$w(S) = \sup_{\|x\|=1} |\langle Sx, x\rangle|.$$

For any operator $S \in \mathcal{B}(\mathcal{H})$, we have

$$w(S) \leq \|S\| \leq 2w(S), \tag{1.1}$$

which establishes the equivalence between these two quantities on $\mathcal{B}(\mathcal{H})$.

A particularly effective method for deriving such bounds is the Cartesian decomposition of an operator. For any $S \in \mathcal{B}(\mathcal{H})$, the Cartesian decomposition is

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given by

$$S = S_{Re} + iS_{Im},$$

where $S_{Re} = \frac{S+S^*}{2}$ and $S_{Im} = \frac{S-S^*}{2i}$ denote the real and imaginary parts of S, respectively. The norms of S_{Re} and S_{Im} offer valuable insight into the behaviour of S and play a crucial role in deriving the main results of this work. Notably, Kittaneh [12] refined the inequalities in (1.1) by utilizing Cartesian decomposition, leading to the following result:

$$\frac{1}{4}\|S^*S + SS^*\| \leq w^2(S) \leq \frac{1}{2}\|S^*S + SS^*\|. \tag{1.2}$$

Additionally, Bhunia and Paul [5] developed lower bounds for the numerical radius that incorporate the real and imaginary parts of an operator S, as given by

$$w(S) \geq \frac{1}{2}\|S\| + \frac{1}{2}\left|\|S_{Re}\| - \|S_{Im}\|\right|,$$

$$w^2(S) \geq \frac{1}{4}\left\|S^*S + SS^*\right\| + \frac{1}{2}\left|\|S_{Re}\|^2 - \|S_{Im}\|^2\right|,$$

which refine the upper bound in (1.1) and the lower bound in (1.2).

Further notable refinements based on Cartesian decomposition can be found in [2, 7, 4, 1].

This paper presents new inequalities that refine the upper bound in (1.1) and the lower bound in (1.2). Specifically, we prove the following results:

$$\|S\| \leq w(S) + \min\{\|S_{Re}\|, \|S_{Im}\|\},$$

and

$$\frac{1}{2}\|SS^* + S^*S\| \leq w(S)^2 + \min\{\|S_{Re}\|, \|S_{Im}\|\}^2.$$

Additionally, we improve the inequality $w(S) \geq \frac{1}{2}\left\|S + \frac{\lambda+\mu}{2}S^*\right\|$, established in [8], by demonstrating that:

$$w(S) + \frac{1}{2}\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \geq \left\|S + \frac{\lambda+\mu}{2}S^*\right\|,$$

for all $\lambda, \mu \in \{z \in \mathbb{C} : |z| = 1\}$.

2. Numerical Radius Inequalities via Cartesian Decomposition

We begin by introducing a fundamental theorem that refines the classical triangle inequality. This result, from [13], will be instrumental in establishing our refinement of numerical radius inequalities.

Theorem 2.1 (L. Maligranda)**.** *Let x and y be non-zero elements in a normed space $X = (X, \|\cdot\|)$. Then the following inequality holds:*

$$\|x + y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\min\{\|x\|, \|y\|\} \leq \|x\| + \|y\|.$$

Additionally, we employ a well-known lemma (see [15]) that provides a helpful characterization of the numerical radius.

Lemma 2.2. *If $S \in \mathcal{B}(\mathcal{H})$, then*

$$w(S) = \sup_{\theta \in \mathbb{R}} \left\| (e^{i\theta} S)_{Re} \right\|. \tag{2.1}$$

Equipped with these foundational results, we present our refined inequality for the numerical radius.

Theorem 2.3. *If $S \in \mathcal{B}(\mathcal{H})$, then*

$$\|S\| \le w(S) + \min\{\|S_{Re}\|, \|S_{Im}\|\}. \tag{2.2}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. We will prove this theorem by considering two cases.
Case 1: If $S_{Re} = 0$ or $S_{Im} = 0$, it can be easily verified that S is a normal operator. Hence, we have $w(S) = \|S\|$, which leads to the following inequality:

$$\|S\| = w(S) \le w(S) + \min\{\|S_{Re}\|, \|S_{Im}\|\}.$$

Case 2: Suppose that $S_{Re} \neq 0$ and $S_{Im} \neq 0$. By Lemma 2.2, we have

$$w(S) \ge \|S_{Re}\| \quad \text{and} \quad w(S) \ge \|S_{Im}\|,$$

which implies

$$\begin{aligned} 2w(S) \ge& 2\max\{\|S_{Re}\|, \|S_{Im}\|\} \\ &= \|S_{Re}\| + \|S_{Im}\| + |\|S_{Re}\| - \|S_{Im}\||. \end{aligned}$$

Applying Theorem 2.1 with $x = S_{Re}$ and $y = iS_{Im}$, we obtain

$$\begin{aligned} 2w(S) \ge& \|S_{Re} + iS_{Im}\| + |\|S_{Re}\| - \|S_{Im}\|| \\ &+ \left(2 - \left\| \frac{S_{Re}}{\|S_{Re}\|} + i\frac{S_{Im}}{\|S_{Im}\|} \right\|\right) \min\{\|S_{Re}\|, \|S_{Im}\|\} \\ =& \|S\| + |\|S_{Re}\| - \|S_{Im}\|| + 2\min\{\|S_{Re}\|, \|S_{Im}\|\} \\ &- \left\| \frac{S_{Re}}{\|S_{Re}\|} + i\frac{S_{Im}}{\|S_{Im}\|} \right\| \min\{\|S_{Re}\|, \|S_{Im}\|\}. \end{aligned}$$

Since

$$\begin{aligned} 2\min\{\|S_{Re}\|, \|S_{Im}\|\} &= \|S_{Re}\| + \|S_{Im}\| - |\|S_{Re}\| - \|S_{Im}\|| \\ &\ge \|S_{Re} + iS_{Im}\| - |\|S_{Re}\| - \|S_{Im}\|| \\ &= \|S\| - |\|S_{Re}\| - \|S_{Im}\||, \end{aligned}$$

it follows that

$$\begin{aligned} 2w(S) &\ge 2\|S\| - \left\| \frac{S_{Re}}{\|S_{Re}\|} + i\frac{S_{Im}}{\|S_{Im}\|} \right\| \min\{\|S_{Re}\|, \|S_{Im}\|\} \\ &\ge 2\|S\| - 2\min\{\|S_{Re}\|, \|S_{Im}\|\} \end{aligned}$$

Hence

$$w(S) + \min\{\|S_{Re}\|, \|S_{Im}\|\} \ge \|S\|.$$

In both cases, the inequality holds. Consequently, for all $S \in \mathcal{B}(\mathcal{H})$, we conclude

$$w(S) + \min\{\|S_{Re}\|, \|S_{Im}\|\} \ge \|S\|.$$

□

Corollary 2.4. *If $S \in \mathcal{B}(\mathcal{H})$, then*

$$\|S\| \leq w(S) + \inf_{\theta \in \mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|(e^{i\theta}S)_{Im}\|\}. \tag{2.3}$$

Proof. By substituting S with $e^{i\theta}S$ in (2.2) and then taking the infimum over all $\theta \in \mathbb{R}$, we obtain the desired inequality. □

The following proposition is an immediate consequence of Corollary 2.4.

Proposition 2.5. *If $\inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|\Im(e^{i\theta}S)\|\} = 0$, then $w(S) = \|S\|$.*

Proof. Suppose $\inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|(e^{i\theta}S)_{Im}\|\} = 0$. Then, by Corollary 2.4, we have

$$\|S\| \leq w(S) + \inf_{\theta \in \mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|(e^{i\theta}S)_{Im}\|\} = w(S) + 0 = w(S).$$

The result shows that $\|S\| = w(S)$, as required. □

Example 2.6. Consider the 2-dimensional Hilbert space $\mathbb{C}^2$ and define the operator S by

$$S = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

The adjoint S^* of the operator S is given by

$$S^* = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}.$$

For any $\theta \in \mathbb{R}$, we have

$$e^{i\theta}S = e^{i\theta}\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ie^{i\theta} \\ e^{i\theta} & 0 \end{pmatrix},$$

and

$$e^{-i\theta}S^* = e^{-i\theta}\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\theta} \\ -ie^{-i\theta} & 0 \end{pmatrix}.$$

Thus, the real and imaginary parts of $e^{i\theta}S$ are

$$(e^{i\theta}S)_{Re} = \frac{1}{2}\begin{pmatrix} 0 & ie^{i\theta} + e^{-i\theta} \\ e^{i\theta} - ie^{-i\theta} & 0 \end{pmatrix},$$

and

$$(e^{i\theta}S)_{Im} = \frac{1}{2i}\begin{pmatrix} 0 & ie^{i\theta} - e^{-i\theta} \\ e^{i\theta} + ie^{-i\theta} & 0 \end{pmatrix}.$$

By calculating the norms, we find that

$$\left\|(e^{i\theta}S)_{Re}\right\| = \sqrt{1 - \sin 2\theta},$$

and

$$\left\|(e^{i\theta}S)_{Im}\right\| = \sqrt{1 + \sin 2\theta}.$$

Therefore, we have

$$\begin{aligned}\|S\| &\le w(S) + \inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|(e^{i\theta}S)_{Im}\|\}\\ &= w(S) + \inf_{\theta\in\mathbb{R}} \min\{\sqrt{1-\sin 2\theta}, \sqrt{1+\sin 2\theta}\}\\ &\le w(S) + \min\{\sqrt{1-\sin\left(2\frac{\pi}{4}\right)}, \sqrt{1+\sin\left(2\frac{\pi}{4}\right)}\}\\ &= w(S) + \min\{0, \sqrt{2}\} = w(S).\end{aligned}$$

Hence, we conclude that $w(S) = \|S\|$.

Proposition 2.7. *If* $\inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|\Im(e^{i\theta}S)\|\} = 0$, *then* S *is normal.*

Proof. Consider the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Let us define the function f as follows:

$$\begin{aligned}&f : \mathbb{T} \longrightarrow \mathbb{R}_+,\\ &\quad \lambda \mapsto f(\lambda) = \min\{\|(\lambda S)_{Re}\|, \|(\lambda S)_{Im}\|\}.\end{aligned}$$

For any $\lambda_1, \lambda_2 \in \mathbb{T}$, we have

$$\begin{aligned}f(\lambda_1 - \lambda_2) &= \min\{\|((\lambda_1-\lambda_2)S)_{Re}\|, \|((\lambda_1-\lambda_2)S)_{Im}\|\}\\ &\le \|((\lambda_1-\lambda_2)S)_{Re}\|\\ &= \left\|\frac{(\lambda_1-\lambda_2)S + \overline{(\lambda_1-\lambda_2)}S^*}{2}\right\|\\ &\le |\lambda_1-\lambda_2|\|S\|.\end{aligned}$$

The result demonstrates that f is Lipschitz continuous and, consequently, continuous. Since f is continuous and the unit circle $\mathbb{T}$ is a nonempty compact set, it follows that f is bounded, and there exists $p \in \mathbb{T}$ such that

$$f(p) = \inf_{\lambda\in\mathbb{T}} f(\lambda).$$

Now, assume that

$$\inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|\Im(e^{i\theta}S)\|\} = 0.$$

It follows that there exists $p \in \mathbb{T}$ such that $f(p) = 0$. Hence, either $(pS)_{Re} = 0$ or $(pS)_{Im} = 0$. Without loss of generality, assume $(pS)_{Re} = 0$, since the case $(pS)_{Im} = 0$ can be handled using similar reasoning. Then we have

$$(pS)_{Re}S = \frac{pS^2 + \bar{p}SS^*}{2} = 0, \tag{2.4}$$

and

$$S(pS)_{Re} = \frac{pS^2 + \bar{p}S^*S}{2} = 0. \tag{2.5}$$

By subtracting equation (2.4) from equation (2.5), we obtain

$$SS^* = S^*S.$$

The result shows that S is normal. □

In the following theorem, we establish a new numerical radius inequality that improves Kittaneh's inequality (1.2).

Theorem 2.8. *If $S \in \mathcal{B}(\mathcal{H})$, then*
$$\frac{1}{2}\|SS^* + S^*S\| \le w(S)^2 + \min\{\|S_{Re}\|, \|S_{Im}\|\}^2. \tag{2.6}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. To prove this result, we consider the following cases:
Case 1: If $S_{Re} \neq 0$ and $S_{Im} \neq 0$. we have $w(S)^2 \ge \|S_{Re}\|^2$ and $w(S)^2 \ge \|S_{Im}\|^2$. Thus,
$$\begin{aligned} 2w(S)^2 &\ge 2\max\{\|S_{Re}\|^2, \|S_{Im}\|^2\} \\ &= \|S_{Re}\|^2 + \|S_{Im}\|^2 + |\|S_{Re}\|^2 - \|S_{Im}\|^2| \\ &= \|S_{Re}^2\| + \|S_{Im}^2\| + |\|S_{Re}\|^2 - \|S_{Im}\|^2|. \end{aligned}$$
By applying Theorem 2.1 with $x = S_{Re}^2$ and $y = S_{Im}^2$, we obtain the following result:
$$\begin{aligned} 2w(S)^2 \ge{}& \|S_{Re}^2 + S_{Im}^2\| + |\|S_{Re}\|^2 - \|S_{Im}\|^2| \\ &+ \left(2 - \left\|\frac{S_{Re}^2}{\|S_{Re}^2\|} + \frac{S_{Im}^2}{\|S_{Im}^2\|}\right\|\right)\min\{\|S_{Re}^2\|, \|S_{Im}^2\|\} \\ ={}&\frac{1}{2}\|SS^* + S^*S\| + |\|S_{Re}\|^2 - \|S_{Im}\|^2| \\ &+ 2\min\{\|S_{Re}\|^2, \|S_{Im}\|^2\} - \left\|\frac{S_{Re}^2}{\|S_{Re}^2\|} + \frac{S_{Im}^2}{\|S_{Im}\|^2}\right\|\min\{\|S_{Re}^2\|, \|S_{Im}^2\|\}. \end{aligned}$$
Since
$$2\min\{\|S_{Re}\|^2, \|S_{Im}\|^2\} + |\|S_{Re}\|^2 - \|S_{Im}\|^2| = \|S_{Re}^2\| + \|S_{Im}^2\| \ge \frac{1}{2}\|SS^* + S^*S\|,$$
it follows that
$$\begin{aligned} 2w(S)^2 &\ge \|SS^* + S^*S\| - \left\|\frac{S_{Re}^2}{\|S_{Re}\|^2} + \frac{S_{Im}^2}{\|S_{Im}\|^2}\right\|\min\{\|S_{Re}^2\|, \|S_{Im}^2\|\} \\ &\ge \|SS^* + S^*S\| - 2\min\{\|S_{Re}^2\|, \|S_{Im}^2\|\}. \end{aligned}$$
Thus,
$$w(S)^2 + \min\{\|S_{Re}\|, \|S_{Im}\|\}^2 \ge \frac{1}{2}\|SS^* + S^*S\|.$$
Case 2: If $S_{Re} = 0$ or $S_{Im} = 0$, S is a normal operator, so we have $\|S\| = w(S)$. Therefore,
$$\frac{1}{2}\|SS^* + S^*S\| = \frac{1}{2}\|2SS^*\| = \|SS^*\| = \|S\|^2 = w(S)^2,$$
which implies that
$$\frac{1}{2}\|SS^* + S^*S\| = w(S)^2 \le w(S)^2 + \min\{\|S_{Re}\|, \|S_{Im}\|\}^2.$$
Therefore, the inequality (2.6) holds for all $S \in \mathcal{B}(\mathcal{H})$. □

If we replace S by $e^{i\theta}S$ in (2.6) and take the infimum over all $\theta \in \mathbb{R}$, we obtain the following corollary.

Corollary 2.9. *Let $S \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{2}\|SS^* + S^*S\| \leq w(S)^2 + \inf_{\theta\in\mathbb{R}} \min\{\|(e^{i\theta}S)_{Re}\|, \|(e^{i\theta}S)_{Im}\|\}^2. \tag{2.7}$$

3. A Numerical Radius Inequality via Generalized Cartesian Decomposition

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In [8], the authors introduced the generalized Cartesian decomposition of a linear operator S, given by

$$S = \frac{S + \lambda S^*}{2} + i\frac{S - \lambda S^*}{2i}, \quad \text{for every } \lambda \in \mathbb{T}.$$

The expressions $\frac{S+\lambda S^*}{2}$ and $\frac{S-\lambda S^*}{2i}$ are referred to as the generalized real part and generalized imaginary part of S, and are denoted by $\mathrm{Re}_\lambda(S)$ and $\mathrm{Im}_\lambda(S)$, respectively. Furthermore, it has been shown that the numerical radius of S satisfies

$$w(S) = \max_{\lambda\in\mathbb{T}} \| \mathrm{Re}_\lambda(S)\|. \tag{3.1}$$

Theorem 3.1. *Let $S \in \mathcal{B}(\mathcal{H})$. For all $\lambda, \mu \in \mathbb{T}$, the following inequality holds:*

$$w(S) + \frac{1}{2}\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \geq \left\|S + \frac{\lambda + \mu}{2}S^*\right\|. \tag{3.2}$$

Proof. Let $\lambda, \mu \in \mathbb{T}$. From (3.1), we know that

$$2w(S) \geq \|S + \lambda S^*\| \quad \text{and} \quad 2w(S) \geq \|S + \mu S^*\|.$$

Consequently, we have

$$\begin{aligned} 4w(S) &\geq 2\max\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \\ &= \|S + \lambda S^*\| + \|S + \mu S^*\| + \left|\|S + \lambda S^*\| - \|S + \mu S^*\|\right|. \end{aligned}$$

Case 1: $S+\lambda S^* \neq 0$ and $S+\mu S^* \neq 0$. By applying Theorem 2.1 with $x = S+\lambda S^*$ and $y = S + \mu S^*$, it follows that

$$\begin{aligned} 4w(S) \geq{} & \|2S + (\lambda + \mu)S^*\| + \left|\|S + \lambda S^*\| - \|S + \mu S^*\|\right| \\ &+ \left(2 - \left\|\frac{S + \lambda S^*}{\|S + \lambda S^*\|} + \frac{S + \mu S^*}{\|S + \mu S^*\|}\right\|\right) \min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\}. \end{aligned}$$

Since

$$2\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} + \left|\|S + \lambda S^*\| - \|S + \mu S^*\|\right| \geq \|2S + (\lambda + \mu)S^*\|,$$

we obtain

$$\begin{aligned} 4w(S) &\geq 2\|2S + (\lambda + \mu)S^*\| - \left\|\frac{S + \lambda S^*}{\|S + \lambda S^*\|} + \frac{S + \mu S^*}{\|S + \mu S^*\|}\right\| \min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \\ &\geq 2\|2S + (\lambda + \mu)S^*\| - 2\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \\ &\geq 4\|S + \frac{\lambda + \mu}{2}S^*\| - 2\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\}. \end{aligned}$$

Hence, we conclude that

$$w(S) + \frac{1}{2}\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\} \geq \|S + \frac{\lambda + \mu}{2}S^*\|.$$

Case 2: $S + \lambda S^* = 0$ or $S + \mu S^* = 0$. We observe that

$$\|S + \frac{\lambda + \mu}{2}S^*\| = \left\|\frac{S + \lambda S^*}{2} + \frac{S + \mu S^*}{2}\right\| \leq \left\|\frac{S + \lambda S^*}{2}\right\| + \left\|\frac{S + \mu S^*}{2}\right\|.$$

Thus, it follows that

$$\|S + \frac{\lambda + \mu}{2}S^*\| \leq w(S) \leq w(S) + \frac{1}{2}\min\{\|S + \lambda S^*\|, \|S + \mu S^*\|\}.$$

Therefore, the inequality holds for every $S \in \mathcal{B}(\mathcal{H})$. $\square$

Remark 3.2. The inequality presented in Theorem 3.1 provides an improvement over the inequality $w(S) \geq \frac{1}{2}\left\|S + \frac{\lambda+\mu}{2}S^*\right\|$, as established in [8].

Remark 3.3. If we set $\lambda = -\mu$ in inequality (3.2), we obtain inequality (2.2). Hence, Theorem 3.1 is a generalization of Theorem 2.3.

Conflicts of Interest. The authors have no conflicts of interest to declare.

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