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# EXISTENCE RESULT FOR NONLINEAR PANTOGRAPH FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING GENERALIZED CAPUTO DERIVATIVE

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Abstract. In this paper, we are interested in proving the existence, uniqueness and stability of solutions to nonlinear pantograph fractional differential equations involving the generalized Caputo derivative with nonlocal conditions. To obtain the main results, we examine two classical theorems of functional analysis, the Schaefer fixed point theorem and the Banach contraction principle. Furthermore, we explore the stability of the solutions by analogy with the Ulam-Hyers theory. Finally, as applications of the theoretical results, we provide specific examples to demonstrate the practicality and validity of the key findings.

## 1. Introduction

Fractional calculus is a branch of mathematics concerned with integrals and derivatives of non-integer order, dates back to the XVII-th century in the works of Leibniz and L'Hôpital. Since, the concept has rapidly evolved, becoming a fundamental tool in various applied fields such as fractional geometry and fractional dynamics. (see [18, 2, 21] and the references therein). Fractional calculus research is gaining attention due to its ability to accurately characterize various dynamical models compared to integer-order derivatives, it offers more realistic models, revealing hidden characteristics in mathematical and physical models [13, 4, 7]. In recent decades, several fractional operators have been developed, such as Riemann-Liouville, Caputo, Riesz-Caputo, Erdélyi-Kober, Katugampola, Hadamard, $\psi$-Hilfer and others, see [8].

Fractional differential equations (FDEs) are used in a variety of domains, including dynamical system control, aerodynamics, polymer rheology, physics, chemistry, engineering, economics and finance (see [14, 13, 9]). Many academics are interested in investigating many aspects of fractional differential equations, such as techniques for determining the existence and uniqueness of solutions, exact solutions, stability of solutions, and explicit and numerical solutions [16, 8, 22]. Iterative procedures, numerical methods, fixed point theorems, and upper-lower

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solutions are commonly used to demonstrate the existence and uniqueness of solutions. Stability theory in the sense of Ulam-Hyers, a concept of data dependence for solution stability, is a significant topic in the study of fractional differential equations [11].

Pantograph fractional differential equations ($\mathtt{PFDEs}$) are a type of delay differential equation that occur in deterministic conditions, they take the following form:

$$\begin{cases}\mathfrak{D}^{\theta}\zeta(\mathtt{t}) = \mathtt{f}\left(\mathtt{t}, \zeta(\mathtt{t}),\ \zeta(\gamma\mathtt{t})\right), \quad \mathtt{t} \in [0, \mathtt{T}],\ \theta > 0,\ \gamma \in ]0, 1[,\\ \zeta(0) = \zeta_0.\end{cases}$$

They are extremely important in mathematics, physics and engineering applications, such as probability theory, astrophysics, nonlinear dynamical systems, electrodynamics, biology, economics, quantum mechanics and other technological and industrial applications. For more details, we recommend the reader to [3, 20, 5]. However, we notice the deficiency of Scientific productions and studies on this type of fractional differential equations, among which we mention:
In [1], Abdeljawad and al. established the existence and uniqueness results and discussed two sorts of Ulam-Hyers stability for the pantograph fractional differential equations ($\mathtt{PFDEs}$) involving Atangana–Baleanu–Caputo derivative and nonlocal conditions of the form

$$\begin{cases}{}^{\mathtt{ABC}}\mathfrak{D}^{\theta}_{\mathtt{a}^+}\zeta(\mathtt{t}) = \mathtt{f}(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\gamma\mathtt{t})), \quad \mathtt{t} \in [\mathtt{a}, \mathtt{d}],\ (\gamma, \theta) \in ]0, 1[\times]0, 1]\\ \zeta(\mathtt{a}) = \sum\limits_{k=1}^{p} \mathtt{c}_k \zeta\left(\tau_k\right), \quad \tau_k \in ]\mathtt{a}, \mathtt{d}[\end{cases}$$

Harikrishman and al. [10] proved the existence and uniqueness solution of pantograph $\psi$-Hilfer fractional differential equations with nonlocal initial value problem, represented by:

$$\begin{cases}{}^{\mathtt{H}}\mathfrak{D}_{\mathtt{a}^+}{}^{(\alpha,\beta);\psi}\zeta(\mathfrak{s}) = \mathtt{f}(\mathfrak{s}, \zeta(\mathfrak{s}), \zeta(\gamma\mathfrak{s})), \quad \mathfrak{s} \in ]\mathtt{a}, \mathtt{d}],\ \mathtt{d} > \mathtt{a},\ \gamma \in ]0, 1[,\\ \mathfrak{I}^{1-\gamma;\psi}_{\mathtt{a}^+}\zeta(\mathtt{a}) = \sum\limits_{j=1}^{k} \mathtt{c}_j \zeta\left(\tau_j\right), \quad \tau_j \in ]\mathtt{a}, \mathtt{d}[,\ \gamma = \alpha + \beta - \alpha\beta,\end{cases}$$

The last study is innovative in that, given varying values of the $\psi$ function and the parameters $\alpha$ and $\beta$, the $\psi$-Hilfer derivative generalises the known fractional derivatives like Riemann-Liouville (for $\psi(\mathtt{t}) = \mathtt{t}$ and $\alpha = \beta = 0$), $\psi$-Riemann-Liouville (for $\alpha = \beta = 0$), Caputo (for $\psi(\mathtt{t}) = \mathtt{t}$, Hilfer (for $\psi(\mathtt{t}) = \mathtt{t}$) and $\alpha = \beta = 1$), Katugampola (for $\psi(\mathtt{t}) = \mathtt{t}^{\varrho}$), and Hilfer-Hadamard derivative (for $\psi(\mathtt{t}) = \log(\mathtt{t})$), see [19].

Inspired and encouraged by the works listed above, we chose to investigate the following problem

$$\begin{cases}{}^{\mathtt{C}}\mathfrak{D}^{\beta;\varrho}_{\mathtt{a}}\zeta(\mathtt{t}) = \mathtt{f}\left(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu\left(\mathtt{t} - \mathtt{a}\right))\right), \quad \mathtt{t} \in [\mathtt{a}, \mathtt{d}],\ \mathtt{a} \geq 0,\ \mu \in ]0, 1[,\\ \zeta(\mathtt{a}) = \sum\limits_{k=1}^{p} \lambda_k \zeta\left(\tau_k\right), \quad \tau_k \in ]\mathtt{a}, \mathtt{d}[\end{cases} \tag{1.1}$$

where, the generalized Caputo derivative ${}^{\mathtt{C}}\mathfrak{D}^{\beta;\varrho}_{\mathtt{a}}$ of orders $(\beta, \varrho) \in ]0, 1]\times]0, \infty[$ is independent of the $\psi$-Hilfer for different values of the parameters, $\mathtt{f}$ is a continuous function defined on $[\mathtt{a},\mathtt{d}]\times\mathbb{R}\times\mathbb{R}$ with values in $\mathbb{R}\backslash\{0\}$, also $\mathtt{f}(\mathtt{a},\zeta(\mathtt{a}),\zeta(\mu\mathtt{a}))=0$, and the constants $\lambda_k$ satisfies the condition $\sum_{k=1}^{p}\lambda_k\neq 1$.

The body of this paper is planned like this. After the introduction, we present a simple review of some definitions, properties and lemmas that appear necessary for our work in Section 2. In Section 3, we aim to prove the existence and uniqueness of the solution to the pantograph FDEs (1.1), using two well-known fixed point theorems. In the fourth section, we'll talk about Ulam-Hyers stability solutions. Finally, in the final section, we propose using two examples to concretise and validate the key findings.

## 2. Basic Mathematical Tools

In this section, we provide several definitions, lemmas on generalized Caputo fractional operators and some fixed point theorems that we will utilize throughout our study.
Let $(\mathtt{a},\mathtt{d})\in\mathbb{R}^2$, we denote $\Omega:=[\mathtt{a},\mathtt{d}]$ and

$$\mathcal{C}^0(\Omega)=\{f:[\mathtt{a},\mathtt{d}]\longrightarrow\mathbb{R},\ \text{continuous}\ \}.$$

Equipped with the sup metric norm, $(\mathcal{C}^0(\Omega),\|.\|)$ is a Banach space.

Let $\beta>0$ and $\varrho>0$, two positive real numbers:

**Definition 2.1** (Generalized fractional integral [15]).
If $\mathtt{f}\in\mathcal{C}^0(\Omega)$, the generalized fractional integral of orders $(\beta,\varrho)$, denoted by $\mathfrak{I}^{\beta;\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t})$, is explicitly given by the formula

$$\mathfrak{I}^{\beta;\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t})=\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathfrak{t}}\mathfrak{s}^{\varrho-1}\left(\mathfrak{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}\mathtt{f}(\mathfrak{s})\,d\mathfrak{s},\quad \mathfrak{t}>\mathtt{a}$$

For the rest of this section, we denote $m\in\mathbb{N}^*$ the ceiling of $\beta$. Furthermore, we denote $\mathcal{C}^{\mathtt{m}}([\mathtt{a},\mathtt{d}])$ the set of all functions of class $\mathcal{C}^{\mathtt{m}}$ on $\Omega$.

**Lemma 2.2.** *If* $\mathtt{f}\in\mathcal{C}^0(\Omega)$,

$$\lim_{\mathfrak{t}\to\mathtt{a}}\mathfrak{I}^{\beta;\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t},\zeta(\mathfrak{t}),\zeta(\mathtt{a}+\mu(\mathfrak{t}-\mathtt{a})))=0.$$

**Definition 2.3** (Generalized Riemann-Liouville fractional derivative (GRL) [15]).

If $\mathtt{f}\in\mathcal{C}^{\mathtt{m}}([\mathtt{a},\mathtt{d}])$, the GRL fractional derivative denoted by ${}^{\mathtt{GRL}}\mathfrak{D}^{\beta,\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t})$, of order $\beta\in]\mathtt{m}-1,\mathtt{m}]$, is given by the following expression:

$$\begin{aligned}{}^{\mathtt{GRL}}\mathfrak{D}^{\beta,\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t})&=\left(\mathfrak{t}^{1-\varrho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathtt{m}}\mathfrak{I}^{\mathtt{m}-\beta,\varrho}_{\mathtt{a}^+}\mathtt{f}(\mathfrak{t}),\quad \mathfrak{t}>\mathtt{a}\geq 0\\&=\frac{\varrho^{\beta-\mathtt{m}+1}}{\Gamma(\mathtt{m}-\beta)}\left(\mathfrak{t}^{1-\varrho}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)^{\mathtt{m}}\int_{\mathtt{a}}^{\mathfrak{t}}\mathfrak{s}^{\varrho-1}\left(\mathfrak{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\mathtt{m}-\beta-1}\mathtt{f}(\mathfrak{s})\,d\mathfrak{s}\end{aligned}$$

**Definition 2.4** (Generalized Caputo fractional derivative [17]).
If $\mathtt{f} \in \mathcal{C}^{\mathtt{m}}([\mathtt{a},\mathtt{d}])$, the generalized Caputo fractional derivative (GCFD) denoted by ${}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}\mathtt{f}(\mathtt{t})$, of order $\beta \in ]\mathtt{m}-1,\mathtt{m}]$, is given by the following expression:

$$ {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}\mathtt{f}(\mathtt{t}) = {}^{\mathtt{RL}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho}\left[\mathtt{f}(\mathfrak{s}) - \sum_{\mathtt{n}=0}^{\mathtt{m}-1}\frac{\mathtt{f}^{(\mathtt{n})}(\mathtt{a})}{\mathtt{n}!}(\mathfrak{s}-\mathtt{a})^{\mathtt{n}}\right]\Bigg|_{\mathfrak{s}=\mathtt{t}}, \quad \mathtt{t} > \mathtt{a} \geq 0. $$

**Definition 2.5** (Generalized Caputo fractional derivative [17]).
If $\mathtt{f} \in \mathcal{C}^{\mathtt{m}}([\mathtt{a},\mathtt{d}])$, the generalized Caputo fractional derivative operator ${}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho}$, of order $\beta \in ]\mathtt{m}-1,\mathtt{m}]$, is given by the following expression:

$$ \begin{aligned} \left({}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho} f\right)(\mathtt{t}) &= \mathfrak{I}_{\mathtt{a}^+}^{\mathtt{m}-\beta,\varrho}\left[\left(\mathtt{t}^{1-\varrho}\frac{\mathtt{d}}{\mathtt{d}t}\right)^{\mathtt{m}}\mathtt{f}(\mathtt{t})\right], \quad \mathtt{t} > \mathtt{a} \geq 0 \\ &= \frac{\varrho^{\beta-\mathtt{m}+1}}{\Gamma(\mathtt{m}-\beta)}\int_{\mathtt{a}}^{\mathtt{t}} \mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\mathtt{m}-\beta-1}\left(\mathfrak{s}^{1-\varrho}\frac{\mathtt{d}}{\mathtt{d}s}\right)^{\mathtt{m}}\mathtt{f}(\mathfrak{s})d\mathfrak{s} \end{aligned} \tag{2.1} $$

where $\mathtt{a} \geq 0$, and $\beta \in ]\mathtt{m}-1,\mathtt{m}]$.

*Remark* 2.6. In case of $\beta \in ]0,1]$, if $\mathtt{f} \in \mathcal{C}^{1}([\mathtt{a},\mathtt{d}])$,

$$ {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}\mathtt{f}(\mathtt{t}) = \frac{\varrho^{\beta}}{\Gamma(1-\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{-\beta} f'(\mathfrak{s})\, d\mathfrak{s}, \quad \mathtt{t} > \mathtt{a} \geq 0 $$

**Lemma 2.7.** [17] *Let* $\mathtt{f} \in \mathcal{C}^{0}(\Omega)$ *and* $\beta,\ \varrho > 0$, *then:*

$$ {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho}\left[\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}(\mathtt{t})\right] = \mathtt{f}(\mathtt{t}), \quad \mathtt{t} \in [\mathtt{a},\mathtt{d}]. \tag{2.2} $$

**Lemma 2.8.** [17] *Let* $\mathtt{f} \in \mathcal{C}^{\mathtt{m}}([\mathtt{a},\mathtt{d}])$, *where* $\mathtt{m} = \lceil\beta\rceil$ *the ceiling of* $\beta$, *then*

$$ \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\left[{}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho}\mathtt{f}(\mathtt{t})\right] = \mathtt{f}(\mathtt{t}) - \sum_{\mathtt{n}=0}^{\mathtt{m}-1}\frac{(\mathtt{t}^{\varrho}-\mathtt{a}^{\varrho})^{\mathtt{n}}}{\varrho^{\mathtt{n}}\mathtt{n}!}\left[\left(\mathfrak{s}^{1-\varrho}\frac{\mathtt{d}}{\mathtt{d}\mathfrak{s}}\right)^{\mathtt{n}}\mathtt{f}(\mathfrak{s})\right]\Bigg|_{\mathfrak{s}=\mathtt{a}} $$

*Remark* 2.9. In case of $\beta \in (0,1]$,

$$ \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\left({}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta,\varrho}\mathtt{f}(\mathtt{t})\right) = \mathtt{f}(\mathtt{t}) - \mathtt{f}(\mathtt{a}). \tag{2.3} $$

Establishing the existence theory for the problem (1.1) is mostly dependent on fixed point theorems. Here, we gather the two fixed point theorems that are applied in this work.

**Lemma 2.10** (Arzelà-Ascoli theorem).
*Let* $\mathtt{S}$ *be a subset of* $\mathcal{C}^{0}(\Omega)$. *Suppose that*

(i) *there is some* $\mathtt{A} > 0$ *such that, for all* $\mathtt{f} \in \mathtt{S}$ *and* $\mathtt{t} \in [\mathtt{a},\mathtt{d}]$

$$ |\mathtt{f}(\mathtt{t})| \leq \mathtt{A}; $$

(ii) $\mathtt{S}$ *is equicontinuous;*

*Then, the closure of* $\mathtt{S}$ *is compact.*

**Lemma 2.11** (Schaefer's fixed point theorem[6]).
*Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, $\mathcal{N} : \mathcal{B} \to \mathcal{B}$ a continuous mapping which is compact on each bounded subset $\mathtt{S}$ of $\mathcal{B}$. If for some $\lambda \in ]0,\ 1[$, the set*

$$\mathfrak{S} = \{\zeta \in \mathcal{B} \ : \ \zeta = \lambda \mathcal{N}(\zeta)\}$$

*is bounded. Then, $\mathcal{N}$ has a fixed point in $\mathcal{B}$.*

**Lemma 2.12** (Banach contraction principle).
*Let $\mathcal{N} : \mathtt{X} \to \mathtt{X}$ be a Lipschitz mapping on a complete metric space $\mathtt{X}$ with Lipschitz constant $\kappa \in ]0,1[$. Then, $\mathcal{N}$ has a unique fixed point $\zeta^* \in \mathtt{X}$, and for each $\zeta \in \mathtt{X}$, we have $\mathcal{N}^{\mathtt{n}}(\zeta) \to \zeta^*$ as $\mathtt{n} \to +\infty$.*
*Moreover, the sequence $\{\zeta_{\mathtt{n}}\}_{\mathtt{n}=0}^{\infty} \subset \mathtt{X}$ associated to $\mathcal{N}$, defined by*

$$\zeta_0 \in \mathtt{M} \text{ and } \zeta_{\mathtt{n}+1} = \mathcal{N}(\zeta_{\mathtt{n}}), \mathtt{n} = 0, 1, 2, 3, \ldots,$$

*converges to $\zeta^*$, for any initial guess $\zeta_0$.*

## 3. Main Results

In this section, we use the fixed point theorems of Banach and Schafer to demonstrate the existence and uniqueness of the solution for (1.1).

### 3.1. Equivalence of (1.1) with integral equation.

This section is devoted to reformulating the generalized Caputo pantograph problem (1.1) differently.

**Lemma 3.1.** *Let $\beta \in ]0,1]$ and $\sum_{k=1}^{p} \lambda_k \neq 1$. If $\mathtt{h} \in \mathcal{C}^0(\Omega)$ is a continuous function on $[\mathtt{a}, \mathtt{d}]$, with $\mathtt{h}(\mathtt{a}) = 0$. A function $\zeta \in \mathcal{C}^0(\Omega)$ is a solution of the problem*

$$\begin{cases} {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}\zeta(\mathtt{t}) = \mathtt{h}\left(\mathtt{t}\right) \ , \quad \mathtt{t} \in \Omega := [\mathtt{a}, \mathtt{d}], \\ \zeta(\mathtt{a}) = \sum\limits_{k=1}^{p} \lambda_k \zeta\left(\tau_k\right), \quad \tau_k \in ]\mathtt{a}, \mathtt{d}[ \end{cases} \tag{3.1}$$

*if and only if $\zeta$ is a solution of the fractional integral equation* (FIE)

$$\zeta(\mathtt{t}) = \frac{1}{1 - \sum\limits_{k=1}^{p} \lambda_k} \sum_{k=1}^{p} \lambda_k \cdot \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right) + \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\mathtt{t}} \mathfrak{s}^{\varrho-1} \left(\mathtt{t}^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{h}\left(\mathfrak{s}\right) d\mathfrak{s}. \tag{3.2}$$

*Proof.* Let $\zeta \in \mathcal{C}^0(\Omega)$ be a solution of the first equation of (3.1). From Lemma 2.8, we have

$$\zeta(\mathtt{t}) = \zeta(\mathtt{a}) + \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\mathtt{t}} \mathfrak{s}^{\varrho-1} \left(\mathtt{t}^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{h}\left(\mathfrak{s}\right) d\mathfrak{s}. \tag{3.3}$$

By substituting $\mathtt{t} = \tau_k$ in (3.3), we get

$$\lambda_k \zeta\left(\tau_k\right) = \lambda_k \zeta(\mathtt{a}) + \lambda_k \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\tau_k} \mathfrak{s}^{\varrho-1} \left(\mathtt{t}^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{h}\left(\mathfrak{s}\right) d\mathfrak{s}.$$

According to the nonlocal condition (second equation of (3.1)), we obtain

$$\zeta(\mathtt{a}) = \sum_{k=1}^{p} \frac{\lambda_k}{1-\sum\limits_{k=1}^{p}\lambda_k} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right) \tag{3.4}$$

By merging the equations (3.3) and (3.4), we get the integral equation (3.2)

$$\zeta(\mathtt{t}) = \sum_{k=1}^{p} \frac{\lambda_k}{1-\sum\limits_{k=1}^{p}\lambda_k} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right) + \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\mathtt{t}} \mathfrak{s}^{\varrho-1} \left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{h}\left(\mathfrak{s}\right) \, d\mathfrak{s}. \tag{3.2}$$

For the converse, if $\mathtt{y} \in \mathcal{C}^0(\Omega)$ satisfies (3.2). Applying derivative operator ${}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}$ on both sides of (3.2), then using Lemma 2.7, and from the fact ${}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho}(\mathtt{c}) = 0$, for $\mathtt{c}$ constant, we find that

$$\begin{aligned} {}^{\mathtt{C}}\mathfrak{D}_{a}^{\beta,\varrho}\zeta(\mathtt{t}) =& {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho} \left[ \sum_{k=1}^{p} \frac{\lambda_k}{1-\sum\limits_{k=1}^{p}\lambda_k} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right) \right] \\ &+ {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho} \left[ \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\mathtt{t}} \mathfrak{s}^{\varrho-1} \left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{h}\left(\mathfrak{s}\right) d\mathfrak{s} \right] \\ =& {}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}}^{\beta;\varrho} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\mathtt{t}\right) \\ =& \mathtt{h}\left(\mathtt{t}\right). \end{aligned}$$

Substitute $\mathtt{t} = \tau_k$, multiply by $\lambda_k$, and sum in (3.2). We get

$$\sum_{k=1}^{\mathtt{p}} \lambda_k \zeta\left(\tau_k\right) = \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \cdot \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right).$$

On the other hand, by taking $\mathtt{t} \to \mathtt{a}$ on both sides of (3.2), then using the fact that $\mathtt{f}(\mathtt{a}, \zeta(\mathtt{a}), \zeta(\mu\mathtt{a})) = 0$ and $\lim\limits_{\mathtt{t}\to\mathtt{a}} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{f}(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a}+\mu\left(\mathtt{t}-\mathtt{a}\right))) = 0$, we get

$$\zeta(\mathtt{a}) = \sum_{k=1}^{p} \frac{\lambda_k}{1-\sum\limits_{k=1}^{p}\lambda_k} \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} \mathtt{h}\left(\tau_k\right).$$

It follows from the last two equalities that $\zeta(\mathtt{a}) = \sum\limits_{k=1}^{\mathtt{p}} \lambda_k \zeta\left(\tau_k\right)$. □

As a result of Lemma 3.1, we have the following lemma.

**Lemma 3.2.** *Let* $\mathtt{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function. For each* $\zeta \in \mathcal{C}^0(\Omega)$*, if* $\sum\limits_{k=1}^{p} \lambda_k \neq 1$ *and* $\mathtt{f}(\mathtt{a}, \zeta(\mathtt{a}), \zeta(\mu\mathtt{a})) = 0$*, the generalized Caputo pantograph fractional differentiel equation* (FDEs) (1.1) *are equivalent to the following generalized* *pantograph integral equation :*

$$\begin{aligned}\zeta(\mathtt{t}) =& \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p}\lambda_k\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta(\cdot),\zeta(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\tau_k)\\ &+\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}\mathtt{f}(\mathfrak{s},\zeta(\mathfrak{s}),\zeta(\mathtt{a}+\mu\left(\mathfrak{s}-\mathtt{a}\right)))\,d\mathfrak{s},\quad \mathtt{t}\in\Omega.\end{aligned}\tag{3.5}$$

*Remark* 3.3. According to the previous lemma 3.2, we can transform the integral equation (3.5) into the operator $\mathcal{N}$ defined on $\mathcal{C}^0(\Omega)$ by

$$\begin{aligned}(\mathcal{N}\zeta)(\mathtt{t}) =& \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p}\lambda_k\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta(\cdot),\zeta(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\tau_k)\\ &+\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}\mathtt{f}(\mathfrak{s},\zeta(\mathfrak{s}),\zeta(\mathtt{a}+\mu\left(\mathfrak{s}-\mathtt{a}\right)))\,d\mathfrak{s},\quad \mathtt{t}\in\Omega.\end{aligned}\tag{3.6}$$

3.2. **Existence results.**
In accordance with Lemma 3.2 and Remark 3.3, the generalized Caputo pantograph FDEs (1.1) admits a solution if and only if, the operator $\mathcal{N}:\mathcal{C}^0(\Omega)\to\mathcal{C}^0(\Omega)$ defined in the previous remark admits a fixed point. In the course of searching for results in accordance with adopted fixed point theorems cited above, we are led to formulate the following hypotheses:

$(\mathbf{H_1})$ Suppose that, there exists $\mathtt{M}_0>0$, such that

$$|\mathtt{f}(\mathtt{t},0,0)|\leq\mathtt{M}_0,\quad\forall\mathtt{t}\in\Omega.$$

$(\mathbf{H_2})$ The function $\mathtt{f}:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is continuous, and there exists $\mathtt{L}_{\mathtt{f}}>0$ such that, for all $\mathtt{t}\in\Omega$ and $\zeta_1,\ \zeta_2,\ \xi_1,\ \xi_2\in\mathbb{R}$,

$$|\mathtt{f}(\mathtt{t},\ \zeta_1,\ \zeta_2)-\mathtt{f}(\mathtt{t},\ \xi_1,\ \xi_2)|\leq\mathtt{L}_{\mathtt{f}}\left(|\zeta_1-\xi_1|+|\zeta_2-\xi_2|\right)$$

**Lemma 3.4.** *The mapping $\mathcal{N}$, defined above, is completely continuous.*
*Proof.*

Let $\mathtt{S} \subset \mathcal{C}^0(\Omega)$ be a bounded set ($\exists\ \mathtt{R} > 0$ such that $\|\zeta\| \leq \mathtt{R}$ for all $\zeta \in \mathtt{S}$). Then, for $\zeta \in \mathtt{S}$ and $\mathtt{t} \in \Omega$, we have

$$
\begin{aligned}
|(\mathcal{N}\zeta)(\mathtt{t})| \leq{}& \frac{1}{\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\cdot\Big[\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}|\mathtt{f}(\tau_k,\zeta(\tau_k),\zeta(\mathtt{a}+\mu(\tau_k-\mathtt{a})))-\mathtt{f}(\tau_k,0,0)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad +\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}|\mathtt{f}(\tau_k,0,0)|\Big] \\
&+\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}|\mathtt{f}(\mathfrak{s},\zeta(\mathfrak{s}),\zeta(\mathtt{a}+\mu(\mathfrak{s}-\mathtt{a})))-\mathtt{f}(\mathfrak{s},0,0)|\,d\mathfrak{s} \\
&\qquad\qquad\qquad +\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}|\mathtt{f}(\mathfrak{s},0,0)|\,d\mathfrak{s} \\
\leq{}& \frac{2\mathtt{L}_{\mathtt{f}}\varrho^{-\beta}}{\Gamma(\beta+1)\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}\|\zeta\|+\frac{\mathtt{M}_0\varrho^{-\beta}}{\Gamma(\beta+1)\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|} \\
&\times\sum_{k=1}^{p}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}+\frac{2\mathtt{L}_{\mathtt{f}}\varrho^{-\beta}}{\Gamma(\beta+1)}\left(\mathtt{d}^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}\|\zeta\|+\frac{\mathtt{M}_0\varrho^{-\beta}}{\Gamma(\beta+1)}\left(\mathtt{d}^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}.
\end{aligned}
$$

For brevity, if we put $\lambda_{p+1} = \left|1-\sum\limits_{k=1}^{p}\lambda_k\right|$ and $\tau_{p+1} = \mathtt{d}$, we get

$$
\|(\mathcal{N}\zeta)\| \leq \omega_1\mathtt{R}+\omega_2, \text{ for all } \zeta \in \mathtt{S},
$$

where,

$$
\omega_1 = \frac{2\mathtt{L}_{\mathtt{f}}\varrho^{-\beta}}{\Gamma(\beta+1)\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p+1}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}, \tag{3.7}
$$

and

$$
\omega_2 = \frac{\mathtt{M}_0\varrho^{-\beta}}{\Gamma(\beta+1)\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p+1}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}. \tag{3.8}
$$

Therefore, the set $\mathcal{N}(\mathtt{S})$ is uniformly bounded on $\Omega = [\mathtt{a}, \mathtt{d}]$.
In order to apply the Arzelà-Ascoli theorem Lemma 2.10, we must demonstrate the uniform equicontinuousness of $\mathcal{N}(\mathtt{S})$. For this, let $\mathtt{a} < \upsilon_1 < \upsilon_2 < \mathtt{d}$, for all

$\zeta \in \mathtt{S}$, we get

$$
\begin{aligned}
|(\mathcal{N}\zeta)(\upsilon_2) - (\mathcal{N}\zeta)(\upsilon_1)| \leq\ & \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \left| \int_{\mathtt{a}}^{\upsilon_2} \mathfrak{s}^{\varrho-1} \left(\upsilon_2^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{f}(\mathfrak{s}, \zeta(\mathfrak{s}), \zeta(\mathtt{a} + \mu\,(\mathfrak{s} - \mathtt{a})))\, d\mathfrak{s} \right. \\
& \left. - \int_{\mathtt{a}}^{\upsilon_1} \mathfrak{s}^{\varrho-1} \left(\upsilon_1^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \mathtt{f}(\mathfrak{s}, \zeta(\mathfrak{s}), \zeta(\mathtt{a} + \mu\,(\mathfrak{s} - \mathtt{a})))\, d\mathfrak{s} \right| \\
\leq\ & \frac{\varrho^{-\beta}}{\Gamma(\beta)} \int_{\mathtt{a}}^{\upsilon_1} (\mathfrak{s}^{\varrho})' \left( \left(\upsilon_2^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} - \left(\upsilon_1^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \right) [|\mathtt{f}(\mathfrak{s}, 0, 0)| \\
& + |\mathtt{f}(\mathfrak{s}, \zeta(\mathfrak{s}), \zeta(\mathtt{a} + \mu\,(\mathfrak{s} - \mathtt{a}))) - \mathtt{f}(\mathfrak{s}, 0, 0)|]\, d\mathfrak{s} \\
& + \frac{\varrho^{-\beta}}{\Gamma(\beta)} \int_{\upsilon_1}^{\upsilon_2} (s^{\varrho})' \left(\upsilon_2^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} [|\mathtt{f}(\mathfrak{s}, 0, 0)| \\
& + |\mathtt{f}(\mathfrak{s}, \zeta(\mathfrak{s}), \zeta(\mathtt{a} + \mu\,(\mathfrak{s} - \mathtt{a}))) - \mathtt{f}(\mathfrak{s}, 0, 0)|]\, d\mathfrak{s} \\
\leq\ & \frac{\varrho^{-\beta}\left(\mathtt{M}_0 + 2\mathtt{L}_{\mathtt{f}}\|\zeta\|\right)}{\Gamma(\beta)} \int_{\mathtt{a}}^{\upsilon_1} (s^{\varrho})' \left( \left(\upsilon_2^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} - \left(\upsilon_1^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1} \right) d\mathfrak{s} \\
& + \frac{\varrho^{-\beta}\left(\mathtt{M}_0 + 2\mathtt{L}_{\mathtt{f}}\|\zeta\|\right)}{\Gamma(\beta)} \int_{\upsilon_1}^{\upsilon_2} (s^{\varrho})' \left(\upsilon_2^{\varrho} - \mathfrak{s}^{\varrho}\right)^{\beta-1}\ d\mathfrak{s} \\
\leq\ & \frac{\varrho^{-\beta}\left(\mathtt{M}_0 + 2\mathtt{L}_{\mathtt{f}}\|\zeta\|\right)}{\Gamma(\beta+1)} \Big( -\left(\upsilon_2^{\varrho} - \upsilon_1^{\varrho}\right)^{\beta} + \left(\upsilon_2^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta} \\
& \qquad - \left(\upsilon_1^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta} + \left(\upsilon_2^{\varrho} - \upsilon_1^{\varrho}\right)^{\beta} \Big) \\
\leq\ & \frac{\varrho^{-\beta}\left(\mathtt{M}_0 + 2\mathtt{L}_{\mathtt{f}}\|\zeta\|\right)}{\Gamma(\beta+1)} \left( \left(\upsilon_2^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta} - \left(\upsilon_1^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta} \right).
\end{aligned}
\tag{3.9}
$$

As the function $\upsilon_2 \longmapsto \left(\upsilon_2^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta} - \left(\upsilon_1^{\varrho} - \mathtt{a}^{\varrho}\right)^{\beta}$ is continuous on $\Omega$, the inequality's right side (3.9) tends to 0, as $\upsilon_2 \to \upsilon_1$ independently of $\zeta \in \mathtt{S}$. Since $\mathcal{N}(\mathtt{S}) \subseteq \mathtt{S}$ and $\mathtt{S}$ is bounded, referring to Arzelaŝ-Ascoli theorem (Lemma 2.10), the mapping $\mathcal{N} : \mathtt{S} \to \mathtt{S}$ is compact, and the demonstration was completed. □

Our first existence result is based on Schaefer's fixed point theorem (Lemma 2.11).

**Theorem 3.5.** *Suppose there is a continuous function $\psi : [\mathtt{a}, \mathtt{d}] \longmapsto ]0, +\infty[$, such that*

$$|\mathtt{f}\,(\mathtt{t},\ \zeta(\mathtt{t}),\ \xi(\mathtt{t}))| \leq \psi(\mathtt{t}) \quad \textit{for all}\ \ \mathtt{t} \in \Omega = [\mathtt{a}, \mathtt{d}]. \tag{3.10}$$

*If the above assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are verified, the generalized Caputo pantograph (FDEs) (1.1) has at least one continuous solution defined on $\Omega$.*

*Proof.*
For $\lambda \in ]0, 1[$, let us show that the set $\mathfrak{S} = \{\zeta \in \mathcal{C}^0(\Omega) : \zeta = \lambda\mathcal{N}(\zeta)\}$ is bounded. Given $\zeta \in \mathfrak{S}$ we have, for each $\mathtt{t} \in [\mathtt{a}, \mathtt{d}]$,

$$
\begin{aligned}
|\zeta(\mathtt{t})| &\leq \lambda \cdot |(\mathcal{N}\zeta)(\mathtt{t})| \\
&\leq |(\mathcal{N}\zeta)(\mathtt{t})|.
\end{aligned}
$$

Adopting the same steps as the proof of the lemma 3.4, we obtain

$$\begin{aligned}
|\zeta(\mathtt{t})| \leq & \frac{1}{\left|1-\sum_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}|\mathtt{f}\left(\cdot,\zeta(\cdot),\zeta(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)|\,(\tau_k) \\
& + \frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}|\mathtt{f}(\mathfrak{s},\zeta(\mathfrak{s}),\zeta(\mathtt{a}+\mu\left(\mathfrak{s}-\mathtt{a}\right)))|\,d\mathfrak{s} \\
& \leq \frac{\|\psi\|}{\left|1-\sum_{k=1}^{p}\lambda_k\right|}\left[\frac{\varrho^{-\beta}}{\Gamma(\beta+1)}\right]\sum_{k=1}^{p}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}+\frac{\|\psi\|\varrho^{-\beta}}{\Gamma(\beta+1)}\left(\mathtt{d}^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta} \\
& \leq \frac{\|\psi\|}{\left|1-\sum_{k=1}^{p}\lambda_k\right|}\left[\frac{\varrho^{-\beta}}{\Gamma(\beta+1)}\right]\sum_{k=1}^{p+1}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta},
\end{aligned}$$

where $\lambda_{p+1}=\left|1-\sum_{k=1}^{p}\lambda_k\right|$, $\tau_{p+1}=\mathtt{d}$, and $\|\psi\|=\sup_{\mathtt{t}\in\Omega}|\psi(\mathtt{t})|<\infty$. Therfore,

$$\|y\| \leq \frac{\|\psi\|}{2\mathrm{L}_{\mathtt{f}}}\omega_1,$$

which proves that the set $\mathfrak{S}$ is bounded. According to Lemma 3.4 and Schaefer's fixed point Theorem (Lemma 2.11) we conclude that, the operator $\mathcal{N}$ has at least one fixed point according to . The proof is therefore complete since there exists at least one solution $\zeta(.)$ to the problem (1.1) on $\Omega=[\mathtt{a},\mathtt{d}]$. □

3.3. **Uniqueness result.**

Now, we reveal the circumstances in which the Banach contraction principle theorem ensures that there is only one solution to the pantograph `FDEs` (1.1).

**Theorem 3.6.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *are verified. If the constant* $\omega_1$ *provided by (3.7) meets the following inequality*

$$\omega_1 < 1, \tag{3.11}$$

*there is only one continuous solution to the generalized Caputo pantograph (*`FDEs`*) (1.1) on* $\Omega$.

*Proof.* We developed the proof in two steps:
(i) *Construction of a* $\mathcal{N}$*-stable subspace of* $\mathcal{C}^0(\Omega)$.
For $\zeta\in\mathcal{C}^0(\Omega)$. Adopting the same steps as the proof of the lemma 3.4, it follows

$$|(\mathcal{N}\zeta)(t)| \leq \omega_1\|y\|+\omega_2,$$

where, $\omega_1$ and $\omega_2$ are constants given in (3.7) and (3.8) respectively.

Let $\mathtt{R}\geqslant\frac{\omega_2}{1-\omega_1}$. Consider $\mathtt{B}_{\mathtt{R}}$ the closed ball of radius $\mathtt{R}$, defined as

$$\mathtt{B}_{\mathtt{R}}=\left\{\zeta(.)\in\mathcal{C}^0(\Omega)\ :\ \|\zeta\|\leq\mathtt{R}\right\}.$$

For each $\zeta \in \mathtt{B}_{\mathtt{R}}$, clearly we have

$$\begin{aligned}\|\mathcal{N}(y)\| &\leq \omega_1\|y\| + (1-\omega_1)\mathtt{R}\\ &\leq \omega_1\mathtt{R} + (1-\omega_1)\mathtt{R}\\ &\leq \mathtt{R}.\end{aligned}$$

Then, the mapping $\mathcal{N}$ carries $\mathtt{B}_{\mathtt{R}}$ into itself. Moreover, the closed ball $\mathtt{B}_{\mathtt{R}}$ is bounded, convex, and nonempty.
(ii) *Now, let us show that $\mathcal{N}$ is a contraction mapping.*
For $\zeta,\ \xi \in \mathcal{C}^0(\Omega)$, we have

$$\begin{aligned}|(\mathcal{N}\zeta)(\mathtt{t}) - (\mathcal{N}\xi)(\mathtt{t})| \leq{}& \frac{1}{\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\cdot|\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}(\tau_k,\zeta(\tau_k),\zeta(\mathtt{a}+\mu(\tau_k-\mathtt{a})))\\ &\qquad\qquad - \mathtt{f}(\tau_k,\xi(\tau_k),\xi(\mathtt{a}+\mu(\tau_k-\mathtt{a})))|\\ &+\frac{\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\mathtt{t}}\mathfrak{s}^{\varrho-1}\left(\mathtt{t}^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}|\mathtt{f}(\mathfrak{s},\zeta(\mathfrak{s}),\zeta(\mathtt{a}+\mu(\mathfrak{s}-\mathtt{a})))\\ &\qquad\qquad - \mathtt{f}(\mathfrak{s},\xi(\mathfrak{s}),\xi(\mathtt{a}+\mu(\mathfrak{s}-\mathtt{a})))|\,d\mathfrak{s}\\ \leq{}& \frac{1}{\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\cdot\frac{2\mathrm{L}_{\mathtt{f}}\,\varrho^{1-\beta}}{\Gamma(\beta+1)}\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}\|\zeta-\xi\|\\ &+\frac{2\mathrm{L}_{\mathtt{f}}\,\varrho^{1-\beta}}{\Gamma(\beta+1)}\left(\mathtt{d}^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}\|\zeta-\xi\|\\ \leq{}& \frac{2\mathrm{L}_{\mathtt{f}}\,\varrho^{1-\beta}}{\Gamma(\beta+1)}\left[\frac{1}{\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}+\left(\mathtt{d}^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}\right]\cdot\|\zeta-\xi\|\\ \leq{}& \omega_1\cdot\|\zeta-\xi\|.\end{aligned}$$

Therefore,

$$\|(\mathcal{N}\zeta)-(\mathcal{N}\xi)\| \leq \omega_1\cdot\|\zeta-\xi\|$$

According to the inequality (3.11), $\mathcal{N}$ is a contraction map. By the Banach fixed point theorem (Lemma 2.12) we deduce that $\mathcal{N}$ has a fixed point, which is the unique solution of the generalized Caputo pantograph FDEs (1.1). The proof is finished. □

## 4. Stability in the sense of Ulam Hyers

In this part, we discuss the stability of solutions for pantograph's equations FDEs (1.1). For $\varepsilon > 0$, we consider the following inequation:

$$\left|{}^{\mathtt{C}}\mathfrak{D}_{\mathtt{a}^+}^{\beta;\varrho}\zeta(\mathtt{t}) - \mathtt{f}(\mathtt{t},\zeta(\mathtt{t}),\zeta(\mathtt{a}+\mu(\mathtt{t}-\mathtt{a})))\right| \leq \varepsilon, \quad \mathtt{t}\in\Omega := [\mathtt{a},\mathtt{d}]. \tag{4.1}$$

**Definition 4.1** (Ulam Hyers stability)**.**
The generalized Caputo pantograph FDEs (1.1) are Ulam Hyers stable if there

exists $\delta > 0$ such that, for all $\varepsilon > 0$ and each $\zeta \in \mathcal{C}^0(\Omega)$ solution of (4.1), there exists a unique $\zeta^* \in \mathcal{C}^0(\Omega)$ solution of (1.1), with

$$\sup_{\mathtt{t}\in\Omega} |\zeta(\mathtt{t}) - \zeta^*(\mathtt{t})| \leq \delta\varepsilon.$$

*Remark* 4.2. A function $\zeta \in \mathcal{C}^0(\Omega)$ is a solution of inequality (4.1) if and only if there exists a function $\varpi_\zeta \in \mathcal{C}^0(\Omega)$ which depends on $\zeta$ such that

(a) $|\varpi_\zeta(\mathtt{t})| \leq \varepsilon, \quad \mathtt{t} \in \Omega,$

(b) ${}^{\mathtt{c}}\mathfrak{D}_{a^+}^{\beta;\varrho}\zeta(\mathtt{t}) = \mathtt{f}(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu(\mathtt{t} - \mathtt{a}))) + \varpi_\zeta(\mathtt{t}), \quad \mathtt{t} \in \Omega.$

**Lemma 4.3.** *Given $\zeta \in \mathcal{C}^0(\Omega)$. If the function $\zeta$ is a solution of the inequation (4.1), then $\zeta$ is also solution of the following inequation,*

$$\left|\zeta(\mathtt{t}) - \left[\mathtt{Q}_\zeta + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu(\mathtt{t} - \mathtt{a})))\right]\right| \leq \eta\varepsilon \tag{4.2}$$

*with,* $\mathtt{Q}_\zeta := \frac{1}{1-\sum_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \, \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot, \zeta(\cdot), \zeta(\mathtt{a} + \mu(\cdot - \mathtt{a}))\right)(\tau_k)$ *and*

$$\eta := \frac{\varrho^{-\beta}}{\Gamma(\beta+1)\left(1 - \sum\limits_{k=1}^{p}\lambda_k\right)} \sum_{k=1}^{p+1} \lambda_k \left(\tau_k^\varrho - \mathtt{a}^\varrho\right)^\beta.$$

*Proof.* In view of Remark 4.2, we have:

$$\begin{cases} {}^{\mathtt{c}}\mathfrak{D}_{a^+}^{\beta;\varrho}\zeta(\mathtt{t}) = \mathtt{f}\left(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu(\mathtt{t} - \mathtt{a}))\right) + \varpi_\zeta(\mathtt{t}), \quad \mathtt{t} \in \Omega \\ \zeta(\mathtt{a}) = \sum\limits_{k=1}^{p} \lambda_k \zeta(\tau_k), \quad \tau_k \in \overset{\circ}{\Omega}. \end{cases}$$

Then, by Lemma 3.1, we get

$$\begin{aligned} \zeta(\mathtt{t}) &= \frac{1}{1 - \sum\limits_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \cdot \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}[\mathtt{f}\left(\cdot, \zeta(\cdot), \zeta(\mathtt{a} + \mu(\cdot - \mathtt{a}))\right) + \varpi_\zeta(\cdot)](\tau_k) \\ &\quad + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}[\mathtt{f}\left(\cdot, \zeta(\cdot), \zeta(\mathtt{a} + \mu(\cdot - \mathtt{a}))\right) + \varpi_\zeta(\cdot)](\mathtt{t}) \\ &= \frac{1}{1 - \sum\limits_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\tau_k, \zeta(\tau_k), \zeta(\mathtt{a} + \mu(\tau_k - \mathtt{a}))\right) \\ &\quad + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu(\mathtt{t} - \mathtt{a}))\right) + \frac{1}{1 - \sum\limits_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\varpi_\zeta(\tau_k) + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\varpi_\zeta(\mathtt{t}) \\ &= \left[\mathtt{Q}_\zeta + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\mathtt{t}, \zeta(\mathtt{t}), \zeta(\mathtt{a} + \mu(\mathtt{t} - \mathtt{a}))\right)\right] + \frac{1}{1 - \sum\limits_{k=1}^{p}\lambda_k} \sum_{k=1}^{p} \lambda_k \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\varpi_\zeta(\tau_k) \\ &\qquad + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\varpi_\zeta(\mathtt{t}). \end{aligned} \tag{4.3}$$

From this it follows that

$$\begin{aligned}
\left|\zeta(\mathtt{t})-\left[\mathtt{Q}_{\zeta}+\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\mathtt{t},\zeta(\mathtt{t}),\zeta(\mathtt{a}+\mu(\mathtt{t}-\mathtt{a}))\right)\right]\right| &\leq \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p}\lambda_k\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}|\varpi_{\zeta}|(\tau_k) \\
&\quad + \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\varpi_{\zeta}\left(\mathtt{t}\right) \\
&\leq \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p+1}\lambda_k\left(\frac{\|\varpi_{\zeta}\|\varrho^{1-\beta}}{\Gamma(\beta)}\int_{\mathtt{a}}^{\tau_k}\mathfrak{s}^{\varrho-1}\left(\tau_k^{\varrho}-\mathfrak{s}^{\varrho}\right)^{\beta-1}\,d\mathfrak{s}\right) \\
&\leq \frac{\|\varpi_{\zeta}\|\varrho^{-\beta}}{\Gamma(\beta+1)\left(1-\sum\limits_{k=1}^{p}\lambda_k\right)}\sum_{k=1}^{p+1}\lambda_k\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta} \\
&\leq \varepsilon\cdot\frac{\varrho^{-\beta}}{\Gamma(\beta+1)\left(1-\sum\limits_{k=1}^{p}\lambda_k\right)}\sum_{k=1}^{p+1}\lambda_k\left(\tau_k^{\varrho}-\mathtt{a}^{\varrho}\right)^{\beta}.
\end{aligned}$$

Hence, the proof is established. □

**Theorem 4.4.** *Under the same conditions as in the previous Theorem 3.6, the generalized Caputo pantograph* (FDEs) (1.1) *are stable according to the Ulam-Hyers theory.*

*Proof.* Let $\zeta$ and $\zeta^*$ be two elements of the space $\mathcal{C}^0(\Omega)$ solutions of the problems (4.1) and (1.1) respectively. For $\varepsilon > 0$, according to Lemma 3.2, we get

$$\begin{aligned}
\zeta^*(\mathtt{t}) =& \frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p}\lambda_k\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta^*(\cdot),\zeta^*(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\tau_k) \\
&+ \mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta^*(\cdot),\zeta^*(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\mathtt{t}), \quad \mathtt{t}\in\Omega.
\end{aligned} \tag{4.4}$$

Applying hypothesis $(\mathbf{H_2})$ and the result of Lemma 4.3, it follow

$$
\begin{aligned}
|\zeta(\mathtt{t})-\zeta^*(\mathtt{t})| &\leq \left|\zeta(\mathtt{t})-\left[\mathtt{Q}_\zeta+\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} f(\mathtt{t},\zeta(\mathtt{t}),\zeta(\mathtt{a}+\mu(\mathtt{t}-\mathtt{a})))\right]\right| \\
&\quad+\left|\mathtt{Q}_\zeta-\frac{1}{1-\sum\limits_{k=1}^{p}\lambda_k}\sum_{k=1}^{p}\lambda_k\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta^*(\cdot),\zeta^*(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\tau_k)\right| \\
&\quad+\left|\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho} f(\mathtt{t},\zeta(\mathtt{t}),\zeta(\mathtt{a}+\mu(\mathtt{t}-\mathtt{a})))-\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\mathtt{f}\left(\cdot,\zeta^*(\cdot),\zeta^*(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)(\mathtt{t})\right| \\
&\leq \eta\varepsilon+\frac{1}{\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\sum_{k=1}^{p}|\lambda_k|\cdot\mathfrak{I}_{\mathtt{a}^+}^{\beta;\varrho}\big|\mathtt{f}\left(\cdot,\zeta^*(\cdot),\zeta(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right) \\
&\quad-\mathtt{f}\left(\cdot,\zeta(\cdot),\zeta(\mathtt{a}+\mu(\cdot-\mathtt{a}))\right)\big|(\tau_k)+\frac{\mathtt{L}_\mathtt{f}\,\varrho^{-\beta}}{\Gamma(\beta+1)}\sup_{\mathtt{t}\in\Omega}|\zeta(\mathtt{t})-\zeta^*(\mathtt{t})| \\
&\leq \Lambda\varepsilon+\frac{\mathtt{L}_\mathtt{f}\,\varrho^{-\beta}}{\Gamma(\beta+1)\left|1-\sum\limits_{k=1}^{p}\lambda_k\right|}\left(\sum_{k=1}^{p+1}|\lambda_k|\left(\tau_k^\varrho-\mathtt{a}^\varrho\right)^\beta\right)\sup_{\mathtt{t}\in\Omega}|\zeta(\mathtt{t})-\zeta^*(\mathtt{t})| \\
&\leq \Lambda\varepsilon+\frac{\omega_1}{2}\sup_{\mathtt{t}\in\Omega}|\zeta(\mathtt{t})-\zeta^*(\mathtt{t})|.
\end{aligned}
\tag{4.5}
$$

Passing to the supremum on $\mathtt{t}\in\Omega$ in the right side of the previous inequality, we obtain $\sup\limits_{\mathtt{t}\in\Omega}|\zeta(\mathtt{t})-\zeta^*(\mathtt{t})|\leq\delta\varepsilon$, where

$$
\delta:=\frac{2\eta}{2-\omega_1}.
$$

Hence, the proof is established. □

## 5. Practical examples

**Example 5.1.** In this example, we justify the validity of the first result (Theorem 3.5). For $\mu=\frac{2}{3}\in]\mathtt{0},\mathtt{1}[$, we consider the following the generalized Caputo pantograph fractional differential equation with nonlocal conditions described by ::

$$
\begin{cases}
{}^{\mathtt{C}}\mathfrak{D}_{0^+}^{\frac{2}{3};\frac{1}{2}}\zeta(\mathtt{t})=\mathtt{f}\left(\mathtt{t},\zeta(\mathtt{t}),\zeta\left(1+\frac{2}{3}(\mathtt{t}-1)\right)\right), & \mathtt{t}\in[1,\mathtt{e}] \\
\zeta(1)=\frac{1}{10}\zeta\left(\dfrac{3}{2}\right)+\frac{3}{10}\zeta(2)+\frac{1}{2}\zeta\left(\dfrac{5}{2}\right).
\end{cases}
\tag{5.1}
$$

Where, $\mathtt{f}:[\mathtt{a},\mathtt{d}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function defined as,

$$
\mathtt{f}(\mathtt{t},\zeta,\xi))=\ln(\mathtt{t})\left[\frac{e^{-\mathtt{t}}+\sin(\mathtt{t})}{1+5^{\mathtt{t}}}\left(\frac{|\zeta|}{1+|\zeta|}\right)+\frac{\sqrt{\mathtt{t}}}{5}\left(\frac{1}{1+\exp(-|\xi|)}\right)\right].
$$

Clearly, we have $\mathtt{f}(1,\zeta(1),\zeta(\mu))=\mathtt{0}$. According to Lemma 3.2, the pantograph problem (5.1) is equivalent to the integral equation $\zeta=\mathcal{N}(\zeta)$, with $\zeta\in\mathcal{C}^0(\Omega)$.

Now, we have

$$\sup_{\mathtt{t}\in[1,\mathtt{e}]}|\mathtt{f}(\mathtt{t},0,0)| = \sup_{\mathtt{t}\in[1,\mathtt{e}]}\left|\ln(\mathtt{t})\cdot\frac{\sqrt{\mathtt{t}}}{10}\right| = \frac{\sqrt{\mathtt{e}}}{10}$$

Then, the hypothesis $(\mathbf{H_1})$ holds with $\mathtt{M}_0 = \frac{\sqrt{\mathtt{e}}}{10}$. Furthermore, for $\mathtt{t}\in[1,\mathtt{e}]$, $(\zeta_1,\ \zeta_2)\in\mathbb{R}^2$ and $(\xi_1,\ \xi_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}|\mathtt{f}(\mathtt{t},\zeta_1,\zeta_2)-\mathtt{f}(\mathtt{t},\xi_1,\xi_2)| \le{}& \frac{\ln(\mathtt{t})\left(e^{-\mathtt{t}}+\sin(\mathtt{t})\right)}{1+5^{\mathtt{t}}}\left|\frac{|\zeta_1|}{1+|\zeta_1|}-\frac{|\xi_1|}{1+|\xi_1|}\right| \\ &+\frac{\sqrt{\mathtt{t}}\ln(\mathtt{t})}{5}\left|\frac{1}{1+\exp(-|\zeta_2|)}-\frac{1}{1+\exp(-|\xi_2|)}\right|.\end{aligned}$$

The Logistic Sigmoid function $(\mathtt{t},\xi)\mapsto\frac{1}{1+\exp(-|\xi|)}$ is bounded and $\frac{1}{4}$- Lipschitz on $\mathbb{R}^2$, then

$$\begin{aligned}|\mathtt{f}(\mathtt{t},\zeta_1,\zeta_2)-\mathtt{f}(\mathtt{t},\xi_1,\xi_2)| \le{}& \sup_{\mathtt{t}\in[1,\mathtt{e}]}\frac{\ln(\mathtt{t})\left(e^{-\mathtt{t}}+\sin(\mathtt{t})\right)}{1+5^{\mathtt{t}}}|\zeta_1-\xi_1| \\ &+\sup_{\mathtt{t}\in[1,\mathtt{e}]}\frac{\sqrt{\mathtt{t}}\ln(\mathtt{t})}{20}|\zeta_2-\xi_2| \\ \le{}& \max\left\{\frac{1+\mathtt{e}}{2\mathtt{e}},\frac{\sqrt{\mathtt{e}}}{20}\right\}\left(|\zeta_1-\xi_1|+|\zeta_2-\xi_2|\right)\end{aligned}$$

Therefore, the hypothesis $(\mathbf{H_2})$ holds with $\mathtt{L_f} = \frac{1+\mathtt{e}}{2\mathtt{e}}$. Moreover, we have

$$\sup_{\substack{\mathtt{t}\in[1,\mathtt{e}]\\(\zeta,\xi)\in\mathbb{R}^2}}|\mathtt{f}(\mathtt{t},\zeta,\xi)| \le \ln(\mathtt{t})\left(\frac{e^{-\mathtt{t}}+1}{(1+5^{\mathtt{t}})}+\frac{\sqrt{\mathtt{t}}}{5}\right).$$

The function $\psi:\mathtt{t}\mapsto\ln(\mathtt{t})\left(\frac{e^{-\mathtt{t}}+1}{1+5^{\mathtt{t}}}+\frac{\sqrt{\mathtt{t}}}{5}\right)$ is continuous on $[1,\mathtt{e}]$. All conditions of Theorem 3.5 hold. Thus, there is at least one defined and continuous solution on $[1,\mathtt{e}]$ for the generalized Caputo pantograph $(\mathtt{FDEs})$ (5.1).

**Example 5.2.** In this example, we justify the validity of the second result (Theorem 3.6). For $\mu=\frac{1}{2}\in]0,1[$, we consider the following the generalized Caputo pantograph fractional differential equation with nonlocal conditions described by :

$$\begin{cases}{}^{\mathtt{C}}\mathfrak{D}_{0^+}^{\frac{1}{2};1}\zeta(\mathtt{t}) = \dfrac{\mathtt{t}^2}{10}\left(\dfrac{\exp\left(-|\zeta(\mathtt{t})|\right)}{e^{\mathtt{t}+1}}+\dfrac{1}{3+\ln(1+\mathtt{t})}\cdot\left(\dfrac{\left|\zeta\left(\frac{1}{2}\mathtt{t}\right)\right|}{5+\left|\zeta\left(\frac{1}{2}\mathtt{t}\right)\right|}\right)\right), & \mathtt{t}\in[0,1]\\ \zeta(0) = \frac{1}{10}\zeta\left(\dfrac{1}{3}\right)+\frac{3}{10}\zeta\left(\dfrac{3}{5}\right)+\frac{1}{2}\zeta\left(\dfrac{3}{4}\right),\end{cases} \tag{5.2}$$

where $\Omega=[0,1]$, $\varrho=1$, $\beta=\frac{1}{2}$, $\lambda_1=\frac{1}{10}$, $\lambda_2=\frac{3}{10}$, $\lambda_3=\frac{1}{2}$, $\tau_1=\frac{1}{3}$, $\tau_2=\frac{3}{5}$, $\tau_3=\frac{3}{4}$ $(p=3)$, and

$$\mathtt{f}(\mathtt{t},\zeta,\xi) = \frac{\mathtt{t}^2}{10}\left(\frac{\exp(-|\zeta|)}{e^{\mathtt{t}+1}}+\frac{1}{3+\ln(1+\mathtt{t})}\cdot\left(\frac{|\zeta|}{5+|\zeta|}\right)\right).$$

Clearly, we have $\mathtt{f}(0,\zeta(0),\zeta(0))=0$. According to Lemma 3.2, the pantograph problem (5.2) is equivalent to the integral equation $\zeta=\mathcal{N}(\zeta)$, with $\zeta\in\mathcal{C}^0(\Omega)$. Moreover,

$$\sup_{\mathtt{t}\in[0,1]}|\mathtt{f}(\mathtt{t},0,0)|=\sup_{\mathtt{t}\in[0,1]}\frac{\mathtt{t}^2}{10}\left|\frac{1}{e^{\mathtt{t}+1}}\right|=\frac{1}{10\mathtt{e}}$$

Then, the hypothesis $(\mathbf{H_1})$ holds with $\mathtt{M}_0=\frac{1}{10e}$.
Let $\mathtt{t}\in[0,1]$, for all $(\zeta_1,\ \zeta_2)\in\mathbb{R}^2$ and $(\xi_1,\ \xi_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}|\mathtt{f}\left(\mathtt{t},\zeta_1,\zeta_2\right)-\mathtt{f}\left(\mathtt{t},\xi_1,\xi_2\right)| &\leq \frac{\mathtt{t}^2}{10(e^{t+1})}\left|\exp\left(-|\zeta_1|\right)-\exp\left(-|\xi_1|\right)\right| \\ &\quad+\frac{\mathtt{t}^2}{10(3+\ln\left(1+\mathtt{t}\right))}\left|\frac{|\zeta_2|}{5+|\zeta_2|}-\frac{|\xi_2|}{5+|\xi_2|}\right| \\ &\leq \max\left\{\frac{1}{30},\frac{1}{10\mathtt{e}}\right\}\left(|\zeta_1-\xi_1|+|\zeta_2-\xi_2|\right)\end{aligned}$$

Therefore, hypothesis $(\mathbf{H_2})$ holds with $\mathtt{L}_\mathtt{f}=\dfrac{1}{10\mathtt{e}}$. Moreover, we have

$$\sup_{\substack{\mathtt{t}\in[1,\mathtt{e}]\\(\zeta,\xi)\in\mathbb{R}^2}}|\mathtt{f}(\mathtt{t},\zeta,\xi)|\leq\frac{\mathtt{t}^2}{10}\left(\frac{1}{e^{\mathtt{t}+1}}+\frac{1}{25}\cdot\frac{1}{3+\ln\left(1+\mathtt{t}\right)}\right).$$

The function $\psi:\mathtt{t}\mapsto\frac{\mathtt{t}^2}{10}\left(\frac{1}{e^{\mathtt{t}+1}}+\frac{1}{25}\cdot\frac{1}{3+\ln\left(1+\mathtt{t}\right)}\right)$ is continuous on $[0,1]$. According to Theorem 3.5, the generalized Caputo pantograph (FDEs) (5.2) have at least one solution.
By substituting the values noted above, in formulas (3.7) and (3.8), we obtain

$$\begin{aligned}\omega_1 &= \frac{1/5\mathtt{e}}{0.89}\cdot\left(\sqrt{\frac{1}{3}}+3\sqrt{\frac{3}{5}}+5\sqrt{\frac{3}{4}}+1\right) \\ &\approx 0.65\end{aligned}$$

$$\begin{aligned}\omega_2 &\approx \frac{1/30\mathtt{e}}{\Gamma(1.5)}\cdot\left(\sqrt{\frac{1}{3}}+3\sqrt{\frac{3}{5}}+5\sqrt{\frac{3}{4}}+1\right) \\ &\approx 0.14\end{aligned}$$

Since $\omega_1<1$, according to Theorem (3.6) the generalized Caputo pantograph (FDEs) (5.2) has a unique solution $\zeta^*$ defined and continuous on $[0,1]$. The generalized Caputo pantograph problem (5.2) is also Ulam-Hyers stable with reference to Theorem 4.4.

## 6. Conclusion

Pantograph equations are powerful tools for modeling systems where scaling, self-similarity, and proportional delays are key features. In this paper, we looked at the existence of solutions to a set of these equations using the generalized Caputo derivative and special nonlocal boundary value requirements. The availability of solutions was proven using Schafer's fixed point theorem, and under additional hypotheses, Banach's theorem guarantees a unique solution, which is

stable according to Ulam-Hyers theory.
In the future, we plan to conduct a qualitative study on the behaviour of the solutions, and we may even try to produce an approximate solution using a numerical approach.

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