Gulf Journal of Mathematics ISSN: 2309-4966
Vol 19, Issue 1 (2025) 374-391 https://doi.org/10.56947/gjom.v19i1.2576

DYNAMICS OF DEPENDENCY: FINITE DIFFERENCE SOLUTIONS TO THE BLACK-SCHOLES PDE WITH COPULAS

HERMAN TIEMTORÉ[1] and REMI GUILLAUME BAGRÉ[2*]

Abstract. The copula function is a tool for modeling any form of dependency that may exist between several random variables. Given n random variables, there exists an n- dimensional copula that captures their dependence (Sklar's Theorem). In this paper, we present a method for obtaining a class of copulas that models the dependence between the underlyings of a bivariate option over time until the option expires. This class of copulas verifies the 2-dimensional Black-Scholes partial differential equation. Given the difficulty of obtaining an analytical solution to such an equation, We use a Finite Difference approximation to obtain an approximated solution. We show that the numerical scheme obtained is conditionally convergent. Numerical results are also presented using MATLAB software.

1. Introduction

The financial risk is a menace to the world of financial markets, and in particular to investors in options markets. It is therefore important to provide oneself with the appropriate tools to manage it, in order to avoid failures or insolvency. Managing the overall risk of a multi-asset investment requires a good understanding of the joint behavior of all assets. When pricing options with multiple underlyings, we need to take account of time and the joint evolution of the underlyings. To meet this need, we can use the copula, a flexible statistical tool for modeling dependence between several random variables, whatever their nature. To do this, we need to find the most suitable copula to minimize estimation bias. Given two random variables x and y with joint distribution function H_{xy} with margins F and G,respectively, there exists a copula C_{xy} such that $H_{xy}(x, y) = C_{xy}(F(x), H(y))$. From this result, we understand that the goodness of fit of the joint distribution function H_{xy} depends strongly on the quality of the copula chosen. For an option with two underlyings S_1 and S_2, its price $V = V(S_1, S_2, t)$, where $t \in [0;T]$ and T the option expiry date, satisfies the partial differential equation (PDE), also

Date: Received: Jan 2, 2025; Accepted: Jan 14, 2025.

* Corresponding author.

2020 *Mathematics Subject Classification.* 35A25, 62H05, 62H20, 65C20.

Key words and phrases. Black-Scholes PDE, Copulas, Finite Difference Method, Bivariate option.

known as the 2-dimensional Black-Scholes equation, given by (see [4]):

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{l,p=1}^{2}\sigma_{l,p}S_lS_p\frac{\partial^2 V}{\partial S_l\partial S_p}+\sum_{l=1}^{2}(r-q_l)S_l\frac{\partial V}{\partial S_l}-rV=0 \tag{1.1}$$

Where $(\sigma_{l,p})_{(l,p)\in\{1,2\}}$ are the coefficients of the variance-covariance matrix of the random vector (S_1,S_2), q_l the dividend rate of the underlying $S_l, l=1,2$, and $r>0$ the risk-free interest rate. In the literature, equation (1.1) has a final condition $V=V(S_1,S_2,T)$. The suitable copula for modeling the joint behavior of the random vector (S_1,S_2) at each time $t\in[0;T]$ would be the evolutive copula $\mathbf{C}=\mathbf{C}(x,\nu,t)$ satisfying equation (1.1), where x and ν are uniform transformations on $[0;1]$ of the variables S_1 and S_2 respectively. More precisely, we have:

$$\frac{\partial \mathbf{C}}{\partial t}+\frac{1}{2}\left(\sigma_1^2x^2\frac{\partial^2\mathbf{C}}{\partial x^2}+2\sigma_{1,2}x\nu\frac{\partial^2\mathbf{C}}{\partial x\partial\nu}+\sigma_2^2\nu^2\frac{\partial^2\mathbf{C}}{\partial\nu^2}\right)+(r-q_1)x\frac{\partial\mathbf{C}}{\partial x}+ \\ (r-q_2)\nu\frac{\partial\mathbf{C}}{\partial\nu}-r\mathbf{C}=0 \tag{1.2}$$

where $\mathbf{C}=\mathbf{C}(x,\nu,t),\forall(x,\nu,t)\in(0;1)^2\times(0;T]$.

In this work, we study the case where the dependence between the underlyings is non-linear ($\sigma_{1,2}=0$) and we assume that the underlying assets do not pay a dividend during the option period ($q_l=0; l=1,2$). Thus, equation (1.2) becomes as follows:

$$\frac{\partial \mathbf{C}}{\partial t}+\frac{1}{2}\left(\sigma_1^2x^2\frac{\partial^2\mathbf{C}}{\partial x^2}+\sigma_2^2\nu^2\frac{\partial^2\mathbf{C}}{\partial\nu^2}\right)+rx\frac{\partial\mathbf{C}}{\partial x}+r\nu\frac{\partial\mathbf{C}}{\partial\nu}-r\mathbf{C}=0 \tag{1.3}$$

Since the basic model is defined with a final condition, we apply the change of variable $\tau=T-t$ in order to reduce the problem to an initial condition problem. With the change of variable, we have:

$$\frac{\partial \mathbf{C}}{\partial t}=\frac{\partial\tau}{\partial t}.\frac{\partial\mathbf{C}}{\partial\tau}=-\frac{\partial\mathbf{C}}{\partial\tau}$$

Thus, we rewrite the problem (1.3) as follows:

$$\frac{\partial \mathbf{C}}{\partial \tau}=\frac{1}{2}\left(\sigma_1^2x^2\frac{\partial^2\mathbf{C}}{\partial x^2}+\sigma_2^2\nu^2\frac{\partial^2\mathbf{C}}{\partial\nu^2}\right)+rx\frac{\partial\mathbf{C}}{\partial x}+r\nu\frac{\partial\mathbf{C}}{\partial\nu}-r\mathbf{C} \tag{1.4}$$

Now with a given initial condition $\mathbf{C}_0(x,\nu)=\mathbf{C}(x,\nu,0)$ representing the dependency between the two underlying assets before the option is signed. Thus, we pose $\mathbf{C}_0(x,\nu)=f(x,\nu)$, where f defines a copula. Furthermore, by definition, the function $\mathbf{C}$ must satisfy the so-called boundary conditions and 2-growth condition (see Section 2). More precisely, for all $x,\nu\in[0;1]$ and $\tau\in[0;T]$ we have:

$$\mathbf{C}(0,\nu,\tau)=\mathbf{C}(x,0,\tau)=0,\mathbf{C}(1,\nu,\tau)=\nu,\mathbf{C}(x,1,\tau)=x$$

and

$$\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial\nu\partial x}\geq 0$$

However, adding the 2-growth condition to the problem is not necessary since the stability of the numerical scheme, shown in Proposition 3.6, and the fact that $\mathbf{C}_0$ is a copula guarantee this condition over time. Thus, the aim of this article is to construct the family of temporal copulas that are solutions of the problem $(\mathcal{P})$ defined for all $(x,\nu,t) \in [0;1]^2 \times [0;T]$ by:

$$(\mathcal{P}) : \begin{cases} \dfrac{\partial \mathbf{C}}{\partial \tau} = \dfrac{1}{2}\left(\sigma_1^2 x^2 \dfrac{\partial^2 \mathbf{C}}{\partial x^2} + \sigma_2^2 \nu^2 \dfrac{\partial^2 \mathbf{C}}{\partial \nu^2}\right) + rx\dfrac{\partial \mathbf{C}}{\partial x} + r\nu \dfrac{\partial \mathbf{C}}{\partial \nu} - r\mathbf{C} \\ \mathbf{C}(0,\nu,\tau) = \mathbf{C}(x,0,\tau) = 0 \\ \mathbf{C}(1,\nu,\tau) = \nu \text{ and } \mathbf{C}(x,1,\tau) = x, \\ \mathbf{C}_0(x,\nu) = f(x,\nu) \end{cases}$$

The family of copulas of the problem $(\mathcal{P})$ models the dependence between the option's two underlyings at each time $\tau \in [0;T]$. Knowing the joint random behavior of the two underlying assets instantaneously enables decisions to be taken at the opportune moment to avoid substantial losses. However, we note that the analytical solution to this type of problem is not generally known. Thus, numerical methods can be used to obtain an approximated solution by considering a finite number of points in the problem domain (discretization process(see [3],[2],[10],[9])). For our study, we use a finite difference approach, which is widely used and easy to implement. This method involves first subdividing the domain of each variable into a finite number of regular intervals whose amplitudes are called **step**, in order to form a space grid on $[0;1]^2$ at each regular instant. The nodes of the grid are the points at which we calculate the copula values for the instant in question. Finally, we approach the differential operators by truncated Taylor developments. We note that, in general, these class of copulas are suitable for modeling dynamic dependencies.

2. Preliminaries

For the rest, I denotes the unit interval and I^2 denotes the unit square ($I = [0;1]$ and $I^2 = I \times I = [0;1] \times [0;1]$).

Definition 2.1. [5, 8] An absolutely continuous bivariate copula is a two-variable function $\mathbf{C}$ satisfying :

(1) For all $x, \nu \in I$

$$\mathbf{C}(0,\nu) = \mathbf{C}(x,0) = 0 \tag{2.1}$$

and

$$\mathbf{C}(1,\nu) = \nu \text{ and } \mathbf{C}(x,1) = x \tag{2.2}$$

(2) For all $(x,\nu) \in I^2$,

$$\frac{\partial^2 \mathbf{C}(x,\nu)}{\partial \nu \partial x} \geq 0 \tag{2.3}$$

The conditions (2.1) and (2.2) are the boundary conditions and the (2.3) designates the 2-growth condition .

By this definition, we deduce that evolution copula $\mathbf{C}(.,.,t)$ satisfies (2.1),(2.2) and (2.3). More concretely, we write the follow (see [10]):

(1) For all $(x, \nu, t) \in I^2 \times [0; T]$

$$\mathbf{C}(0, \nu, t) = \mathbf{C}(x, 0, t) = 0, \tag{2.4}$$

and

$$\mathbf{C}(1, \nu, t) = \nu \text{ and } \mathbf{C}(x, 1, t) = x \tag{2.5}$$

(2) For all $(x, \nu, t) \in I^2 \times [0, T]$, we have

$$\frac{\partial^2 \mathbf{C}(x, \nu, t)}{\partial \nu \partial x} \geq 0$$

Theorem 2.2. *[Sklar's Theorem](see [8])*
Let Λ be a two-variable distribution whose marginal distributions are Λ_1 and Λ_2. There exist a two-variable copula $\mathbf{C}$ verifying :

$$\forall\, (\lambda_1, \lambda_2) \in \mathbb{R}^2,\ \Lambda(\lambda_1, \lambda_2) = \mathbf{C}(\Lambda_1(\lambda_1), \Lambda_2(\lambda_2)) \tag{2.6}$$

Moreover, the copula $\mathbf{C}$ is unique if the marginals Λ_1 and Λ_2 are continuous. If not, it is uniquely defined on Range$\Lambda_1 \times$ RangeΛ_2.

From Theorem 2.2, we understand that when the behavior of random variables evolves with time, equation (2.6) can be written as :

$$\forall\, (\lambda_1, \lambda_2) \in \mathbb{R}^2,\ \Lambda(\lambda_1, \lambda_2, t) = \mathbf{C}(\Lambda_1(\lambda_1), \Lambda_2(\lambda_2), t) \tag{2.7}$$

Definition 2.3. [1, 7, 11] Let $s > 0$ and $\varphi : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ be a sufficiently regular function. We say that a finite-difference operator $D_{s,k}$ of **step** s is a consistent approximation of order p of $\dfrac{\partial^l \varphi}{\partial x_k^l}$ for the norm $\|\ .\ \|$ if we have :

$$\left\| (D_{s,k}\varphi)(\mathbf{x}) - \frac{\partial^l \varphi}{\partial x_k^l}(\mathbf{x}) \right\| \leq L s^p$$

where L is a constant independent of s and $\mathbf{x} = (x_1, ..., x_m)$.

Definition 2.4. [1, 7] Let P be a problem with initial data $\mathbf{C}^0$. A finite difference scheme is stable for a given norm $\|\ .\ \|$, if there exists a constant L the norm if there exists a constant L such that

$$\|\mathbf{C}^s\| \leq L \|\mathbf{C}^0\| \text{ for all } s \geq 1 \tag{2.8}$$

for all initial value $\mathbf{C}^0$ where $\mathbf{C}^s = (\mathbf{C}^s_{p,l})_{0 \leq p,l \leq M}$.

Theorem 2.5 (Lax Theorem)**.** [3, 6] *Given a properly posed initial value problem, a finite difference scheme is convergent if and only if the scheme is consistent and stable.*

For the rest, we will use the infinite norm given by

$$\|\varphi(x)\|_\infty = \sup_x \{\varphi(x)\} \text{ or } \|V\|_\infty = \max_p \{V_p\} \tag{2.9}$$

when φ is a scalar function and V a vector.

Theorem 2.6. [8] *Let $\mathbf{C}_N$ denote the empirical copula for the sample $\{(x_k, y_k)\}_{1\leq k\leq N}$. If ρ denotes the sample version of the Spearman's rho, then*

$$\rho = \frac{12}{N^2-1}\sum_{p=1}^{N}\sum_{l=1}^{N}\left[\mathbf{C}_N\left(\frac{p}{N},\frac{l}{N}\right) - \frac{p}{N}\frac{l}{N}\right]$$

3. The main contributions of the paper

In this section, we present the finite difference approximation of problem $(\mathcal{P})$. We demonstrate some results derived from the approximation. Using MATLAB software we present graphical solutions of our problem.

3.1. **Theoretical results.** As described in the introduction, we consider the discretization below.
Let be M, L be integers not equal to zero and let put,

$$h = \frac{1}{M}$$

the mesh in the u and v directions.

$$\Delta\tau = \frac{T}{L}$$

the mesh for discretization of the time, and

$$u_p = ph = \frac{p}{M},\ \text{for } p \in [|0; M|]$$

$$\nu_l = lh = \frac{l}{M},\ \text{for } l \in [|0; M|]$$

$$\tau^s = s\,\Delta\tau = s\frac{T}{L},\ \text{for } s \in [|0; L|]$$

Proposition 3.1. *Let $\mathbf{C} \in \mathcal{C}^{4,2}(I^2 \times [0; T])$, a copula verified problem $(\mathcal{P})$. Let M, L, h and $\Delta\tau$ be defined above. By posing $\mathbf{C}^s_{p,l} = \mathbf{C}(ph, lh, s\Delta\tau)$ $\forall(p, l, s) \in [|0; M|]^2 \times [|0; L|]$, the approximation of $\mathbf{C}(x, \nu, \tau)$, then a finite difference scheme of the main equation can be given as follows:*

$$\forall(p, l, s) \in [|0; M|]^2 \times [|0; L|],$$
$$\mathbf{C}^{s+1}_{p,l} = \Delta\tau\alpha_p\mathbf{C}^s_{p-1,l} + \Delta\tau\beta_l\mathbf{C}^s_{p,l-1} + (1-\Delta\tau\gamma_{p,l})\mathbf{C}^s_{p,l} + \Delta\tau\mu_p\mathbf{C}^s_{p+1,l} + \Delta\tau\eta_l\mathbf{C}^s_{p,l+1} \tag{3.1}$$

Where

$$\alpha_p = \frac{\sigma_1^2x_p^2}{2h^2} - \frac{rx_p}{2h};\quad \beta_l = \frac{\sigma_2^2\nu_l^2}{2h^2} - \frac{r\nu_l}{2h};\quad \mu_p = \frac{\sigma_1^2x_p^2}{2h^2} + \frac{rx_p}{2h};\quad \eta_l = \frac{\sigma_2^2\nu_l^2}{2h^2} + \frac{r\nu_l}{2h}$$

$$\gamma_{p,l} = \alpha_p + \mu_p + \beta_l + \eta_l + r = \frac{\sigma_1^2x_p^2}{h^2} + \frac{\sigma_2^2\nu_l^2}{h^2} + r$$

Proof. Applying Taylor's formula to the copula $\mathbf{C}$ in the direction of τ at order 1, we have :

$$\mathbf{C}(x,\nu,\tau+\Delta\tau)=\mathbf{C}(x,\nu,\tau)+\Delta\tau\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial\tau}+\epsilon(\Delta\tau) \tag{3.2}$$

Rewriting equation (3.2), we obtain

$$\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial\tau}=\frac{\mathbf{C}(x,\nu,\tau+\Delta\tau)-\mathbf{C}(x,\nu,\tau)}{h}-\epsilon(\Delta\tau) \tag{3.3}$$

Equation (3.3) can be approximated by finite difference as follows:

$$\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial\tau}\simeq\frac{\mathbf{C}^{s+1}_{p,l}-\mathbf{C}^{s}_{p,l}}{\Delta\tau} \tag{3.4}$$

Applying again Taylor's formula to order 2 to the copula $\mathbf{C}$ in the x direction, we have:

$$\mathbf{C}(x+h,\nu,\tau)=\mathbf{C}(x,\nu,\tau)+h\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial x}+\frac{h^2}{2}\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial x^2}+\epsilon(h^2) \tag{3.5}$$

and

$$\mathbf{C}(x-h,\nu,\tau)=\mathbf{C}(x,\nu,\tau)-h\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial x}+\frac{h^2}{2}\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial x^2}+\epsilon(h^2) \tag{3.6}$$

Summing equations (3.5) and (3.6), we have the following:

$$\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial x^2}=\frac{\mathbf{C}(x+h,\nu,\tau)-2\mathbf{C}(x,\nu,\tau)+\mathbf{C}(x-h,\nu,\tau)}{h^2}+\epsilon(h^2) \tag{3.7}$$

And doing the difference of Equations (3.5) and (3.6), we get:

$$\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial x}=\frac{\mathbf{C}(x+h,\nu,\tau)-\mathbf{C}(x-h,\nu,\tau)}{2h} \tag{3.8}$$

The approximation by finite difference of Equation (3.7) is given as

$$\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial x^2}\backsimeq\frac{\mathbf{C}^{s}_{p+1,l}-2\mathbf{C}^{s}_{p,l}+\mathbf{C}^{s}_{p-1,l}}{h^2} \tag{3.9}$$

and the approximation by finite difference of Equation (3.8) is given as

$$\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial x}\simeq\frac{\mathbf{C}^{s}_{p+1,l}-\mathbf{C}^{s}_{p-1,l}}{2h} \tag{3.10}$$

By reasoning similarly, we obtain that :

$$\frac{\partial^2\mathbf{C}(x,\nu,\tau)}{\partial \nu^2}\backsimeq\frac{\mathbf{C}^{s}_{p,l+1}-2\mathbf{C}^{s}_{p,l}+\mathbf{C}^{s}_{p,l-1}}{h^2} \tag{3.11}$$

and

$$\frac{\partial\mathbf{C}(x,\nu,\tau)}{\partial \nu}\simeq\frac{\mathbf{C}^{s}_{p,l+1}-\mathbf{C}^{s}_{p,l-1}}{2h} \tag{3.12}$$

By putting the finite difference approximations of the partial derivatives given by Equations (3.4), (3.9), (3.10), (3.11) and (3.12) in the main equation of problem $(\mathcal{P})$, we have

$$\frac{\mathbf{C}_{p,l}^{s+1}-\mathbf{C}_{p,l}^{s}}{\Delta\tau}=\frac{\sigma_1^2x_p^2}{2}\frac{\mathbf{C}_{p+1,l}^{s}-2\mathbf{C}_{p,l}^{s}+\mathbf{C}_{p-1,l}^{s}}{h^2}+\frac{\sigma_2^2\nu_l^2}{2}\frac{\mathbf{C}_{p,l+1}^{s}-2\mathbf{C}_{p,l}^{s}+\mathbf{C}_{p,l-1}^{s}}{h^2}+ rx_p\frac{\mathbf{C}_{p+1,l}^{s}-\mathbf{C}_{p-1,l}^{s}}{2h}+r\nu_l\frac{\mathbf{C}_{p,l+1}^{s}-\mathbf{C}_{p,l-1}^{s}}{2h}-r\mathbf{C}_{p,l}^{s} \quad (3.13)$$

After some operations, we have,

$$\mathbf{C}_{p,l}^{s+1}=\mathbf{C}_{p,l}^{s}+\Delta\tau\left[\left(\frac{\sigma_1^2x_p^2}{2h^2}-\frac{rx_p}{2h}\right)\mathbf{C}_{p-1,l}^{s}+\left(\frac{\sigma_2^2\nu_l^2}{2h^2}-\frac{r\nu_l}{2h}\right)\mathbf{C}_{p,l-1}^{s}-\left(\frac{\sigma_1^2x_p^2}{h^2}+\frac{\sigma_2^2\nu_l^2}{h^2}+r\right)\mathbf{C}_{p,l}^{s}+\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{rx_p}{2h}\right)\mathbf{C}_{p+1,l}^{s}+\left(\frac{\sigma_2^2\nu_l^2}{2h^2}+\frac{r\nu_l}{2h}\right)\mathbf{C}_{p,l+1}^{s}\right]$$

By posing,

$$\alpha_p = \frac{\sigma_1^2x_p^2}{2h^2}-\frac{rx_p}{2h};\quad \beta_l=\frac{\sigma_2^2\nu_l^2}{2h^2}-\frac{r\nu_l}{2h};\quad \mu_p=\frac{\sigma_1^2x_p^2}{2h^2}+\frac{rx_p}{2h};\quad \eta_l=\frac{\sigma_2^2\nu_l^2}{2h^2}+\frac{r\nu_l}{2h}$$

$$\gamma_{p,l} = \frac{\sigma_1^2x_p^2}{h^2}+\frac{\sigma_2^2\nu_l^2}{h^2}+r,$$

we deduce the result. □

For all $(p,l,s)\in[|0;M|]^2\times[|0;L|]$, the approximation of the boundary conditions can be given as follow :

$$\begin{cases}\mathbf{C}_{p,0}^{s}=\mathbf{C}_{0,l}^{s}=0;\\ \mathbf{C}_{p,M}^{s}=\dfrac{p}{M}\text{ and }\mathbf{C}_{M,l}^{s}=\dfrac{l}{M}\end{cases}\quad (3.14)$$

And the initial condition is given as follow :

$$\mathbf{C}_{p,l}^{0}=f\left(\frac{p}{M},\frac{l}{M}\right) \quad (3.15)$$

So the approximate problem of problem $(\mathcal{P})$ can be given by :

$$(\mathcal{P}_{h,\Delta\tau}):\begin{cases}\mathbf{C}_{p,l}^{s+1}=\Delta\tau\alpha_p\mathbf{C}_{p-1,l}^{s}+\Delta\tau\beta_l\mathbf{C}_{p,l-1}^{s}+(1-\Delta\tau\gamma_{p,l})\mathbf{C}_{p,l}^{s}+\Delta\tau\mu_p\mathbf{C}_{p+1,l}^{s}+\Delta\tau\eta_l\mathbf{C}_{p,l+1}^{s}\\ \mathbf{C}_{p,0}^{s}=\mathbf{C}_{0,l}^{s}=0;\\ \mathbf{C}_{p,M}^{s}=\dfrac{p}{M};\\ \mathbf{C}_{M,l}^{s}=\dfrac{l}{M};\\ \mathbf{C}_{p,l}^{0}=f\left(\frac{p}{M},\frac{l}{M}\right).\end{cases}$$

Proposition 3.2. *Let $s \in \{0,1,...,L\}$ and $\mathbf{C}^s$ be a solution at time τ^s of the problem $(\mathcal{P}_{h,\Delta\tau})$. The density function c^s of the copula $\mathbf{C}^s$ can be approximated as given in the system (S).*

$$(S):\begin{cases} c^s_{p,l} = \dfrac{\mathbf{C}^s_{p+1,l+1} - \mathbf{C}^s_{p+1,l} - \mathbf{C}^s_{p,l+1} + \mathbf{C}^s_{p,l}}{h^2} & \textit{if } (p,l) \in \{0,1,...,M-1\}^2 \\ c^s_{M,l} = \dfrac{\mathbf{C}^s_{M,l+1} - \mathbf{C}^s_{M,l} - \mathbf{C}^s_{M-1,l+1} + \mathbf{C}^s_{M-1,l}}{h^2} & \textit{if } l \in \{0,1,...,M-1\} \\ c^s_{p,M} = \dfrac{\mathbf{C}^s_{p+1,M} - \mathbf{C}^s_{p+1,M-1} - \mathbf{C}^s_{p,M} + \mathbf{C}^s_{p,M-1}}{h^2} & \textit{if } p \in \{0,1,...,M-1\} \\ c^s_{M,M} = \dfrac{\mathbf{C}^s_{M,M} - \mathbf{C}^s_{M,M-1} - \mathbf{C}^s_{M-1,M} + \mathbf{C}^s_{M-1,M-1}}{h^2} & \end{cases}$$

Proof. Our study concerns absolutely copulas. So, we have

$$c^s(x,\nu) = \frac{\partial^2 \mathbf{C}^s(x,\nu)}{\partial\nu\partial x} \tag{3.16}$$

- To obtain the first equation, we apply a Taylor development off-center to the right in the directions of x and ν of Equation (3.16). More precisely, we have:

$$\begin{aligned} c^s(x,\nu) &= \frac{\partial}{\partial\nu}\left(\frac{\mathbf{C}^s(x+h,\nu) - \mathbf{C}^s(x,\nu)}{h}\right) \\ &= \frac{1}{h}\left(\frac{\partial\mathbf{C}^s(x+h,\nu)}{\partial\nu} - \frac{\partial\mathbf{C}^s(x,\nu)}{\partial\nu}\right) \\ &= \frac{\mathbf{C}^s(x+h,\nu+h) - \mathbf{C}^s(x+h,\nu)}{h^2} - \frac{\mathbf{C}^s(x,\nu+h) - \mathbf{C}^s(x,\nu)}{h^2} \end{aligned}$$

In the discrete case, we have :

$$c^s_{p,l} = \frac{\mathbf{C}^s_{p+1,l+1} - \mathbf{C}^s_{p+1,l} - \mathbf{C}^s_{p,l+1} + \mathbf{C}^s_{p,l}}{h^2} \tag{3.17}$$

In our work, $\mathbf{C}^s$ is a matrix of size $(M+1)^2$ with indexing starting with 0. So, equation (3.17) is valid if $(p,l) \in \{0,1,...,M-1\}^2$.
- Using similar reasoning to the above point with off-center Taylor developments, to the left in the direction of x and to the right in the direction of ν, then to the right in the direction of x and to the left in the direction of ν, we obtain the second equation and the third equation respectively.
- Finally, off-center Taylor expansions to the left in the directions of x and ν of Equation (3.16) are used to obtain the last equation. □

Proposition 3.3. *Let $T > 0$ and $\mathbf{C} \in \mathcal{C}^{4,2}(I^2 \times [0;T])$. The finite difference scheme (Equation (3.1)) is consistent of order 1 in time and order 2 in space.*

Proof. Let $\mathbf{D}_{h,\Delta\tau}$ be the finite difference operator of the discretized equation (Equation (3.1)) and $\mathbf{D}$ the partial derivative operator (Equation (1.4)), we have

$$\mathbf{D}_{h,\Delta\tau}\mathbf{C}(x,\nu,\tau) = \frac{\mathbf{C}(x,\nu,\tau+\Delta\tau)-\mathbf{C}}{\Delta\tau} - \frac{\sigma_1^2 x^2}{2}\frac{\mathbf{C}(x+h,\nu,\tau)-2\mathbf{C}+\mathbf{C}(x-h,\nu,\tau)}{h^2} -$$

$$\frac{\sigma_2^2\nu^2}{2}\frac{\mathbf{C}(x,\nu+h,\tau)-2\mathbf{C}+\mathbf{C}(x,\nu-h,\tau)}{h^2} - rx\frac{\mathbf{C}(x+h,\nu,\tau)-\mathbf{C}(x-h,\nu,\tau)}{h} -$$

$$r\nu\frac{\mathbf{C}(x,\nu+h,\tau)-\mathbf{C}(x,\nu-h,\tau)}{h} + r\mathbf{C} \quad (3.18)$$

and

$$\mathbf{D}\,\mathbf{C}(x,\nu,\tau) = \frac{\partial \mathbf{C}}{\partial\tau} - \frac{\sigma_1^2x^2}{2}\frac{\partial^2\mathbf{C}}{\partial x^2} - \frac{\sigma_2^2\nu^2}{2}\frac{\partial^2\mathbf{C}}{\partial \nu^2} - rx\frac{\partial \mathbf{C}}{\partial x} - r\nu\frac{\partial \mathbf{C}}{\partial \nu} + r\mathbf{C}$$

As $\mathbf{C}\in \mathcal{C}^{4,2}(I^2,[0;T])$, we write the followings :

$$\mathbf{C}(x,\nu,\tau+\Delta\tau) = \mathbf{C} + \Delta\tau\frac{\partial \mathbf{C}}{\partial\tau} + \frac{(\Delta\tau)^2}{2}\frac{\partial^2\mathbf{C}(x,\nu,\tilde{\tau})}{\partial\tau^2} \quad (3.19)$$

$$\mathbf{C}(x+h,\nu,\tau) = \mathbf{C} + h\frac{\partial \mathbf{C}}{\partial x} + \frac{h^2}{2}\frac{\partial^2\mathbf{C}}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3\mathbf{C}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4\mathbf{C}(\tilde{x},\nu,\tau)}{\partial x^4} \quad (3.20)$$

$$\mathbf{C}(x-h,\nu,\tau) = \mathbf{C} - h\frac{\partial \mathbf{C}}{\partial x} + \frac{h^2}{2}\frac{\partial^2\mathbf{C}}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3\mathbf{C}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4\mathbf{C}(\tilde{x},\nu,\tau)}{\partial x^4} \quad (3.21)$$

$$\mathbf{C}(x,\nu+h,\tau) = \mathbf{C} + h\frac{\partial \mathbf{C}}{\partial \nu} + \frac{h^2}{2}\frac{\partial^2\mathbf{C}}{\partial \nu^2} + \frac{h^3}{6}\frac{\partial^3\mathbf{C}}{\partial \nu^3} + \frac{h^4}{24}\frac{\partial^4\mathbf{C}(x,\tilde{\nu},\tau)}{\partial \nu^4} \quad (3.22)$$

$$\mathbf{C}(x,\nu-h,\tau) = \mathbf{C} - h\frac{\partial \mathbf{C}}{\partial \nu} + \frac{h^2}{2}\frac{\partial^2\mathbf{C}}{\partial \nu^2} - \frac{h^3}{6}\frac{\partial^3\mathbf{C}}{\partial \nu^3} + \frac{h^4}{24}\frac{\partial^4\mathbf{C}(x,\tilde{\nu},\tau)}{\partial \nu^4} \quad (3.23)$$

Introducing the results of Equations (3.19)-(3.23) into Equation (3.18), we obtain the following:

$$\mathbf{D}_{h,\Delta\tau}\mathbf{C} = \frac{\partial \mathbf{C}}{\partial\tau} - \frac{\sigma_1^2x^2}{2}\frac{\partial^2\mathbf{C}}{\partial x^2} - \frac{\sigma_2^2\nu^2}{2}\frac{\partial^2\mathbf{C}}{\partial \nu^2} - rx\frac{\partial \mathbf{C}}{\partial x} - r\nu\frac{\partial \mathbf{C}}{\partial \nu} + r\mathbf{C} + \frac{\Delta\tau}{2}\frac{\partial^2\mathbf{C}(x,\nu,\tilde{\tau})}{\partial\tau^2} -$$

$$h^2\frac{rx}{3}\frac{\partial^3\mathbf{C}(x,\nu,\tau)}{\partial x^3} - h^2\frac{r\nu}{24}\frac{\partial^3\mathbf{C}(x,\nu,\tau)}{\partial \nu^3} - h^2\frac{\sigma_1^2x^2}{24}\frac{\partial^4\mathbf{C}(\tilde{x},\nu,\tau)}{\partial x^4} - h^2\frac{\sigma_2^2\nu^2}{24}\frac{\partial^4\mathbf{C}(x,\tilde{\nu},\tau)}{\partial \nu^4}$$

As a result, we have

$$\begin{aligned}\|\mathbf{D}_{h,\Delta\tau}\mathbf{C}-\mathbf{D}\,\mathbf{C}\|_\infty &\leq \frac{\Delta\tau}{2}\left\|\frac{\partial^2\mathbf{C}(x,\nu,\tilde\tau)}{\partial\tau^2}\right\|_\infty + h^2\frac{\sigma_1^2x^2}{24}\left\|\frac{\partial^4\mathbf{C}(\tilde x,\nu,\tau)}{\partial x^4}\right\|_\infty + \\ &\qquad h^2\frac{\sigma_2^2\nu^2}{24}\left\|\frac{\partial^4\mathbf{C}(x,\tilde\nu,\tau)}{\partial\nu^4}\right\|_\infty \\ &\leq \frac{\Delta\tau}{2}\left\|\frac{\partial^2\mathbf{C}(x,\nu,\tilde\tau)}{\partial\tau^2}\right\|_\infty + \frac{h^2}{24}\left\|\sigma_1^2\frac{\partial^4\mathbf{C}(\tilde x,\nu,\tau)}{\partial x^4}\right\|_\infty + \\ &\qquad \frac{h^2}{24}\left\|\sigma_2^2\frac{\partial^4\mathbf{C}(x,\tilde\nu,\tau)}{\partial\nu^4}\right\|_\infty\end{aligned}$$

Consequently, the finite difference scheme (Equation (3.1)) is consistent of order 1 in time and order 2 in space. □

Proposition 3.4. *Let* $\mathbf{C}$ *be a copula verifying the main equation of Problem* $(\mathcal{P})$. *The finite difference scheme given by Equation (3.1) is stable if we have following,*

(1) $\min(\sigma_1,\sigma_2)\geq r^{\frac{1}{2}}$

(2) $\dfrac{\Delta\tau}{h^2}\leq\dfrac{1}{x_{\max}^2\sigma_1^2+\nu_{\max}^2\sigma_2^2+rh^2}$

where $x_{\max}=\max\limits_{1\leq p\leq M}\{x_p\}$ *and* $\nu_{\max}=\max\limits_{1\leq l\leq M}\{\nu_l\}$.

Proof. For all $s\in\{0,1,...,N\}$, let us pose,

$$\mathbf{C}^s={}^t\,(\mathbf{C}^s_{1,1},\mathbf{C}^s_{2,1},...,\mathbf{C}^s_{M,1},\mathbf{C}^s_{1,2},...,\mathbf{C}^s_{M,2},\mathbf{C}^s_{1,M},...,\mathbf{C}^s_{M,M})={}^t\,(\mathbf{C}^s_k)_{1\leq k\leq M^2}$$

From Equation (3.1), we can write the following:

$$\begin{aligned}\left|\mathbf{C}^{s+1}_{p,l}\right| &\leq \left|\Delta\tau\alpha_p\mathbf{C}^s_{p-1,l}+\Delta\tau\beta_l\mathbf{C}^s_{p,l-1}+(1-\Delta\tau\gamma_{p,l})\mathbf{C}^s_{p,l}\right|+\left|\Delta\tau\mu_p\mathbf{C}^s_{p+1,l}+\Delta\tau\eta_l\mathbf{C}^s_{p,l+1}\right| \\ &\leq \Delta\tau\left|\alpha_p\right|\mathbf{C}^s_{p-1,l}+\Delta\tau\left|\beta_l\right|\mathbf{C}^s_{p,l-1}+\left|1-\Delta\tau\gamma_{p,l}\right|\mathbf{C}^s_{p,l}+\left|\Delta\tau\mu_p\mathbf{C}^s_{p+1,l}+\Delta\tau\eta_l\mathbf{C}^s_{p,l+1}\right|\end{aligned}$$

If (1) holds, then $\alpha_p\geq 0$ and $\beta_l\geq 0$

$$\begin{cases}\sigma_1^2\geq r\\[2ex] \sigma_2^2\geq r\end{cases}\iff\begin{cases}\dfrac{\sigma_1^2x_p^2}{2h^2}\geq\dfrac{rx_p}{2h}\\[2ex] \dfrac{\sigma_2^2\nu_l^2}{2h^2}\geq\dfrac{r\nu_l}{2h}\end{cases}\iff\begin{cases}\alpha_p\geq 0\\ \beta_l\geq 0\end{cases}$$

and if (2) holds, then

$$\begin{aligned}1-\Delta\tau\left(\frac{\sigma_1^2x_{\max}^2}{h^2}+\frac{\sigma_2^2\nu_{\max}^2}{h^2}+r\right)\geq 0 &\iff 1-\Delta\tau\left(\frac{\sigma_1^2x_p^2}{h^2}+\frac{\sigma_2^2\nu_l^2}{h^2}+r\right)\geq 0\\ &\iff 1-\Delta\tau\gamma_{p,l}\geq 0\end{aligned} \tag{3.24}$$

So, we have :

$$\begin{aligned}\max_{1\le k\le M^2}\left|\mathbf{C}_k^{s+1}\right| &\le (|1-\Delta\tau(\alpha_p+\beta_l-\gamma_{p,l})|+\Delta\tau\,|\mu_p+\eta_l|)\max_{1\le k\le M^2}|\mathbf{C}_k^s| \\ &\le \left(\left|1-\Delta\tau\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right)\right|+\right. \\ &\qquad \left.\Delta\tau\left|\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right|\right)\max_{1\le k\le M^2}|\mathbf{C}_k^s|\end{aligned}$$

We have,

$$\gamma_{p,l}-\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right)=r+\frac{\sigma_1^2x_p^2-rhx_p}{2h^2}+\frac{\sigma_2^2\nu_l^2-rh\nu_l}{2h^2}$$

For all $p,l\in\{1,...,M-1\}, x_p\ge x_1=h$ and $\nu_l\ge\nu_1=h$. Furthermore, if (1) holds, we have

$$\sigma_1^2x_p^2-rhx_p\ge 0 \text{ and } \sigma_2^2\nu_l^2-rh\nu_l\ge 0$$

As a result, we can write,

$$1-\Delta\tau\gamma_{p,l}\le 1-\Delta\tau\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right)$$

Using (3.24), we deduce that

$$1-\Delta\tau\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right)\ge 0$$

and therefore,

$$\left|1-\Delta\tau\left(\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right)\right|+\Delta\tau\left|\frac{\sigma_1^2x_p^2}{2h^2}+\frac{\sigma_1^2\nu_l^2}{2h^2}+\frac{rx_p}{2h}+\frac{r\nu_l}{2h}\right|=1$$

By using the definition of the infinite norm (Equation (2.9)), we have

$$\|\mathbf{C}^{s+1}\|_\infty\le\|\mathbf{C}^s\|_\infty \tag{3.25}$$

Summing (3.25) over s, we obtain:

$$\|\mathbf{C}^s\|_\infty\le\|\mathbf{C}^0\|_\infty \tag{3.26}$$

□

Since the finite difference scheme (Equation (3.1)) is consistent (Proposition 3.3) and conditionally stable (Proposition 3.4), we deduce from Theorem 2.5 that it is conditionally convergent.

Proposition 3.5. *Let $(\mathbf{C}^s)_{s>0}$ be the copula sequence, solution of problem $(\mathcal{P}_{h,\Delta\tau})$. $(\mathbf{C}^s)$ converges uniformly to the copula $\mathbf{C}^*$ if*

(1) $\min\{\sigma_1,\sigma_2\}\ge r^{\frac{1}{2}}$
(2) $\dfrac{\Delta\tau}{h^2}<\dfrac{1}{x_{\max}^2(\sigma_1^2+\sigma_2^2)+rh^2}$

where $\mathbf{C}^*$ satisfies,

$$\frac{1}{2}\left(\sigma_1^2x^2\frac{\partial^2\mathbf{C}^*}{\partial x^2}+\sigma_2^2\nu^2\frac{\partial^2\mathbf{C}^*}{\partial\nu^2}\right)+rx\frac{\partial\mathbf{C}^*}{\partial x}+r\nu\frac{\partial\mathbf{C}^*}{\partial\nu}-r\mathbf{C}^*=0 \tag{3.27}$$

Proof. Let $\mathbf{C}^*_{p,l}$ be the approximate solution by finite difference of $\mathbf{C}^l$.
Let's say

$$\Delta^s_{p,l} = \mathbf{C}^s_{p,l} - \mathbf{C}^*_{p,l}$$

We have :

$$\frac{\Delta^{s+1}_{p,l} - \Delta^s_{p,l}}{\Delta\tau} = \frac{\mathbf{C}^{s+1}_{p,l} - \mathbf{C}^s_{p,l}}{\Delta\tau} \tag{3.28}$$

When the copula $\mathbf{C}^*$ satisfies equation (3.27), we have :

$$\begin{aligned}
&\frac{\sigma_1^2 x_p^2}{2}\frac{\Delta^s_{p+1,l} - 2\Delta^s_{p,l} + \Delta^s_{p-1,l}}{h^2} + \frac{\sigma_2^2 \nu_l^2}{2}\frac{\Delta^s_{p,l+1} - 2\Delta^s_{p,l} + \Delta^s_{p,l-1}}{h^2} + rx_p\frac{\Delta^s_{p+1,l} - \Delta^s_{p-1,l}}{2h} + \\
&\qquad r\nu_l\frac{\Delta^s_{p,l+1} - \Delta^s_{p,l-1}}{2h} - r\Delta^s_{p,l} \\
&= \frac{\sigma_1^2 x_p^2}{2}\frac{\mathbf{C}^s_{p+1,l} - 2\mathbf{C}^s_{p,l} + \mathbf{C}^s_{p-1,l}}{h^2} + \frac{\sigma_2^2 \nu_l^2}{2}\frac{\mathbf{C}^s_{p,l+1} - 2\mathbf{C}^s_{p,l} + \mathbf{C}^s_{p,l-1}}{h^2} + \\
&\qquad rx_p\frac{\mathbf{C}^s_{p+1,l} - \mathbf{C}^s_{p-1,l}}{2h} + r\nu_l\frac{\mathbf{C}^s_{p,l+1} - \mathbf{C}^s_{p,l-1}}{2h} - r\mathbf{C}^s_{p,l}
\end{aligned} \tag{3.29}$$

If we differentiate equations (3.28) and (3.29), we have the following finite difference scheme:

$$\begin{aligned}
\frac{\Delta^{s+1}_{p,l} - \Delta^s_{p,l}}{\Delta\tau} &= \frac{\sigma_1^2 x_p^2}{2}\frac{\Delta^s_{p+1,l} - 2\Delta^s_{p,l} + \Delta^s_{p-1,l}}{h^2} + \frac{\sigma_2^2 \nu_l^2}{2}\frac{\Delta^s_{p,l+1} - 2\Delta^s_{p,l} + \Delta^s_{p,l-1}}{h^2} + \\
&\qquad rx_p\frac{\Delta^s_{p+1,l} - \Delta^s_{p-1,l}}{2h} + r\nu_l\frac{\Delta^s_{p,l+1} - \Delta^s_{p,l-1}}{2h} - r\Delta^s_{p,l}
\end{aligned} \tag{3.30}$$

If (1) and (2) hold then reasoning similarly to the proof of Proposition 3.4, we write the following:

$$\max_{(x,\nu)\in I^2} |\Delta^s(x,\nu)| \le (1 - r\Delta\tau)^s \max_{(x,\nu)\in I^2} \left|\Delta^0(x,\nu)\right| \tag{3.31}$$

Moreover, if (2) is true, then $0 < 1 - r\Delta\tau < 1$. Thus, we can write :

$$\|\mathbf{C}^s(x,\nu) - \mathbf{C}^*(x,\nu)\|_\infty \le K\Theta^s$$

with $0 < \Theta < 1$ and $0 \le K < \infty$. Therefore, we have :

$$\|\mathbf{C}^s(x,\nu) - \mathbf{C}^*(x,\nu)\|_\infty \longrightarrow 0$$

□

Proposition 3.6. *Given an initial copula $\mathbf{C}^0$ and a finite difference scheme given by (3.1), if $\top_s$ denotes the Spearman correlation coefficient for the copula $\mathbf{C}^s$ for all $s \in \mathbb{N}$, then*

$$\top_{s+1} = (1+\xi)\top_s + \frac{3\Delta\tau(M-1)}{M(M+2)}\lambda - \frac{6\Delta\tau M}{M+2}\chi$$

where $\xi = \Delta\tau(\sigma_1^2 + \sigma_2^2 - 3r)$, $\lambda = (\sigma_1^2 + \sigma_2^2)(M^2 + M - 1) - r(M-3)$ *and*
$\chi = (M\sigma_1^2 - r)\sum_{p=2}^{M}\mathbf{C}^s_{p,M} + (M\sigma_2^2 - r)\sum_{l=2}^{M}\mathbf{C}^s_{M,l}$.

Proof. Let's consider the sample $\{(x_k, \nu_k)\}_{1\leq k\leq M+1}$ obtained by discretization of step $h = \dfrac{1}{M}$ of x and ν on I.

$$x_1 = 0$$

Using Theorem 2.6, we can write

$$\begin{aligned}
\top_{s+1} &= \frac{12}{(M+1)^2-1}\sum_{p=1}^{M+1}\sum_{l=1}^{M+1}(\mathbf{C}_{p,l}^{s+1} - x_p\nu_l)\\
&= \frac{12}{M(M+2)}\sum_{p=2}^{M}\sum_{l=2}^{M}(\mathbf{C}_{p,l}^{s+1} - x_p\nu_l)\\
&= \frac{12}{M(M+2)}\sum_{p=2}^{M}\sum_{l=2}^{M}\big[\Delta\tau\alpha_p\mathbf{C}_{p-1,l}^{s} + \Delta\tau\beta_l\mathbf{C}_{p,l-1}^{s} + (1-\Delta\tau\gamma_{p,l})\mathbf{C}_{p,l}^{s} +\\
&\qquad\qquad \Delta\tau\mu_p\mathbf{C}_{p+1,l}^{s} + \Delta\tau\eta_l\mathbf{C}_{p,l+1}^{s}\big]
\end{aligned}$$

We have,

$$\sum_{p=2}^{M}\sum_{l=2}^{M}(1-\Delta\tau\gamma_{p,l})\mathbf{C}_{p,l}^{s} = \sum_{p=2}^{M}\sum_{l=2}^{M}(1-r\Delta t)\mathbf{C}_{p,l}^{s} - \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}(\alpha_p+\mu_p+\beta_l+\eta_l)\mathbf{C}_{p,l}^{s} \tag{3.32}$$

After simple manipulations, we write the following:

$$\Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\alpha_p\mathbf{C}_{p-1,l}^{s} = \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\alpha_{p+1}\mathbf{C}_{p,l}^{s} - \Delta\tau\sum_{l=2}^{M}\alpha_{M+1}\mathbf{C}_{M,l}^{s} \tag{3.33}$$

$$\Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\mu_p\mathbf{C}_{p+1,l}^{s} = \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\mu_{p-1}\mathbf{C}_{p,l}^{s} + \Delta\tau\sum_{l=2}^{M}\mu_{M}\mathbf{C}_{M+1,l}^{s} \tag{3.34}$$

$$\Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\beta_l\mathbf{C}_{p,l-1}^{s} = \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\beta_{l+1}\mathbf{C}_{p,l}^{s} - \Delta\tau\sum_{p=2}^{M}\beta_{M+1}\mathbf{C}_{p,M}^{s} \tag{3.35}$$

$$\Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\eta_l\mathbf{C}_{p,l+1}^{s} = \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}\eta_{l-1}\mathbf{C}_{p,l}^{s} + \Delta\tau\sum_{p=2}^{M}\eta_{M}\mathbf{C}_{p,M+1}^{s} \tag{3.36}$$

Doing (3.33) + (3.34), we obtain

$$\sum_{p=2}^{M}\sum_{l=2}^{M}(\nu_l\mathbf{C}_{p-1,l}^{s} + \mu_l\mathbf{C}_{p+1,l}^{s}) = \sum_{p=2}^{M}\sum_{l=2}^{M}(\nu_{p+1}+\mu_{p-1})\mathbf{C}_{p,l}^{s} - \sum_{l=2}^{M}\alpha_{M+1}\mathbf{C}_{M,l}^{s} + \sum_{l=2}^{M}\mu_M\mathbf{C}_{M+1,l}^{s} \tag{3.37}$$

and summing Equation (3.35) and Equation (3.36), we have

$$\sum_{p=2}^{M}\sum_{l=2}^{M}(\beta_l\mathbf{C}_{p,l-1}^{s} + \eta_l\mathbf{C}_{p,l+1}^{s}) = \sum_{p=2}^{M}\sum_{l=2}^{M}(\beta_{l-1}+\eta_{l+1})\mathbf{C}_{p,l}^{s} - \sum_{p=2}^{M}\beta_{M+1}\mathbf{C}_{i,M}^{s} + \sum_{p=2}^{M}\eta_M\mathbf{C}_{p,M+1}^{s} \tag{3.38}$$

As a result,

$$
\begin{aligned}
\top_{s+1} &= \frac{12}{M(M+2)}\Bigg[\sum_{p=2}^{M}\sum_{l=2}^{M}(1-r\Delta\tau))\mathbf{C}_{p,l}^{s}+\Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}(\alpha_{p+1}-\alpha_{p}-\mu_{p}+\mu_{p-1})\mathbf{C}_{p,l}^{s}+ \\
&\quad \Delta\tau\sum_{p=2}^{M}\sum_{l=2}^{M}(\beta_{l+1}-\beta_{l}-\eta_{l}+\eta_{l-1})\mathbf{C}_{p,l}^{s}-\Delta\tau\sum_{l=2}^{M}\alpha_{M+1}\mathbf{C}_{M,l}^{s}+\Delta\tau\sum_{l=2}^{M}\mu_{M}\mathbf{C}_{M+1,l}^{s}- \\
&\quad \Delta\tau\sum_{p=2}^{M}\beta_{M+1}\mathbf{C}_{p,M}^{s}+\Delta\tau\sum_{p=2}^{M}\eta_{M}\mathbf{C}_{p,M+1}^{s}-\sum_{p=2}^{M}\sum_{l=2}^{M}x_{p}\nu_{l}\Bigg] \\
&= \frac{12}{M(M+2)}\Bigg[[1+\Delta t(\sigma_1^2+\sigma_2^2-3r)]\sum_{p=2}^{M}\sum_{l=2}^{M}(\mathbf{C}_{p,l}^{s}-x_p\nu_l)+\frac{\Delta t(M-1)}{2}(\mu_M+\eta_M)+ \\
&\quad \Delta\tau(\sigma_1^2+\sigma_2^2-3r)\sum_{p=2}^{M}\sum_{l=2}^{M}x_p\nu_l-\Delta\tau\left(\alpha_{M+1}\sum_{l=2}^{M}\mathbf{C}_{M,l}^{n}+\beta_{M+1}\sum_{p=2}^{M}\mathbf{C}_{p,M}\right)\Bigg] \\
&= [1+\Delta\tau(\sigma_1^2+\sigma_2^2-3r)]\top_s+\frac{3\Delta\tau(M-1)}{M(M+2)}[(\sigma_1^2+\sigma_2^2)(M^2+M-1)-r(M-3)]- \\
&\quad \frac{6\Delta\tau M}{M+2}\left((M\sigma_1^2-r)\sum_{p=2}^{M}\mathbf{C}_{p,M}^{s}+(M\sigma_2^2-r)\sum_{l=2}^{M}\mathbf{C}_{M,l}^{s}\right)
\end{aligned}
$$

By posing

$$
\begin{aligned}
\xi &= \Delta\tau(\sigma_1^2+\sigma_2^2-3r) \\
\lambda &= (\sigma_1^2+\sigma_2^2)(M^2+M-1)-r(M-3) \\
\chi &= (M\sigma_1^2-r)\sum_{p=2}^{M}\mathbf{C}_{p,M}^{s}+(M\sigma_2^2-r)\sum_{l=2}^{M}\mathbf{C}_{M,l}^{s},
\end{aligned}
$$

we obtain the result. □

Proposition 3.7. *Let ξ,λ,χ be the parameters defined in Proposition 3.6 and $\mathbf{C}^s$ a solution of the problem $(\mathcal{P}_{h,\Delta\tau})$ with Spearman correlation coefficient $\top_s$, $s>0$. Then*

$$
\top_s=(1+\xi)^s\left(\top_0+\frac{A}{\xi}\right)-\frac{A}{\xi} \tag{3.39}
$$

where $\top_0$ is the Spearman rho of the initial copula $\mathbf{C}^0$ and

$$
A=\frac{3\Delta\tau(M-1)}{M(M+2)}\lambda-\frac{6\Delta\tau M}{M+2}\chi
$$

Proof. According to Proposition 3.6, we have

$$
\top_{s+1}=(1+\xi)\top_s+A
$$

Let

$$
b=-\frac{A}{\xi}\text{ and }\omega_s=\top_s-b
$$

Thus, we have

$$\begin{array}{rcl}\omega_{s+1} & = & (1+\xi)\top_s + A - b \\ & = & (1+\xi)(\omega_s + b) + A - b \\ & = & (1+\xi)\omega_s\end{array}$$

We deduce that (ω_s) is a geometric suite of reason $1+\xi$ and first term ω_0. So

$$\omega_s = (1+\xi)^s \omega_0$$

As a result, we write the following:

$$\begin{array}{rcl}\top_s & = & (1+\xi)^s\omega_0 + b \\ & = & (1+\xi)^s(\top_0 - b) + b \\ & = & (1+\xi)^s\left(\top_0 + \dfrac{A}{\xi}\right) - \dfrac{A}{\xi}\end{array}$$

□

3.2. **Simulation with MATLAB.** We present here the numerical results using Matlab software. For instants τ we present the copula graph and associated contour plots. These results allow us to clearly visualize the change in dependency between the two underlyings over time, as well as the type and degree of dependency existing between the two underlying assets. Finally, we present the evolution of the Spearman correlation coefficient for several values of the parameters σ_1, σ_2 and r.
We need to choose an initial copula and the parameters σ_1, σ_2 and r for the numerical calculation of the solution. However, the parameters must be chosen in such a way that the stability conditions are satisfied. As parameters and discretization steps are linked, it is difficult to understand the impact of parameters exclusively on the evolution of the copula. Our idea here is to visualize the variation of the copula over time.
For our study, we assume that the initial copula f is equal to the Franck copula defined by :

$$f(u,v) = -\frac{1}{\gamma}\ln\left(1 + \frac{\left(\mathrm{e}^{-\gamma u}-1\right)\left(\mathrm{e}^{-\gamma v}-1\right)}{\mathrm{e}^{-\gamma}-1}\right); \gamma \neq 0$$

Figures 1 to 4 are obtained by setting $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, $r = 0.02$, $T = 1$, $\Delta\tau = \dfrac{1}{100}$ and $h = \dfrac{1}{10}$. We can clearly see the change in the solution over time from moderate to weak negative dependence.

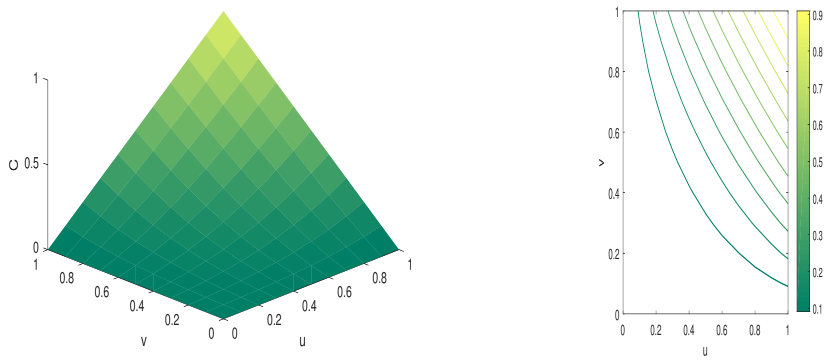


FIGURE 1. Graph (left) and contours (right) of the initial copula ($\tau = 0$) for $\gamma = -3$

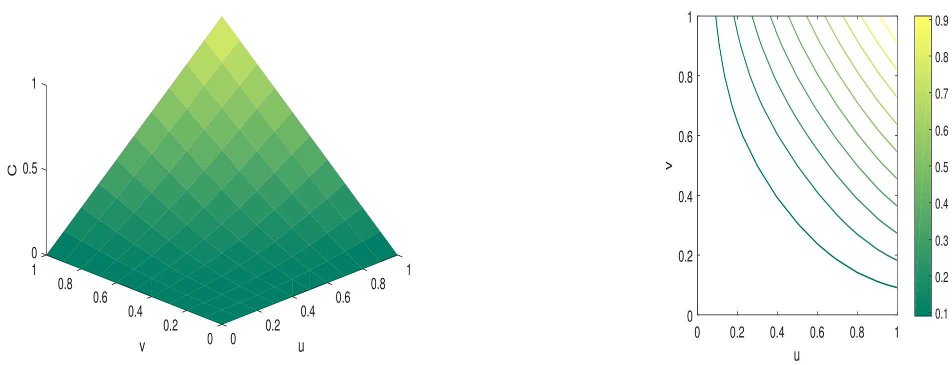


FIGURE 2. Graph (left) and contours (right) of the copula at $\tau = \frac{T}{2}$.

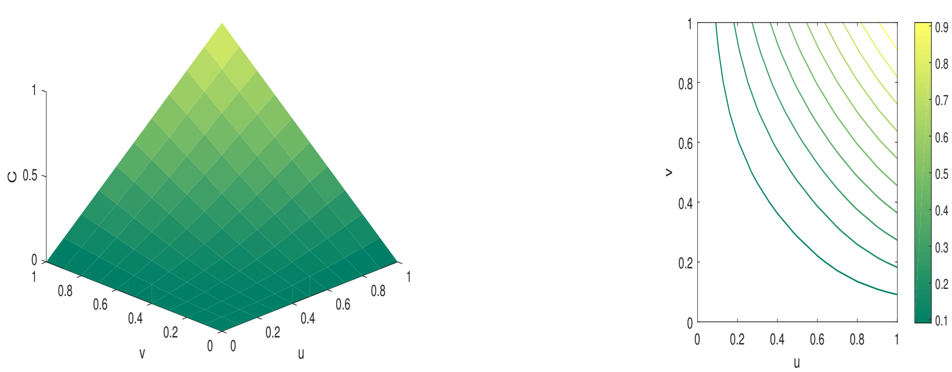


FIGURE 3. Graph (left) and contours (right) of the copula at $\tau = T$.

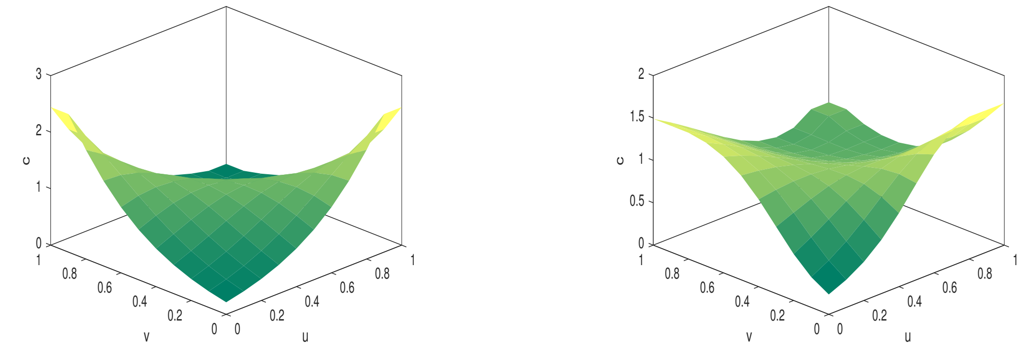
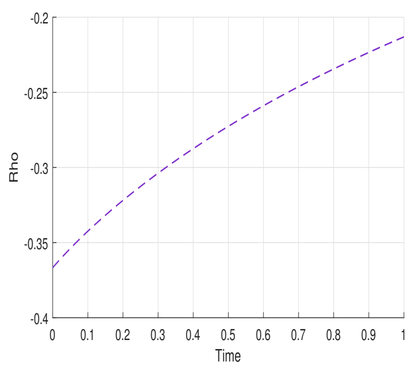
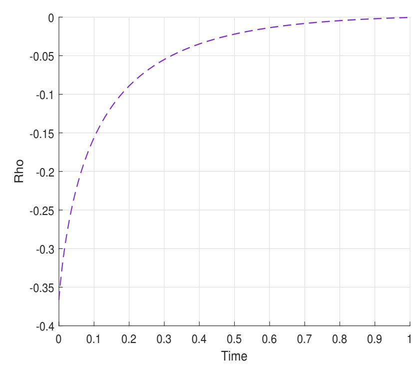


FIGURE 4. Density of the initial solution ($t = 0$) on the left and density of the final solution ($\tau = T$) on the right.

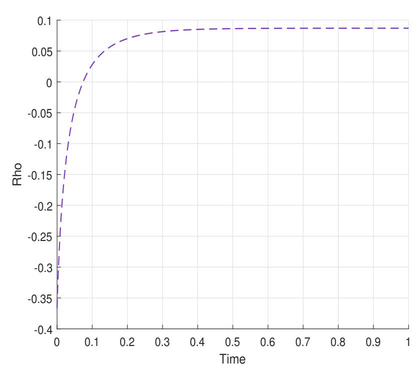
Maintaining the initial copula and space step unchanged, we present in Figure 5a, 5b and 5c, the evolution of Spearman's correlation coefficient as a function of parameters and time step. This illustrates more clearly the evolution of the solution over time.



(A) $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, $r = 0.02$ and $\Delta\tau = \dfrac{1}{100}$



(B) $\sigma_1 = 0.8$, $\sigma_2 = 3$, $r = 0.02$ and $\Delta\tau = \dfrac{1}{1000}$



(C) $\sigma_1 = 4$, $\sigma_2 = 3$, $r = 2$ and $\Delta\tau = \dfrac{1}{10000}$

4. Conclusion

This work consisted to the construction of a flexible evolutive copula for modeling the joint random behavior of the underlying assets of a bivariate option. This class of copula satisfies a 2-dimensional evolution partial differential equation (Black-Scholes equation). It was preferable to use a numerical method, namely the finite difference approach, in order to obtain an approximated solution, since the analytical solution of such a problem is difficult to obtain. The numerical scheme was shown to be convergent under certain conditions.

References

1. Anass Echcherki (2015). Méthodes Numériques pour la Résolution de l'équation de la chaleur non linéaire : Cas de Supraconductivité. Faculté des Sciences et Techniques - FES. https://fr.scribd.com/document/637330195/Untitled
2. Bagré,R.G. and Béré, F. and Somé.K.C and P.D Ibrango.(2021). On Approximation of Copulas, Solution of Elliptic Problem, Using the Finite Difference Method: Generalization. American Journal of Theoretical and Applied Statistics,10(6),257-261. https://www.sciencepublishinggroup.com/article/10.11648/j.ajtas.20211006.15
3. Bagré,R.G., Béré, F. & Loyara. V.Y.B. (2021). Construction of a Class of Copula Using the Finite Difference Method. Journal of Function Spaces, 2021, 8. https://onlinelibrary.wiley.com/doi/10.1155/2021/5271105
4. Chacón-Acosta, Guillermo and Salas, Rubén O. (2022). Projection of the two-dimensional Black-Scholes equation for options with underlying stock and strike prices in two different currencies. Revista Mexicana de Física, 68 (011401), 1-7. https://doi.org/10.31349/RevMexFis.68.011401
5. Chesneau, Christophe. (2023). A Collection of Two-Dimensional Copulas Based on an Original Parametric Ratio Scheme. Symmetry, 15(5), 23. https://www.mdpi.com/2073-8994/15/5/977
6. Lax,P.D & Richtmyer, R.D. (1956). Survey of stability of Linear Finite Difference Equation. Communications on pure and applied Mathematics, 9(2), 267-293. https://doi.org/10.1002/cpa.3160090206
7. Mark H. Holmes.(2006).Introduction to Numerical Methods in Diferential Equations. Springer. https://link.springer.com/book/10.1007/978-0-387-68121-4
8. Nelsen, Roger B. (2006). An introduction to copulas. New York, NY: Springer. https://link.springer.com/book/10.1007/0-387-28678-0
9. Yasukazu Yoshizawa and Naoyuki Ishimura. (2012). Evolution of bivariate copulas in discrete processes. Procedia Economics and Finance, 1, 186-192. https://doi.org/10.1016/S2212-5671(12)00022-6
10. Yasukazu Yoshizawa & Naoyuki Ishimura.(2011). Evolution of bivariate copulas in discrete processes. JSIAM Lett., (3), 77-80. https://api.semanticscholar.org/CorpusID:122154179
11. YAZOUGH.C. (2018). EXISTENCE SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS AND L^1 DATA . Gulf Journal of Mathematics, 6(4), 56-71. https://doi.org/10.56947/gjom.v6i4.249

[1] Department of Mathematics, Norbert ZONGO University, BP 376 Koudougou, Burkina Faso

Email address: hermanteimto94@gmail.com

[2] Department of Mathematics, Norbert ZONGO University, BP 376 Koudougou, Burkina Faso

Email address: baremgui@yahoo.fr