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IMPLICIT FUNCTION THEOREM TO THE DYNAMIC VON KARMAN MODEL OF SHELL

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Abstract. This paper deals with evolutionary von Karman equation with rotational forces not clamped boundary conditions and interior nonlinear damping for shallow shell. We study the existence and uniqueness of a weak solution under weaker conditions to the damping and lower regularity of the internal force. Finally, we employ the finite difference method to approximate the solution to the problem.

1. Introduction

We consider a dynamic von Karman equation with rotational forces ($\alpha > 0$) [4]. From a physical standpoint of view Von-Karman evolution, describes the buckling and flexible phenomenon of small nonlinear vibrations of a thin homogenous. One of the aims of this physical phenomenon, is to reduce the vibrations/oscillations of the structure, by using of a passive and active damping. The mathematical model consists a modeling the physical situation, of nonlinear oscillations to the shell, of vertical displacement to the elastic shallow shell, for displacement u and the Airy stress function ϕ, is of the following form, see for instance [4]:

$$(\mathbb{P})\begin{cases} u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - \alpha \mathrm{div}(d(x)g(\nabla u_t)) + d_0(x)g_0(u_t) & \\ \qquad - [\phi + F_0, u + \theta] = L(u) + p(x) & in \quad \omega \times [0,T] \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u_1 & in \quad \omega \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0,T] \end{cases}$$

and

$$(\mathbb{Q})\begin{cases} \Delta^2\phi + [u, u + 2\theta] = 0 & in \quad \omega \times [0,T], \\ \phi = 0, \ \partial_\nu \phi = 0 & on \quad \Gamma \times [0,T], \end{cases}$$

where ω is the middle surface of the shell, $T > 0$ is a real number, u_0 and u_1 are initial data and $[.;.]$ is the Monge-Ampère operator defined by: [1]

$$[\psi, u] = \partial_{11}\psi\partial_{22}u + \partial_{11}u\partial_{22}\psi - 2\partial_{12}\psi\partial_{12}u. \tag{1.1}$$

$\theta(x; y) \in H^3(\omega)$ [5] is the mapping measuring the deviation of the middle surface of the reference configuration of the shell from a plane, F_0 is an internal force

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and p is the external force. The linear operator source L characterizes the case where the shell interacts with the supersonic flow gaze [4], its simplest canonical form may be given by $L(u) = \rho(x;y)\partial_1 u$, where $\rho(x;y)$ assumed in the direction of (OX) which is depends on the velocity. The terms $g_0(u_t)$, $g(\nabla u_t)$ represent mechanical (potentially nonlinear) damping in the system with damping densities denoted by $d_0(x)$ and $d(x)$, which are assumed nonnegative in ω.

The majority of work in this field considered the dynamics subjected to linear damping and $g_0(.)$ and $g(.)$ are a continuous, monotone, nondecreasing mapping, moreover the majority of authors consider the internal force F_0 be a less regularity in $H^{3+\epsilon}(\omega)$, see [4, 7, 8, 12, 13].

Chueshov and Lasiecka studied the existence and the uniqueness of a weak solution to the problem of the structural interaction coupled with von Karman evolution by using the theory of nonlinear semi-group and the Galerkin method in [4]. For the uniqueness weak solution the authors used the weak continuity of nonlinear terms involving Airy stress function.

The aim of this paper is to give a condition verified by the external loads and the linear operator L in up to have a uniqueness weak solution of the von Karman evolution, with rotational terms nor clamped boundary conditions subject to nonlinear damping energy dissipation. Our approach is based on an implicit function theorem using by P.G.Ciarlet [6] and under assumptions:

- The two operator $g_0(.)$, L are defined and infinitely Fréchet differentiable in $L^2([0,T], L^2(\omega))$ and also $g(.)$ in $L^2([0,T], H^1(\omega))$.
- $g_0^{'}(0)$ and $g^{'}(0)$ are continuous and $\forall (s_1, s_2) \in \mathbb{R}^2$; that is

$$g_0^{'}(0)s_1.s_1 \geq 0 \;\; and \;\; g^{'}(0)(s_1,s_2).(s_1,s_2) \geq 0.$$

- The internal force has a less regularity, to know F_0 is in the space $H^2(\omega)$.

The rest of this document is organized as follows. In section 2 we present the mathematical structure of the model that will be studied in the sequel together with some basic tools and results. Section 3 we use the implicit function theorem for established a uniqueness weak solution of the associated dynamical shell, subject a nonlinear damping dissipation. Finally in section 4 we applied the finite difference method for approached this solution.

2. Preliminary Results and Needed Tools

In this paper, ω denotes a nonempty bounded domain in $\mathbb{R}^2$, with regular boundary $\Gamma = \partial\omega$ and $\alpha > 0$. Let $p \geq 1$ be a real number and $m \geq 1$ be an integer. We denote by $|.|_p$ the classical norm of $L^p(\omega)$ and by $\|.\|_m$ that of $H^m(\omega)$. For $u \in H_0^2(\omega)$ we set $\|u\| := |\Delta u|_2^2 = \displaystyle\int_\omega (\Delta u)^2$ and for the sake of simplicity. We denote by :

$$\|u\|_\alpha := \|u\| + \alpha\,|\nabla u_t|_2^2 + |u_t|_2^2\,. \tag{2.1}$$

Assumptions

- The two operator $d_0(.)g_0(.)$, L are defined and infinitely Fréchet differentiable in $L^2([0,T], L^2(\omega))$ and also $d(.)g(.)$ in $L^2([0,T], H^1(\omega))$.

• $g_0^{'}(0)$ and $g^{'}(0)$ are continuous and $\forall (s_1, s_2) \in \mathbb{R}^2$; that is

$$g_0^{'}(0)s_1.s_1 \geq 0 \ \ and \ \ g^{'}(0)(s_1, s_2).(s_1, s_2) \geq 0. \tag{2.2}$$

• The internal force has a less regularity, to known F_0 is in the space $H^2(\omega)$.
We recall the following, [4, 11]

Theorem 2.1. *Let $f \in L^2(\omega)$. Then the following problem*

$$\begin{cases} \Delta^2 z = f & in \quad \omega, \\ z = 0 & on \quad \Gamma, \\ \partial_\nu z = 0 & on \quad \Gamma, \end{cases}$$

has one and only one solution $z \in H_0^2(\omega) \cap H^4(\omega)$ satisfying that

$$\|z\| \leq c_0 \, |f|_1 \, ,$$

for some constant $c_0 > 0$ depending only on $mes(\omega)$.

The following proposition is of interest.

Proposition 2.2. *Let $u \in L^2\big([0, T]\, , H^4(\omega) \cap H_0^2(\omega)\big)$ be with small norm and $F_0 \in H^2(\omega)$ then there exists $0 < c_1 < 1$ such that*

$$\int_\omega [u, u]\, F_0 \leq c_1 \, \|F_0\|_2^{2(1-r)} \, \|u\| \, ,$$

for all $\dfrac{1}{2} \leq r < 1$.

Proof. We have

$$\int_\omega [u, u]\, F_0 = \int_\omega (2\partial_{11}u\partial_{22}u - 2(\partial_{12}u)^2)F_0 \leq 2\int_\omega |\partial_{11}u\partial_{22}uF_0| + 2\int_\omega \left|(\partial_{12}u)^2 F_0\right| ,$$

and

$$\int_\omega |\partial_{11}u\partial_{22}uF_0| \leq |\partial_{11}u|_2 \, |\partial_{22}uF_0|_2 \, ,$$

moreover ,

$$|\partial_{22}uF_0|_2 = \Big(\int_\omega |\partial_{22}uF_0|^2\Big)^{\frac{1}{2}} \leq |\partial_{22}u|_s^{2r} \, |F_0|_{s^*}^{2(1-r)} \, ,$$

where $\dfrac{1}{2} = \dfrac{r}{s} + \dfrac{1-r}{s^*}$ and $0 \leq r \leq 1$, see [3]. Let $1 < 2r < 2$ and $1 < s < 2r < 2$, we have that $s^* = \dfrac{2s(1-r)}{s-2r} > 2$ and $L^2(\omega) \hookrightarrow L^s(\omega)$, $H_0^1(\omega) \hookrightarrow L^{s^*}(\omega)$. Then there exist $c_0 > 0$ and $c > 0$ sufficiently small, with $\|u\| \leq c$ and $c > 0$

$$\Big(\int_\omega |\partial_{22}uF_0|^2\Big)^{\frac{1}{2}} \leq c_0 \, |\partial_{22}u|_2^{2r} \, \|F_0\|_2^{2(1-r)} \leq c_0 \, \|u\|^r \, \|F_0\|_2^{2(1-r)} \leq c_0 \, \|u\|^r \, \|F_0\|_2^{2(1-r)} ,$$

it follows that

$$\int_\omega |\partial_{11}u\partial_{22}uF_0| \leq c_0 \, \|u\|^{\frac{1}{2}} \, \|u\|^r \, \|F_0\|_2^{2(1-r)} \leq c_0 c^{r-\frac{1}{2}} \, \|u\| \, \|F_0\|_2^{2(1-r)} .$$

By analogous method we deduce that

$$\int_\omega \left|(\partial_{12}u)^2 F_0\right| \le c_0 c^{r-\frac{1}{2}} \|u\| \|F_0\|_2^{2(1-r)},$$

then

$$\int_\omega [u,u]\, F_0 \le 4c_0 c^{r-\frac{1}{2}} \|u\| \|F_0\|_2^{2(1-r)},$$

if we choose $c > 0$, such that $0 < c_1 = 4c_0 c^{r-\frac{1}{2}} < 1$, we have that

$$\int_\omega [u,u]\, F_0 \le c_1 \|u\| \|F_0\|_2^{2(1-r)}.$$

□

We need this theorem, see [4]

Theorem 2.3. *Let (u,v) in $(H^2(\omega))^2$ the following problem*

$$\begin{cases} \Delta^2 B(u,v) = [u,v] \quad in \quad \omega, \\ B(u,v) \in H_0^2(\omega), \end{cases}$$

has one and only one solution $B(u,v) \in H_0^2(\omega)$ and the maping $B : (H^2(\omega))^2 \longrightarrow H_0^2(\omega)$ is defined, continuous and a bilinear operator.

Remark 2.4.

(1) $\forall\, t > 0,\ (u,u_t) \in L^2([0,T], H_0^2(\omega) \times H_0^1(\omega))$ and using the continuous linear operator L we have that

$$2\int_0^t \int_\omega L(u)u_t \le \|L\|^2 \int_0^t \|u\| + \int_0^t |u_t|_2^2.$$

(2) According to Theorem (2.3), the Airy stress function ϕ depend of u and $\dfrac{\partial \phi}{\partial u}(0) = 0$.

Proposition 2.5. *Let $u \in L^2([0,T], H^4(\omega) \cap H_0^2(\omega))$, $(u_t,u_{tt}) \in L^2([0,T], H^2(\omega) \cap H_0^1(\omega) \times H^2(\omega))$ and under assumption in section 2, the operator*

$$Au = u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - \alpha\, div(d(x)g(\nabla u_t)) + d_0(x)g_0(u_t) - [\phi + F_0, u+\theta] - L(u),$$

is defined and infinitely Fréchet differentiable, moreover we have that

$$A^{'}(0)u = u_{tt} - \alpha\Delta u_{tt} - \alpha\, div(d(x)g^{'}(0)\nabla u_t) + d_0(x)g_0^{'}(0)u_t + \Delta^2 u - [F_0,u] - L(u).$$

Proof. For $u \in L^2([0,T], H^4(\omega) \cap H_0^2(\omega))$ and $(u_t,u_{tt}) \in L^2([0,T], H^2(\omega) \cap H_0^1(\omega) \times H^2(\omega))$ we have that

$$\mathrm{div}(d(x)g(\nabla u_t)) + d_0(x)g_0(u_t),$$

is defined and Fréchet differentiable, moreover the linear operator

$$u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - L(u),$$

and the bilinear operator

$$[\phi + F_0, u+\theta],$$

are defined and Fréchet differentiable.
It follows that

$$(u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - L(u))'(0).u = u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - L(u),$$

$$(\mathrm{div}(d(x)g(\nabla u_t)) + d_0(x)g_0(u_t))'(0).u = \mathrm{div}(d(x)g'(0)(\nabla u_t)) + d_0(x)g_0'(0)(u_t),$$

and

$$([\phi + F_0, u + \theta])'(0).u = [u, F_0],$$

then, we deduce that

$$A'(0)u = u_{tt} - \alpha\Delta u_{tt} - \alpha\mathrm{div}(d(x)g'(0)(\nabla u_t)) + d_0(x)g_0'(0)(u_t) + \Delta^2 u - [F_0, u] - L(u).$$

□

Theorem 2.6. *Assume that for $p \in L^2(\omega)$, $F_0 \in H^2(\omega)$, $\alpha > 0$ and $(u_0, u_1) \in H_0^2(\omega) \times H_0^1(\omega)$ the following linearized problem of the problem $(\mathbb{P}_0)$, with (2.2) and $\|L\|$, $\|F_0\|_2$ are small*

$$(\mathbb{P}_l)\begin{cases} A'(0)u = (1-\alpha\Delta)u_{tt} + \Delta^2 u - [u, F_0] - L(u) & \\ \qquad\qquad -\alpha\, div(d(x)g'(0)(\nabla u_t)) + d_0(x)g_0'(0)(u_t) = p & in \quad \omega \times [0,T], \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0,T], \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u_1 & in \quad \omega, \end{cases}$$

has a unique solution $(u, u_t, u_{tt}) \in L^2\big([\,0, T\,], H^4(\omega) \cap H_0^2(\omega) \times H^2(\omega) \cap H_0^1(\omega) \times H^2(\omega)\big)$ and satisfying that

$$\begin{aligned} &|u_t|_2^2 + \alpha\,|\nabla u_t|_2^2 + c_2\,\|u\| + 2\alpha\int_0^t\int_\omega d(x)g'(0)(\nabla u_t)\nabla u_t + 2\int_0^t\int_\omega d_0(x)g_0'(0)(u_t)u_t \\ &\qquad \le \frac{1}{2}\big(e^{2T}+1\big)\Big(\,|u_1|_2^2 + \alpha\,|\nabla u_1|_2^2 + \|u_0\| - \int_\omega [u_0, u_0]F_0 + T\,|p|_2^2\Big) \end{aligned} \tag{2.3}$$

Proof. To show the existence and uniqueness of the solution of our problem considered, we will study the problem $(\mathbb{P}_l)$ by considering the n-order approximate solution and we use the variational problem. One takes $\{\, e_k \,\}$ be a basis in the space $H_0^2(\omega)$. We define an n-order Galerkin approximate solution to the problem $(\mathbb{P}_l)$ with clamped boundary conditions on the interval $[\,0, T\,]$, as a function $u^n(t)$ of the form, see for instance [1, 11].

$$u^n = \sum_{k=1}^{n} h_k(t)e_k \qquad n = 1, 2, 3, ...$$

Where $h_k(t) \in W^{2,+\infty}(0, T, IR)$ and u_{n0}, u_{n1} are chosen such that u_{n0} converge to u_0 in $L^2([0,T], H_0^2(\omega) \times H_0^1(\omega))$ and u_{n1} converge to u_1 in $L^2([0,T], H_0^1(\omega))$. Let the variational problem of $(\mathbb{P}_l)$:

$$\begin{aligned} &\int_\omega u_{tt}^n u_t^n + \alpha\int_\omega \nabla u_{tt}^n \nabla u_t^n + \int_\omega \Delta u^n \Delta u_t^n - \int_\omega [u^n, F_0]\, u_t^n + \alpha\int_\omega d(x)g'(0)(\nabla u_t^n)\nabla u_t^n \\ &\qquad + \int_\omega d_0(x)g_0'(0)(u_t^n)u_t^n - \int_\omega L(u^n)u_t^n = \int_\omega p u_t^n, \end{aligned}$$

$$\frac{1d}{2dt}(|u_t^n|_{2,\omega}^2 + \|u^n\| + \alpha\,|\nabla u_t^n|_{2,\omega}^2 - \int_\omega [u^n, u^n]\, F_0) + \alpha\int_\omega d(x)g'(0)(\nabla u_t^n)\nabla u_t^n$$

$$+\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n-\int_\omega L(u^n)u_t^n=\int_\omega pu_t^n.$$

Now, if we integrate this latter inequality with respect to $t>0$, with (2.1) and by using the full that $u^n_{|t=0}=u_{n0}$ et $(u_t^n)_{|t=0}=u_{n1}$, we deduce that

$$\|u^n\|_\alpha-\int_\omega[u^n,u^n]\,F_0-2\int_0^t\int_\omega L(u^n)u_t^n+2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n$$
$$+2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n=\big(\,|u_{n1}|^2_{2,\omega}+\alpha\,|\nabla u_{n1}|^2_{2,\omega}+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0\big)$$
$$+2\int_0^t\int_\omega pu_t^n.$$

Proposition (2.2) and Remark (2.4) we have that, with $0<c_1<1$ and $\frac{1}{2}<r<1$

$$\|u^n\|_\alpha-c_1\,\|u^n\|\,\|F_0\|_2^{2(1-r)}-\|L\|^2\int_0^t\|u^n\|-\int_0^t|u_t^n|_2^2+2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n$$
$$+2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n\leq\big(\,|u_{n1}|^2_{2,\omega}+\alpha\,|\nabla u_{n1}|^2_{2,\omega}+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0\big)$$
$$+\int_0^t|p|_2^2+\int_0^t|u_t^n|_2^2\,.$$

And for all $0\leq t\leq T$

$$\|u^n\|_\alpha-c_1\,\|u^n\|\,\|F_0\|_2^{2(1-r)}-\|L\|^2\int_0^t\|u^n\|+2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n$$
$$+2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n\leq(|u_{n1}|^2_{2,\omega}+\alpha\,|\nabla u_{n1}|^2_{2,\omega}+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0$$
$$+T\,|p|_2^2)+2\int_0^t|u_t^n|_2^2\,,$$

it follows that, with $c_2=1-c_1\,\|F_0\|_2^{2(1-r)}>0$ and $\|L\|^2\leq c_2$

$$|u_t^n|_2^2+\alpha\,|\nabla u_t^n|_2^2+c_2\,\|u^n\|+2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n+2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n$$
$$\leq(|u_{n1}|^2_{2,\omega}+\alpha\,|\nabla u_{n1}|^2_{2,\omega}+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0+T\,|p|_2^2)+2\int_0^t\Big[\,|u_t^n|_2^2+\alpha\,|\nabla u_t^n|_2^2$$
$$+\,c_2\,\|u^n\|+2\alpha\int_0^s\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n+2\int_0^s\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n\Big]\qquad(2.4)$$

If for any $0\leq s\leq t$, we put

$$I(s)=\int_0^s\Big[\,|u_t^n|_2^2+\alpha\,|\nabla u_t^n|_2^2+c_2\,\|u^n\|+2\alpha\int_0^\sigma\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n$$
$$+2\int_0^\sigma\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n\Big],$$

and

$$J(u_{n0},u_{n1})=|u_{n1}|^2_{2,\omega}+\alpha\,|\nabla u_{n1}|^2_{2,\omega}+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0+T\,|p|_2^2\,.$$

The inequality (2.4), yields

$$e^{-2s}\Big(I(s)-\int_0^s I(\sigma)d\sigma\Big)\leq e^{-2s}J(u_{n0},u_{n1}).$$

Now, we have

$$\frac{d}{ds}\Big(e^{-2s}\int_0^s I(\sigma)d\sigma\Big)=e^{-2s}I(s)-2e^{-2s}\int_0^s I(\sigma)d\sigma=e^{-2s}\Big(I(s)-2\int_0^s I(\sigma)d\sigma\Big),$$

then

$$\frac{d}{ds}\Big(e^{-2s}\int_0^s I(\sigma)d\sigma\Big)\leq e^{-2s}J(u_{n0},u_{n1}),$$

and $J(u_{n0},u_{n1})$ does not depend on s, then

$$\int_0^t \frac{d}{ds}\Big(e^{-2s}\int_0^s I(\sigma)d\sigma\Big)ds\leq (\int_0^t e^{-2s}ds)J(u_{n0},u_{n1}),$$

from which we deduce

$$e^{-2t}\int_0^t I(\sigma)d\sigma\leq\frac{1}{2}(1-e^{-2t})J(u_{n0},u_{n1}).$$

Since

$$\int_0^t\Big[\,|u_t^n|_2^2+\alpha\,|\nabla u_t^n|_2^2+c_2\,\|u^n\|+2\int_0^\sigma\int_\omega\Big(\alpha d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n+d_0(x)g_0^{'}(0)(u_t^n)u_t^n\Big)\Big]$$
$$=\int_0^t I(\sigma)d\sigma.$$

It follows that

$$\int_0^t I(\sigma)d\sigma\leq\frac{(1-e^{-2t})}{2e^{-2t}}J(u_{n0},u_{n1})\leq\frac{1}{2}(e^{2t}-1)J(u_{n0},u_{n1})\leq\frac{1}{2}(e^{2T}-1)J(u_{n0},u_{n1}).$$

This, with (2.4), yields for $0\leq t\leq T$

$$|u_t^n|_2^2+\alpha\,|\nabla u_t^n|_2^2+c_2\,\|u^n\|+2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n+2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n$$
$$\leq J(u_{n0},u_{n1})+\frac{1}{2}(e^{2T}-1)J(u_{n0},u_{n1})\leq\frac{1}{2}(e^{2T}+1)J(u_{n0},u_{n1}),$$
$$\leq\frac{1}{2}(e^{2T}+1)\big(\,|u_{n1}|_{2,\omega}^2+\alpha\,|\nabla u_{n1}|_{2,\omega}^2+\|u_{n0}\|-\int_\omega[u_{n0},u_{n0}]\,F_0+T\,|p|_2^2\big).\quad(2.5)$$

Under asumption we have

$$\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^n)\nabla u_t^n\geq 0$$

and

$$\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^n)u_t^n\geq 0.$$

Then the previous estimate implies that there exist a subsequence u^{n_l} such that $u^{n_l}\rightharpoonup u$ weakly in $L^2([0,T],H_0^2(\omega))$ and $(u^{n_l})_t\rightharpoonup(u)_t$ weakly in $L^2([0,T],H_0^1(\omega))$. For showing that u is a weak solution of the problem $(\mathbb{P}_l)$, we use analogous method in, [11]. Let φ_j $1 \leq j \leq j_0$ are a $C^1(0,T)$ such that $\varphi_j(T) = 0$ and $\psi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j$. After the variational problem, we have

$$-\int_0^T \int_\omega u_t^{nl}\psi_t - \alpha \int_0^T \int_\omega \nabla u_t^{nl} \nabla \psi_t + \int_0^T \int_\omega \Delta u^{nl} \Delta \psi + \alpha \int_0^T \int_\omega d(x) g'(0)(\nabla u_t^{nl}) \nabla \psi$$

$$+ \int_0^T \int_\omega d_0(x) g_0'(0)(u_t^{nl})\psi - \int_0^T \int_\omega [u^{nl}, F_0]\psi - \int_0^T \int_\omega L(u^{nl})\psi = \int_0^T \int_\omega p\psi$$

$$+ \int_\omega u_{nl1}\psi(0) + \alpha \int_\omega \nabla u_{nl1} \nabla \psi(0). \tag{2.6}$$

Now, we can pass with the limit $nl \to +\infty$ in (2.6), we find that for all $\psi \in L^2([0,T], H_0^2(\omega))$ and $\psi_t \in L^2([0,T], H^1(\omega))$, such that $\psi(T) = 0$. With the two linears continuous operators $d(x)g'(0)(\nabla u_t^{nl})$ and $d_0(x)g_0'(0)(u_t^{nl})$, we deduce that

$$-\int_0^T \int_\omega u_t \psi_t - \alpha \int_0^T \int_\omega \nabla u_t \nabla \psi_t + \int_0^T \int_\omega \Delta u \Delta \psi + \alpha \int_0^T \int_\omega d(x) g'(0)(\nabla u_t) \nabla \psi$$

$$+ \int_0^T \int_\omega d_0(x) g_0'(0)(u_t)\psi - \int_0^T \int_\omega [u, F_0]\psi - \int_0^T \int_\omega L(u)\psi = \int_0^T \int_\omega p\psi + \int_\omega u_1 \psi(0)$$

$$+\alpha \int_\omega \nabla u_1 \nabla \psi(0).$$

This show that u is a weak solution of the problem $(\mathbb{P}_l)$, by analogous method in the last proof, we deduce

$$\int_0^T \int_\omega (1-\alpha\Delta)(u_{tt}^{nl} - u_{tt})\psi + \int_0^T \int_\omega (\Delta u^{nl} - \Delta u)\Delta\psi + \alpha \int_0^T \int_\omega d(x) g'(0)(\nabla u_t^{nl} - \nabla u_t)\nabla\psi$$

$$+ \int_0^T \int_\omega d_0(x) g_0'(0)(u_t^{nl} - u_t)\psi - \int_0^T \int_\omega [u^{nl} - u, F_0]\psi - \int_0^T \int_\omega L(u^{nl} - u)\psi = 0$$

so we have

$$(1-\alpha\Delta)u_{tt}^{nl} \text{ converge weakly to } (1-\alpha\Delta)u_{tt} \text{ in } L^2([0,T], L^2(\omega)) \ .$$

By analogous method we can proof of

$$(1-\alpha\Delta)u_t^{nl} \text{ converge weakly to } (1-\alpha\Delta)u_t \text{ in } L^2([0,T], L^2(\omega)) \ .$$

$$L(u^{nl}) \text{ converge weakly to } L(u) \text{ in } L^2([0,T], L^2(\omega)) \ ,$$

$$div(g'(0)(\nabla u_t^{nl})) \text{ converge weakly to } div(g'(0)(\nabla u_t)) \text{ in } L^2([0,T], L^2(\omega)) \ ,$$

$$g_0'(0)(u_t^{nl}) \text{ converge weakly to } g_0'(0)(u_t) \text{ in } L^2([0,T], L^2(\omega)) \ ,$$

$$[u^{nl}, F_0] \text{ converge weakly to } [u, F_0] \text{ in } L^2([0,T], L^2(\omega)) \ .$$

It follows that, u is a solution to the following problem

$$\begin{cases} \Delta^2 u = -u_{tt} + \alpha\Delta u_{tt} + \alpha div(d(x)g'(0)(\nabla u_t)) - d_0(x)g_0'(0)(u_t) + [u, F_0] \\ \qquad\quad +L(u) + p(x) \ \in L^2([0,T], L^2(\omega)), \\ u_{|\Gamma} = \partial_\nu u_{|\Gamma} = 0, \end{cases}$$

moreover u_{tt} is a solution to the problem, with $\alpha > 0$

$$\begin{cases} (1-\alpha\Delta)u_{tt} \in L^2([0,T], L^2(\omega)), \\ u_{tt} \in L^2([0,T], L^2(\omega)), \end{cases}$$

and u_t is a solution to the problem

$$\begin{cases} (1-\alpha\Delta)u_t \in L^2([0,T], L^2(\omega)), \\ u_t \in L^2([0,T], H_0^1(\omega)), \end{cases}$$

this implies that $(u, u_t, u_{tt}) \in L^2([0,T], H^4(\omega)\cap H_0^2(\omega) \times H^2(\omega)\cap H_0^1(\omega) \times H^2(\omega))$, see [1, 4, 10].

Now for showing the inequality (2.3), we can using a similarly last proof of inequality (2.5), with u is a solution to the problem $(\mathbb{P}_l)$.

For the uniqueness, let u_1 and u_2 are two solutions of the problem $(\mathbb{P}_l)$. We use inequality (2.3), to the solution $u^{12} = u_1 - u_2$ of the next problem :

$$\begin{cases} (1-\alpha\Delta)u_{tt}^{12} + \Delta^2 u^{12} - \alpha div(d(x)g^{'}(0)(\nabla u_t^{12}) + d_0(x)g_0^{'}(0)(u_t^{12}) & \\ \qquad -[u^{12}, F_0] - L(u^{12}) = 0 & in \ \ \omega\times[0,T], \\ u^{12} = \partial_\nu u^{12} = 0 & on \ \ \Gamma\times[0,T], \\ u^{12}_{|t=0} = 0, (u_t^{12})_{|t=0} = 0 & in \ \ \omega, \end{cases}$$

it follows that

$$\begin{aligned} \left|u_t^{12}\right|_2^2 + \alpha\left|\nabla u_t^{12}\right|_2^2 + c_2\left\|u^{12}\right\| + 2\alpha\int_0^t\int_\omega d(x)g^{'}(0)(\nabla u_t^{12})\nabla u_t^{12} \\ +2\int_0^t\int_\omega d_0(x)g_0^{'}(0)(u_t^{12})u_t^{12} \le \frac{1}{2}(e^{2T}+1)\big(\left|u_1^{12}\right|_2^2 + \alpha\left|\nabla u_1^{12}\right|_2^2 + \left\|u_0^{12}\right\| \\ - \int_\omega [u_0^{12}, u_0^{12}]F_0\big). \end{aligned}$$

Then, $u_1 = u_2$.

The proof of the Theorem is completed. □

3. Implicit function theorem: The Main Results

In this section, in order to establish the existence and uniqueness of the solution (u, ϕ) satisfying $(\mathbb{P}_0)$ and $(\mathbb{Q})$, we use the following idea of implicit function theorem. Combining the results of the previous section, we shall prove the next theorem which is the original motivation of this paper.

Theorem 3.1. *Let $p \in L^2(\omega)$, $F_0 \in H^2(\omega)$, $\alpha > 0$ and $(u_0, u_1) \in H_0^2(\omega)\times H_0^1(\omega)$ the following problem, with (2.2), $\|\theta\|_3$, $|p|_2$, $\|L\|$ and $\|F_0\|_2$ are small.*

$$(\mathbb{P}_0)\begin{cases} u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - \alpha div(d(x)g(\nabla u_t)) + d_0(x)g_0(u_t) & \\ \qquad -[\phi + F_0, u+\theta] = L(u) + p(x) & in \ \ \omega\times[0,T], \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u_1, & in \ \ \omega, \\ u = \partial_\nu u = 0 & on \ \ \Gamma\times[0,T], \end{cases}$$

and

$$(\mathbb{Q})\begin{cases} \Delta^2\phi + [u, u+2\theta] = 0 & in \ \ \omega\times[0,T], \\ \phi = 0, \ \partial_\nu\phi = 0 & on \ \ \Gamma\times[0,T], \end{cases}$$

has one and only one solution (u,ϕ,u_t,u_{tt}) *in* $L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)$ *and satisfying for any* $0\le t\le T$ *the following energy inequality*

$$|u_t|_2^2+\alpha\,|\nabla u_t|_2^2+c_2\,\|u\|+2\alpha\int_0^t\int_\omega d(x)g(\nabla u_t)\nabla u_t+2\int_0^t\int_\omega d_0(x)g_0(u_t)u_t+\frac{1}{2}\int_\omega|\Delta\phi|^2$$
$$\le\frac{1}{2}(e^{2T}+1)\Big(|u_1|_2^2+\alpha\,|\nabla u_1|_2^2+\|u_0\|-T\int_\omega[u_0+2\theta,u_0]F_0+T\,|p|_2^2+\frac{1}{2}\int_\omega|\Delta\phi_0|^2\Big). \tag{3.1}$$

Proof. Let us put that

$$A_1(u,u_t,u_{tt})=Au+[F_0,\theta],$$

where the operator A is defined in Proposition 2.5. According to Proposition 2.5 with the continuous injection $H^2(\omega)\hookrightarrow C^0(\overline{\omega})$, we deduce that

$$A_1:\ L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)\longrightarrow L^2([0,T],L^2(\omega))$$

is defined and infinitely Fréchet differentiable, and $A_1(0,0,0)=0$. We have that,

$$A_1^{'}(0,0,0)(u,u_t,u_{tt})=A^{'}(0)u,$$

is a continuous linear operator.

According to the Theorem 2.6 the linear problem $(\mathbb{P}_l)$ has one and only one solution (u,u_t,u_{tt}) in $L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)$ and $A_1^{'}(0,0,0)(u,u_t,u_{tt})$ is bijective.

By virtue of the closed graph theorem, we can deduce that $A_1^{'}(0,0,0)\in Isom\Big(L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big), L^2([0,T],L^2(\omega))\Big)$.

According to the implicit function theorem, there exist a neighbourhood W_1 of $(0,0,0)$ in $L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)$ and a neighbourhood W_2 of 0 in $L^2([0,T],L^2(\omega))$ such that for every $f\in W_2$ the problem $A_1(u,u_t,u_{tt})=f$ has one and only one solution $(u,u_t,u_{tt})\in W_1$. Since $|p|_2$, $\|L\|$, $\|\theta\|_3$ and $\|F_0\|_2$ are small, then for all $f=[F_0,\theta]+p$ we have

$$|f|_2\le 4c_0\|F\|_2^r\|\theta\|_3^{2(1-r)}+|p|_2^2$$

and $[F_0,\theta]+p\in W_2$.

We conclude that the problem $A_1(u,u_t,u_{tt})=[F_0,\theta]+p$ has a unique solution in $L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)$ of small norm.

Finally, the nonlinear problem $(\mathbb{P}_0)$ has a unique solution (u,u_t,u_{tt}) in $L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\times(H^2(\omega)\cap H_0^1(\omega))\times H^2(\omega)\big)$.

Now, for all $u\in L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\big)$ and Theorem 2.1, Remark 2.4, the problem $(\mathbb{Q})$ has a uniqueness solution $\phi\in L^2\big([0,T], H^4(\omega)\cap H_0^2(\omega)\big)$.

It remains to show the inequality (3.1). Writing the variational problem associated to $(\mathbb{P}_0)$, we have:

$$\int_\omega u_{tt}u_t+\alpha\int_\omega\nabla u_{tt}\nabla u_t+\int_\omega\Delta u\Delta u_t-\int_\omega[u+\theta,\phi+F_0]\,u_t+\alpha\int_\omega d(x)g(\nabla u_t)\nabla u_t$$
$$+\int_\omega d_0(x)g_0(u_t)u_t-\int_\omega L(u)u_t=\int_\omega pu_t,$$

it follows that

$$\frac{1d}{2dt}(|u_t|^2_{2,\omega}+\|u\|+\alpha\,|\nabla u_t|^2_{2,\omega})-\int_\omega [u+\theta,\phi+F_0]\,u_t+\alpha\int_\omega d(x)g(\nabla u_t)\nabla u_t$$
$$+\int_\omega d_0(x)g_0(u_t)u_t-\int_\omega L(u)u_t=\int_\omega pu_t.$$

Now, if we integrate this latter inequality with respect to $t>0$, with (2.1) and by using the full that $u_{|t=0}=u_0$ et $(u_t)_{|t=0}=u_1$, we deduce that

$$\|u\|_\alpha-2\int_0^t\int_\omega [u+\theta,\phi+F_0]\,u_t-2\int_0^t\int_\omega L(u)u_t+2\alpha\int_0^t\int_\omega d(x)g(\nabla u_t)\nabla u_t$$
$$+2\int_0^t\int_\omega d_0(x)g_0(u_t)u_t=|u_1|^2_{2,\omega}+\alpha\,|\nabla u_1|^2_{2,\omega}+\|u_0\|+2\int_0^t\int_\omega pu_t.$$

On the other hand, one has [4]

$$\int_0^t\int_\omega [u+\theta,\phi+F_0]\,u_t=\int_0^t\int_\omega [u+\theta,\phi]\,u_t+\int_0^t\int_\omega [u+\theta,F_0]\,u_t,$$
$$=\frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}([u+2\theta,u])\phi+\frac{1}{2}\int_0^t\int_\omega\frac{d}{dt}([u+2\theta,F_0]\,u),$$
$$=-\frac{1}{4}\int_\omega|\Delta\phi|^2+\frac{1}{4}\int_\omega|\Delta\phi_0|^2+\frac{1}{2}\int_\omega [u+2\theta,F_0]\,u$$
$$-\frac{1}{2}\int_\omega [u_0+2\theta,F_0]\,u_0.$$

We conclude that

$$\|u\|_\alpha+\frac{1}{2}\int_\omega|\Delta\phi|^2-\int_0^t\int_\omega [u,u]\,F_0-2\int_0^t\int_\omega L(u)u_t+2\alpha\int_0^t\int_\omega d(x)g(\nabla u_t)\nabla u_t$$
$$+2\int_0^t\int_\omega d_0(x)g_0(u_t)u_t=|u_1|^2_{2,\omega}+\alpha\,|\nabla u_1|^2_{2,\omega}+\|u_0\|+\frac{1}{2}\int_\omega|\Delta\phi_0|^2$$
$$-\int_0^t\int_\omega [u_0+2\theta,u_0]\,F_0+2\int_0^t\int_\omega pu_t+2\int_0^t\int_\omega[\theta,F_0]u.$$

Proposition (2.2), Remark (2.4) and for all $0\le t\le T$ we have that, with $0<c_1<1$ and $\frac{1}{2}<r<1$

$$\|u\|_\alpha+\frac{1}{2}\int_\omega|\Delta\phi|^2-c_1\,\|u\|\,\|F_0\|_2^{2(1-r)}-\|L\|^2\int_0^t\|u\|-\int_0^t|u_t|_2^2$$
$$+2\alpha\int_0^t\int_\omega d(x)g(\nabla u_t)\nabla u_t+2\int_0^t\int_\omega d_0(x)g_0(u_t)u_t\le|u_1|^2_{2,\omega}+\alpha\,|\nabla u_1|^2_{2,\omega}+\|u_0\|$$
$$+\frac{1}{2}\int_\omega|\Delta\phi_0|^2-T\int_\omega [u_0+2\theta,u_0]\,F_0+T\,|p|_2^2+\int_0^t|u_t|_2^2+2c_0\|F_0\|_2^r\,\|\theta\|_3^{2(1-r)}\int_0^t\|u\|.$$

We can using a similarly proof of Theorem 2.6 to the following energy inequality

$$|u_t|_2^2+\alpha\,|\nabla u_t|_2^2+c_2\,\|u\|+\frac{1}{2}\int_\omega|\Delta\phi|^2+2\alpha\int_0^t\int_\omega d(x)g(\nabla u_t)\nabla u_t$$

$$+2\int_0^t\int_\omega d_0(x)g_0(u_t)u_t \leq |u_1|_2^2+\alpha\,|\nabla u_1|_2^2+\|u_0\|-T\int_\omega[u_0+2\theta,u_0]F_0+T\,|p|_2^2$$
$$+\frac{1}{2}\int_\omega|\Delta\phi_0|^2+2\int_0^t\Big[\,|u_t|_2^2+\alpha\,|\nabla u_t|_2^2+2c_2\|u\|+\frac{1}{2}\int_\omega|\Delta\phi|^2+2\alpha\int_\omega d(x)g(\nabla u_t)\nabla u_t$$
$$+2\int_\omega d_0(x)g_0(u_t)u_t\Big],$$

with $c_2 = 1-c_1\,\|F_0\|_2^{2(1-r)}>0$, $\|L\|^2\leq c_2$ and $c_0\|F_0\|_2^r\,\|\theta\|_3^{2(1-r)}$ is small. We can deduce the inequality (3.1).
This completes the proof. □

4. **Numerical Application**

In this section let ω be the square $]0,1[\times]0,1[$ in $\mathbb{R}^2$ and $T>0$. In order to solve numerically the problem $(\mathbb{P}_0)$, we introduce a uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with the internal points:

$$x_i=ih,\quad y_j=jh,\quad i,j=1,...N-1,\quad h=\frac{1}{N+1},\quad \Delta t=\frac{1}{T}.$$

Let $\overline{\omega}_h$ be the set of boundary mesh points and u_h be the finite-difference approximation of u.
In [2], Bilbao presented a numerical study of the convergence and stability of the conservative finite difference schemes for the dynamic von Karman plate equations via energy conserving methods.
For approaching the weak unique solution, we will utilize the following discretization method developed by Bilbao and Pereira in [2, 14].

$$(*)\begin{cases}(1-\alpha(\delta_x^2+\delta_y^2))\delta_t^2u_{ij}^n+\Delta_h^2u_{ij}^n=[\,u_{ij}^n+\theta_{ij}\;\,v_{ij}^n+F_{ij}\,]+L_{ij}u_{ij}^n & \\ \qquad +\alpha div((dg)_{ij}(\nabla\delta_t u_{ij}^n))-(d_0g_0)_{ij}(\delta_t u_{ij}^n)+p_{ij} & in\quad \omega_h,\\ \Delta_h^2v_{ij}^n=-[\,u_{ij}^n\;\,u_{ij}^n+2\theta_{ij}\,] & in\quad \omega_h,\\ u_{ij}^0=(u_0)_{ij},\;\delta_t u_{ij}^0=(u_1)_{ij}, & in\quad \omega_h,\\ u_{ij}^n=v_{ij}^n=0 & on\quad \overline{\omega_h},\\ \partial_\nu u_{ij}^n=\partial_\nu v_{ij}^n=0 & on\quad \overline{\omega_h},\end{cases}$$

with the following discrete differential operators:

$$\delta_t^2u_{ij}^n=\frac{u_{ij}^{n+1}-2u_{ij}^n+u_{ij}^{n-1}}{(\Delta t)^2},\qquad \delta_t u_{ij}^n=\frac{u_{ij}^{n+1}-u_{ij}^n}{\Delta t},$$

$$\Delta_h^2u_{ij}^n=h^{-4}\,[\,u_{ij-2}+u_{ij+2}+u_{i-2j}+u_{i+2j}-8(u_{ij-1}+u_{ij+1}+u_{i-1j}+u_{i+1j})$$
$$+2(u_{i-1j-1}+u_{i-1j+1}+u_{i+1j-1}+u_{i+1j+1})-20u_{ij}\,],$$

$$\delta_x^2u_{ij}^n=\frac{u_{i+1j}^n-2u_{ij}^n+u_{i-1j}^n}{(h)^2},\qquad \delta_y^2u_{ij}^n=\frac{u_{ij+1}^n-2u_{ij}^n+u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2u_{ij}^n=\frac{u_{i+1j+1}^n-u_{i+1j-1}^n-u_{i-1j+1}^n+u_{i-1j-1}^n}{(2h)^2},$$

$$[\,u_{ij}^n,\;v_{ij}^n\,]=\delta_x^2u_{ij}^n\delta_y^2v_{ij}^n-2\delta_{xy}^2u_{ij}^n\delta_{xy}^2v_{ij}^n+\delta_y^2u_{ij}^n\delta_x^2v_{ij}^n.$$

After we need two detailed steps as follows:
First step: The 13-point finite difference method developed by Gubta in [9], illustrates the weak solution of the following biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & in \quad \omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Second step: According to the first and second steps, we use the discrete model of von Karman evolution $(*)$ for illustrating the unique solution of the structural interaction model.

Proposition 4.1. *The* 13*-point approximation of the Biharmonic equation is defined by:*

$$(1) \begin{cases} L_h v_{ij} = & h^{-4}\left[v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j})\right. \\ & \left. +2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}) - 20v_{ij}\right] = f_1(x_i, y_j) \end{cases}$$

for $i, j = 1, 2, ..., N-1$*, where we set* $v_{ij} = v(x_i, y_j)$*.*

When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$\begin{aligned} v_{i-2,j} &= \tfrac{1}{2}v_{i+1,j} - v_{ij} + \tfrac{3}{2}v_{i-1,j} - h(\partial_x v)_{i-1,j}, \\ v_{i,j-2} &= \tfrac{1}{2}v_{i,j+1} - v_{ij} + \tfrac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j-1}, \\ v_{i+2,j} &= \tfrac{1}{2}v_{i+1,j} - v_{ij} + \tfrac{3}{2}v_{i-1,j} - h(\partial_x v)_{i+1,j}, \\ v_{i,j+2} &= \tfrac{1}{2}v_{i,j+1} - v_{ij} + \tfrac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j+1}. \end{aligned}$$

4.1. **Numerical Test.** We Consider the following analytical body force and lateral forces:

$$F_0(x,y) = 0.001xe^{-x^2-y^2}, \qquad p(x,y) = 0.001x(x-y)e^{-x^2-y^2},$$
$$u_0 = 1510^{-8}(\sin(\pi x)\sin(\pi y))^2, \qquad u_1 = 1510^{-8}x^2y^2(x-y-1)^2(y-1)^2e^{-x^2-y^2},$$
$$L(u) = 0.001x(e^{-x^2} - e^{-y^2})u, \qquad d_0(x,y)g_0(u_t) = 0.001xe^{-x^2-y^2}(u_t).$$
$$d(x,y)g(\nabla u_t) = 0.001(x-y)^2xe^{-x^2-y^2}(\nabla u_t),$$
$$\theta(x,y) = 10^{-13}x^2y^3(x-1)^2(y-1)^2(e^{-x^2} - e^{-y^2}).$$

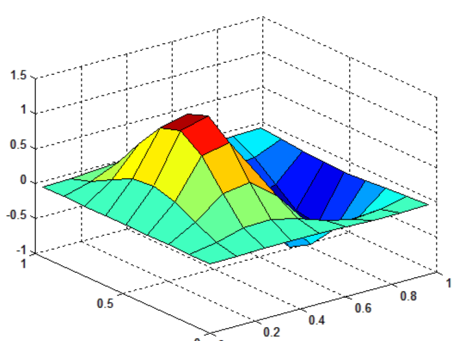

FIGURE 1. Displacement of plate, $t_1 = 4.02s$.

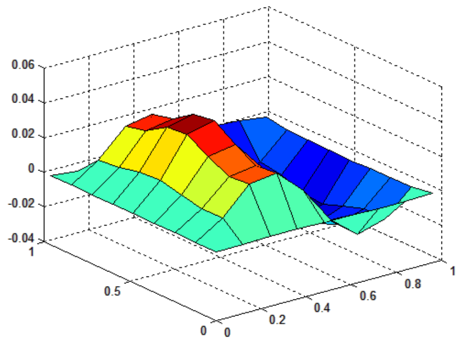

FIGURE 2. Displacement of plate, $t_{20} = 8.22s$.

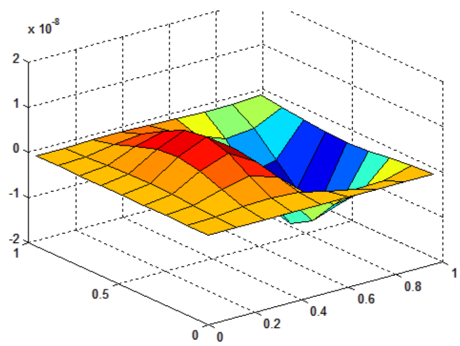

FIGURE 3. Displacement of plate, $t_{47} = 30.22s$.

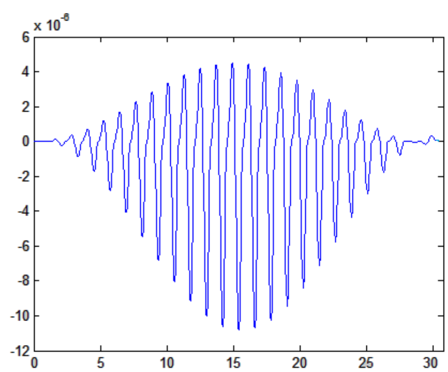

FIGURE 4. Displacement in the point $(4, 6)$, for $t_{47} = 30.22s$.

CONCLUSION

In this paper, we have demonstrated the existence and uniqueness solution of the dynamic von Karman equations of shell using the implicit function theorem. In addition, we use the method of finite difference for approaching the unique solution of the theoretical problem, this reinforces the relevance of paper for applied mathematics and computer mechanics. In the future, we will work on the same problems with free-type boundary conditions.

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