

ON SCI-MODULES

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ABSTRACT. Let R be a commutative ring with unity $1 \neq 0$ and M a unitary left R -module. In this paper, we introduce the *SCI*-modules which is an extension of *SCI*-rings studied in [4]. The aim of this article comes from to the following point. Let $\mathbb{Z}$ be the ring of integers and $M = \bigoplus \mathbb{Z}/p\mathbb{Z}$ a $\mathbb{Z}$ -module with p a prime number. It is easy to see that $M \in \sigma[\mathbb{Z}]$ is co-Hopfian but it is not finitely cogenerated. An R -module M is said to be a *SCI*-module if any co-Hopfian object of $\sigma[M]$ is finitely cogenerated. Firstly, we give some properties of *SCI*-modules. After that, we give two theorems of Characterizations.

1. INTRODUCTION

Let R be a commutative ring with unity and M a unitary left R -module over R . A module M is said to be co-Hopfian if for any injective endomorphism of M is an automorphism. Let M and N be modules over R . N is generated by M if there exists a set Λ and an epimorphism $f : M^{(\Lambda)} \rightarrow N$. A submodule K of N is said subgenerated by M . Thus, the set of modules subgenerated by M constitutes the category $\sigma[M]$. The latter is a full subcategory of $R\text{-Mod}$ introduced by R. Wisbauer in [7]. A module N is said to be finitely cogenerated if its socle is essential in N and finitely generated. A module M is said to be a *SCI*-module if for any co-Hopfian object of $\sigma[M]$ is finitely cogenerated. A module M is said to be a prime module if for every submodule N of M , $\text{Ann}(M) = \text{Ann}(N)$. A module N of $\sigma[M]$ is semiperfect if for every factor module of N has a projective cover in $\sigma[M]$. N is said to be f-semiperfect in $\sigma[M]$ if for every finitely generated submodule K of N , the factor N/K has a projective cover. A module M is said to be faithful if $\text{Ann}(M) = 0$. A module M is a good module if and only if $M/\text{Rad}(M)$ is co-semisimple.

In this paper, it has been showed that if M is a finitely generated, prime module then, M is a *SCI*-module if and only if every object of $\sigma[M]$ is artinian, in particular M . After that, it has been showed that if M is a local, prime module then, M is a *SCI*-module if and only if M is a good and simple module.

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2. SOME PROPERTIES OF SCI-MODULES

In this section, we will give some properties of *SCI*-modules which we will use in this paper.

Proposition 2.1. *Let R be a commutative ring and M a finitely generated *SCI*-module, we have the following assertions:*

- 1- *If M is a prime module then, M is simple;*
- 2- *If M a prime faithful module then, R is a field.*

Proof. (i) As R is a commutative ring and M a finitely generated module then, $\sigma[M] = R/Ann(M)\text{-Mod}$ (see 15.4 of [7]). We can see that every object of $\sigma[M]$ is a module over $R/Ann(M)$. Since M is a *SCI*-module, hence $R/Ann(M)$ is *SCI*-ring. It results from the theorem 2.5 of [4] that $R/Ann(M)$ is an artinian ring. And by 10.18 of [3], any finitely generated module over artinian ring is finitely cogenerated. Therefore M is finitely cogenerated. It is well known that every finitely cogenerated module has a minimal submodule. Let K be that minimal submodule. We can have the following diagram:

$$\begin{array}{ccc} g : R & \longrightarrow & K \\ \downarrow & \nearrow & \\ & R/Ann(K) & \end{array}$$

We can see that $R/Ann(K) \simeq K$. It means that $R/Ann(K)$ is simple. As M is finitely generated let's suppose the following diagram:

$$\begin{array}{ccc} f : R & \longrightarrow & M \\ \downarrow & \nearrow & \\ & R/Ann(M) & \end{array}$$

We can see that $M \simeq R/Ann(M)$. Since M is a prime module then, $Ann(K) = Ann(M)$. Therefore $R/Ann(K) = R/Ann(M)$. Thus, M is simple.

(ii) It has been proved in (1) that $R/Ann(M)$ is simple. Since M is faithful then, R is a field. $\square$

In the proposition 2.2 of [4] it has been proved that if R is a *SCI*-ring then, every prime ideal of R is maximal. We will prove a similar result by supposing that M is a *SCI*-module with certain conditions. Assume the property (P): "A prime ideal containing $Ann(M)$ ".

Proposition 2.2. *Let R be a commutative ring and M a finitely generated *SCI*-module. Then,*

- 1- *Every prime ideal satisfying (P) of R is maximal;*
- 2- *There is a finite number of these prime ideals of R .*

Proof. (i) We know that if M is finitely generated module and R a commutative ring $\sigma[M] = R/Ann(M)\text{-Mod}$. Let J be a prime ideal of $R/Ann(M)$. That implies there exists a prime ideal I containing $Ann(M)$ of R such that $J = I/Ann(M)$. We have seen in (1) of proposition 2.1 that $R/Ann(M)$ is artinian.

Let's consider the factor $(R/Ann(M)/I/Ann(M))$. It is isomorph to R/I . It is well know that the factor of artinian ring is artinian, so is R/I which is also an integral domain ring. It results from the lemma 12.2.5 of [2] that R/I is simple. Therefore I is maximal.

(ii) Let $\Gamma = \{J_\lambda, \lambda \in \Lambda\}$ be the set of prime ideals of $R/Ann(M)$ with $J_\lambda = I_\lambda/Ann(M)$ and I_λ a prime ideal of R .

Let's assume that $P = \bigoplus_{\lambda \in \Lambda} \left((R/Ann(M)) / (I_\lambda/Ann(M)) \right)$. For any $\lambda \in \Lambda$, $(R/Ann(M)) / (I_\lambda/Ann(M))$ is isomorph to R/I_λ . Therefore $P \simeq \bigoplus_{\lambda \in \Lambda} R/I_\lambda$. It has been showed in (1) that, for any $\lambda \in \Lambda$, R/I_λ is simple as module. It is well know that any simple module is co-Hopfian and fully invariant. This implies that $P \simeq \bigoplus_{\lambda \in \Lambda} R/I_\lambda$ is co-Hopfian. As $\bigoplus_{\lambda \in \Lambda} R/I_\lambda$ is in $\sigma[M]$ and M is *SCI*-module, hence $\bigoplus_{\lambda \in \Lambda} R/I_\lambda$ is finitely cogenerated. Thus, Λ is finite. $\square$

Proposition 2.3. *Let R be a commutative ring and M a prime and finitely generated *SCI*-module. Then, the following assertions are verified:*

- 1- *For any object N of $\sigma[M]$, its submodules are maximal;*
- 2- *For any object N of $\sigma[M]$, its submodules are finite;*
- 3- *For any object N of $\sigma[M]$, the factor $N/Rad(N)$ is semisimple.*

Proof. (i) Let N be an object of $\sigma[M]$ and $\{K_j\}_J$ a family of submodules of N . Let's consider $Q = \bigoplus_{j \in J} N/K_j$. Q belongs to $\sigma[M]$ because $\sigma[M]$ is closed under sum direct. We have already seen in proposition 2.1 that M is simple, hence semisimple. It results from 20.3 of [7] that every object of $\sigma[M]$ is semisimple. Therefore Q is semisimple. Hence, for any $j \in J$, N/K_j is simple. Thus, for any $j \in J$, K_j is maximal.

(ii) Let's assume $Q = \bigoplus_{j \in J} N/K_j$. It results from (1) that, for any $j \in J$, N/K_j is simple. It is obvious to see that, for any $j \in J$, N/K_j is co-Hopfian. As N/K_j is fully invariant $j \in J$, then $Q = \bigoplus_{j \in J} N/K_j$ is co-Hopfian too. Therefore, since M is *SCI*-module, Q is finitely cogenerated, hence finitely generated. Therefore J is finite.

(iii) Let N be an objet of $\sigma[M]$. Let's consider $Rad(N) = K_1 \cap K_2 \cap \dots \cap K_n$ with K_j maximal for all $j \in \{1, \dots, n\}$. The following homomorphism:

$$\begin{aligned} f : N/Rad(N) &\rightarrow N/K_1 \times N/K_2 \times \dots \times N/K_n \\ (n + Rad(N)) &\mapsto (n + K_1, n + K_2, \dots, n + K_n) \end{aligned}$$

is an isomorphism. Then, $N/Rad(N) \simeq N/K_1 \times N/K_2 \times \dots \times N/K_n$. By refering to (2), the number of maximal submodules of N is finite therefore, $N/K_1 \times N/K_2 \times \dots \times N/K_n \simeq \bigoplus_{j=1}^n N/K_j$. As N/K_j is simple for any $j \in \{1, \dots, n\}$, therefore $\bigoplus_{j=1}^n N/K_j$ is semisimple. Thus $N/Rad(N)$ is semisimple. $\square$

Corollary 2.4. *Let R be a commutative ring and M a prime and finitely generated *SCI*-module. Then, for every object N of $\sigma[M]$, the following conditions are equivalent:*

- 1- *$N/Rad(N)$ is finitely generated;*
- 2- *$N/Rad(N)$ is noetherian;*
- 3- *$N/Rad(N)$ is artinian;*

4- $N/\text{Rad}(N)$ is finitely cogenerated.

Proof. (i) In the proof of (3) in proposition 2.3 we showed that $N/\text{Rad}(N) \simeq \bigoplus_{j=1}^n N/K_j$. As, for each j , N/K_j is simple, hence it is cyclic. Therefore $\bigoplus_{j=1}^n N/K_j$ is finitely generated since $n < \infty$. (ii),(iii) and (4i) result from 10.16 of [3]. □

Proposition 2.5. *Let M be a R -module then, the following conditions are verified:*

- 1-** Every submodule of a SCI -module is a SCI -module;
- 2-** The homomorphic image of a SCI -module is a SCI -module;
- 3-** Let $M = \prod_{\lambda \in \Lambda} M_\lambda$ be a direct product of submodule of M . We assume that $\sigma[M_\lambda] \cap \sigma[M_\kappa] = 0$ for any $\lambda \neq \kappa$ of Λ . Then, M is a SCI -module if and only if each M_λ is a SCI -module and Λ finite;
- 4-** If M is a SCI -module, then there exists only a finite number of non-isomorphic simple modules in $\sigma[M]$.

Proof. (i) Let N be a submodule of M . N belongs to $\sigma[M]$. The category $\sigma[N]$ is a full subcategory of $\sigma[M]$. Let K be a co-Hopfian object of $\sigma[N]$. K is also an object of $\sigma[M]$. As M is a SCI -module then, K is finitely cogenerated. Hence, N is a SCI -module.

(ii) Let's consider L a homomorphic image of M . Hence, L is generated by M . It means that $L \in \sigma[M]$. It results from (1) that L is a SCI -module.

(iii) Let's suppose that $M = \prod_{\lambda \in \Lambda} M_\lambda$ is a SCI -module. Let's consider the following epimorphic $\phi : \prod_{\lambda \in \Lambda} M_\lambda \rightarrow M_\lambda$. It follows from (1) that, for each $\lambda \in \Lambda$, M_λ is a SCI -module.

Now let's assume that, for every $\lambda \in \Lambda$, M_λ is SCI -module and Λ finite. Let N be a co-Hopfian object of $\sigma[M]$. Since, the direct product is finite therefore, $\prod_{\lambda \in \Lambda} M_\lambda \simeq \bigoplus_{\lambda \in \Lambda} M_\lambda$. Hence, $N \in \sigma[\bigoplus_{\lambda \in \Lambda} M_\lambda]$. It results from the proposition 2.2 of [6] that $N_\lambda \in \sigma[M_\lambda]$ and $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$. As, for any $\lambda \in \Lambda$, N_λ is co-Hopfian hence, it is finitely cogenerated. Therefore N is finitely cogenerated. Thus, M is SCI -module.

(4i) Let $(S_\lambda)_{\lambda \in \Lambda}$ be a complete system of non-isomorphic class of simple modules of $\sigma[M]$. Let's suppose that $L = \bigoplus_{\lambda \in \Lambda} S_\lambda$. L belongs to $\sigma[M]$ because the category $\sigma[M]$ is stable by direct sum. As each S_λ is co-Hopfian and fully invariant, then L is co-Hopfian too. Hence, L is finitely cogenerated since M is a SCI -module. As L is semisimple, it results from 10.16 of [3] that L is finitely generated. Thus, Λ is finite. □

THE MAIN RESULTS

Theorem 2.6. *Let R be a commutative ring and M a prime and finitely generated SCI -module. Then, the following conditions are equivalent:*

- 1-** M is a SCI -module;
- 2-** Every object N of $\sigma[M]$ is artinian, in particular for M .

Proof. (i) $\implies$ (ii) Let N be an object of $\sigma[M]$. For every submodule K of N , the factor module N/K is simple since K is maximal (proposition 2.3). Hence,

N/K is co-Hopfian. Since M is a *SCI*-module, then N/K is finitely cogenerated. It results from the 10.10 of [3] that N is artinian.

(ii) $\implies$ (i) Let N be a co-Hopfian object of $\sigma[M]$. N is also a module over $R/\text{Ann}(M)$ because $\sigma[M] = R/\text{Ann}(M)\text{-Mod}$. As N is artinian and co-Hopfian, it follows from the theorem 9 of [5] that $R/\text{Ann}(M)$ is an artinian principal ideal ring. Thus, it follows from the theorem 2.5 of [4] that $R/\text{Ann}(M)$ is a *SCI*-ring. Thus, M is a *SCI*-module. $\square$

Remark 2.7. Every finitely generated, self-projective and semiperfect module M has a minimal M -generator.

Proposition 2.8. *Let R be a commutative ring and M a prime and finitely generated *SCI*-module. Then,*

- 1- *Every object N of $\sigma[M]$ is semiperfect(f -semiperfect);*
- 2- *There exists a minimal N -generator L in $\sigma[N]$.*

Proof. (i) It has been showed in (1) of proposition 2.1 that $R/\text{Ann}(M)$ is an artinian ring which is also a semiperfect ring. By referring to 27.6 of [3], every simple module over $R/\text{Ann}(M)$ has a projective cover. Let N be an object of $\sigma[M]$. It has already been showed previously that for any submodule K of N the factor N/K is simple. Thus, it results from the definition of semiperfect or f -semiperfect module.

(ii) Let N be an object of $\sigma[M]$. As M is simple(=semisimple) then, every object of $\sigma[M]$ is semisimple. Hence N is semisimple. It results from the theorem 1 that N is artinian, hence N is finitely generated. Moreover, it is well known that every semisimple module is self-projective. Thus, it follows from (1) and the previous remark that N has minimal N -generator. $\square$

Proposition 2.9. *Let R be a commutative ring and M a prime and finitely generated *SCI*-module. Then, every finitely generated (simple, cyclic) submodule of M is a subgenerator in $\sigma[M]$.*

Proof. Let N be a finitely generated (simple, cyclic) submodule of M . Let h be an epimorphism such that $h : R \rightarrow N$. We have $R/\text{Ann}(N) \simeq N$. We have already seen that $R/\text{Ann}(M) \simeq M$. Since, M is a prime module, then $\text{Ann}(M) = \text{Ann}(N)$. That implies $R/\text{Ann}(M) = R/\text{Ann}(N)$. Therefore $M \simeq N$. Thus, N is a subgenerator. $\square$

Corollary 2.10. *Every local module is finitely generated.*

Proof. It results from the definition of local module and 21.6 of [7]. $\square$

Theorem 2.11. *Let R be a commutative ring and M a prime and local module. Then, the following conditions are equivalent:*

- 1- *M is a *SCI*-module;*
- 2- *M is a good and simple module.*

Proof. (i) $\implies$ (ii) By the corollary 2.10, M is finitely generated. Let N be an object of $\sigma[M]$. It has been showed in (3) of proposition 2.3 that $N/\text{Rad}(N)$ is semisimple. It is well know that every semisimple module is co-semisimple.

Hence, N is a good module, in particular for M . M is simple by the proposition 2.1.

(ii) $\implies$ (i) We have seen that $\sigma[M] = R/\text{Ann}(M)\text{-Mod}$. To show that M is a *SCI*-module we are going to prove that $R/\text{Ann}(M)$ is a *SCI*-ring. Indeed, as M is simple hence, it is isomorph to $R/\text{Ann}(M)$. It means that $R/\text{Ann}(M)$ is a field. It is well know that any field is a principal ring. And it results from [4] theorem 2.5 that $R/\text{Ann}(M)$ is *SCI*-ring. Therefore M is a *SCI*-module. $\square$

REFERENCES

1. A. BA, O. DIANKHA *On FGS-Modules* Journal of Mathematics Research, Vol. 5, No. 1; 61-64 (2013) ISSN 1916-9795.
2. A. Chambert-Loir: *Algèbre commutative*, Centre de Mathématique, Ecole polytechnique, 91128 Palaiseau Cedex, Paris 6, 2001.
3. Anderson F.W and Fuller K. (1974) *Rings and categories of modules*, Springer-Verlag.
4. A. Mbaye, M. Sangharé, S. D. Touré *On Commutative SCI-Rings and Commutative SCS-Rings*, International Journal of Algebra, Vol. 4, 2010, no. 12, 585 - 590.
5. Kaidi, A.M et Sangharé, M: Une caractérisation des anneaux artiniens à idéaux principaux. L.Notes in Math. Springer Verlag (1988) pp.245 - 254.
6. Vanaja N. (1996). *All finitely generated M-subgenerated modules are extending*. Comm. Algebra 24(2), 543-572 (1996).
7. Wisbauer R. (1991). *Foundation of Module and Ring theory*. Gordon and Breach Science Publishers (1991).
8. Wisbauer R. (1985). *Decomposition properties in modules categories*. Acta. Univ. Corilia Math. Physica 126(26), 57-68

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