ON SKEW DERIVATIONS IN 3-PRIME NEAR-RINGS

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Abstract. In this paper, we study the relationship between the behavior of skew derivations satisfying certain local properties in near-rings. In particular, our purpose is to extend the results of [1] and [5].

1. Introduction

Throughout this paper, \( N \) will denote a left near-ring. A near-ring \( N \) is called zero symmetric if \( 0x = 0 \) for all \( x \in N \) (recall that in a left near ring \( x0 = 0 \) for all \( x \in N \)). \( N \) is called 3-prime if \( xNy = \{0\} \) implies \( x = 0 \) or \( y = 0 \). The symbol \( Z(N) \) will represent the multiplicative center of \( N \), that is, \( Z(N) = \{ x \in N \mid xy = yx \) for all \( y \in N \} \). For any \( x, y \in N \); as usual \([x, y] = xy - yx \) and \( x \circ y = xy + yx \) will denote the well-known Lie product and Jordan product respectively. Recall that \( N \) is called 2-torsion free if \( 2x = 0 \) implies \( x = 0 \) for all \( x \in N \). For terminologies concerning near-rings we refer to G. Pilz [7]. An additive mapping \( d : N \to N \) is said to be a derivation if \( d(xy) = xd(y) + d(x)y \) for all \( x, y \in N \), or equivalently, as noted in [8], that \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in N \). Let \( g \) be an automorphism of \( N \). An additive mapping \( d : N \to N \) is called a skew derivation of \( N \) if \( d(xy) = d(x)g(y) + xd(y) \) for all \( x, y \in N \). Obviously, any derivation is a skew derivations, but the converse is not true in general. Moreover, if \( g \) is the identity map of \( N \), then all skew derivations associated with \( g \) are certainly derivations of \( N \). An additive mapping \( d \) is said to be commuting if for all \( x \in N \), \([d(x), x] = 0 \). There has been an ongoing interest concerning the relationship between the commutativity of a 3-prime near-ring \( N \) and the behavior of a derivation on \( N \). The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 and serval authors. Motivated by the result of Bell and Daif in [5] which proved that a 2-torsion free prime ring must be commutative if it admits a strong commutativity preserving derivation \( d \), that is a derivation satisfying \([d(x), d(y)] = [x, y] \) for all \( x, y \in R \). Our aim in this paper is to generalize this result in two directions. First of all we will only assume that the commutativity condition is imposed on a 3-prime near-ring \( N \) instead of one ring \( R \). Secondly we will treat the case of two skew derivations instead of one derivation.

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In 2002 Ashraf and Rehman [1], prove that if $R$ is a 2-torsion free prime ring, $I$ is a nonzero ideal of $R$ and $d$ is a nonzero derivation of $R$ such that $d(x)\circ d(y) = x\circ y$ for all $x, y \in I$, then $R$ is commutative. Our result is motivated by the previous results and we here generalized the result obtained in [1], [5]. Moreover, we continue this line of investigation by examining what happens if a 3-prime near-ring $N$ satisfies the identity $d_1(x) \circ d_2(y) = x \circ y$ for all $x, y \in N$ where $d_1$ and $d_2$ are two skew derivations.

2. Some preliminaries

In this paper, we include some well known results which will be used for developing the proof of our main result.

Lemma 2.1. [2, Lemma 1.5] Let $N$ be a 3-prime near ring. If $N \subseteq Z(N)$, then $N$ is a commutative ring.

Lemma 2.2. A near-ring $N$ admits a skew derivation $d$ if and only if it is zero-symmetric.

Proof. Let $N$ be a zero-symmetric near-ring. Then the zero map is a skew derivation $d$ on $N$. Conversely, assume that $N$ has an skew derivation $d$. Let $x, y$ be two arbitrary elements of $N$. By definition of $d$, we have

\[
d(x0y) = d(x(0y)) = d(x)g(0y) + xd(0y) = d(x)g(0)g(y) + xd(0)g(y) + 0d(y) = 0y + (xd(0))g(y) + (x0)d(y) = 0y + (x0)g(y) + (x0)d(y) = 0y + 0g(y) + 0d(y).
\]

On the other hand

\[
d(x0y) = d((x0)y) = d(0y) = d(0)g(y) + 0d(y) = 0g(y) + 0d(y)
\]

By comparing the last two expressions, we find that $0y = 0$ for all $y \in N$, and hence $N$ is a zero-symmetric left near-ring. □ □

Remark 2.3. The above lemma is also true in the case of right near-ring.

Lemma 2.4. Let $d$ be an arbitrary skew derivation on the near ring $N$. Then $N$ satisfies the following partial distributive law

\[(d(x)g(y) + xd(y))z = d(x)g(y)z + xd(y)z \quad \text{for all } x, y, z \in N.\]
Proof. By a simple calculation of \( d(xyz) \) for all \( x, y, z \in N \), we obtain
\[
d(xyz) = d(x)g(yz) + xd(yz)
\]
\[
= d(x)g(y)g(z) + x(yd(y)g(z) + yx) \quad \text{for all } x, y, z \in N.
\]
By another way
\[
d(xyz) = d(xy)g(z) + xyd(z)
\]
\[
= (d(x)g(y) + xd(y))g(z) + yxd(z) \quad \text{for all } x, y, z \in N.
\]
Comparing the last two results, we obtain
\[
(d(x)g(y) + xd(y))g(z) = d(x)g(y)g(z) + xyd(y)g(z) \quad \text{for all } x, y, z \in N.
\]
Since \( g \) is an amorphism, we obtain the required result. \( \square \)

3. Main Results

**Theorem 3.1.** Let \( N \) be a 2-torsion free 3-prime near-ring which admits nonzero skew derivations \( d_1, d_2 \) such that \( d_1 \) is commuting. Then the following assertions are equivalent:

(i) \([d_1(x), d_2(y)] = [x, y] \) for all \( x, y \in N \).

(ii) \( N \) is a commutative ring.

**Proof.** It is easy to verify that (ii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Suppose that \([d_1(x), d_2(y)] = [x, y] \) for all \( x, y \in N \). Replacing \( y \) by \( xy \), we get
\[
[d_1(yx), d_2(y)] = [yx, y] = y[x, y] = y[d_1(x), d_2(y)] \quad \text{for all } x, y \in N
\]
this expression implies that
\[
d_1(yx)d_2(y) - d_2(y)d_1(yx) = yd_1(x)d_2(y) - yd_2(y)d_1(x) \quad \text{for all } x, y \in N \quad (3.1)
\]
Using Lemma 2.4 and (3.1), then for all \( x, y \in N \)
\[
yd_1(x)d_2(y) + d_1(y)g(x)d_2(y) - d_2(y)d_1(y)g(x) - d_2(y)yd_1(x) = yd_1(x)d_2(y) - yd_2(y)d_1(x)
\]
which implies that
\[
d_1(y)[g(x), d_2(y)] = -yd_2(y)d_1(x) + d_2(y)yd_1(x) \quad \text{for all } x, y \in N. \quad (3.2)
\]
Using the fact that \( d_1 \) is commuting, then (3.2) becomes
\[
d_1(y)[g(x), d_2(y)] = d_1(y)d_2(y)[g(x)] \quad \text{for all } x, y \in N. \quad (3.3)
\]
Putting \( xt \) instead of \( x \) in (3.3) and using it again, we find that
\[
d_1(y)[g(x), d_2(y)]d_2(y) = d_1(y)[d_2(y)g(x)]g(t) = d_1(y)g(x)[d_2(y)g(t)] \quad \text{for all } x, y, t \in N.
\]
The latter expression reduces to
\[
d_1(y)[g(N), d_2(x)] = [0] \quad \text{for all } x, y, t \in N. \quad (3.4)
\]
Using the fact that \( g \) is an automorphism and the 3-primeness of \( \mathcal{N} \), (3.4) implies that

\[
d_1(x) = 0 \text{ or } d_2(x) \in Z(g(\mathcal{N})) \quad \text{for all } x \in \mathcal{N}.
\]

If there is \( x_0 \in \mathcal{N} \) such that \( d_1(x_0) = 0 \), then by applying the hypotheses of our theorem, we find that \([x, x_0] = 0\) for all \( x \in \mathcal{N} \) so \( x_0 \in Z(\mathcal{N})\).

In the same way, if there is \( x_0 \in \mathcal{N} \) such that \( d_2(x_0) \in Z(g(\mathcal{N})) \), since \( g \) is an automorphism, then \([x_0, y] = 0\) for all \( y \in \mathcal{N} \) which implies that \( x_0 \in Z(\mathcal{N}) \). In all cases we arrive at \( x \in Z(\mathcal{N}) \) for all \( x \in \mathcal{N} \), then \( \mathcal{N} \subseteq Z(\mathcal{N}) \) and by Lemma 2.1, we conclude that \( \mathcal{N} \) is a commutative ring. \( \Box \)

We now consider differential identities involving anti-commutators instead of commutators. Our result is of a different kind.

**Theorem 3.2.** Let \( \mathcal{N} \) be a 3-prime near-ring with \( Z(\mathcal{N}) \neq \{0\} \). \( \mathcal{N} \) admits no nonzero skew derivations \( d_1, d_2 \) such that \( d_1(x) \circ d_2(y) = x \circ y \) for all \( x, y \in \mathcal{N} \).

**Proof.** Suppose that \( d_1(x) \circ d_2(y) = x \circ y \) for all \( x, y \in \mathcal{N} \). For \( y \in Z(\mathcal{N}) \) and by 2-torsion freeness, we obtain \( d_1(x)d_2(y) = xy \) for all \( x \in \mathcal{N}, y \in Z(\mathcal{N}) \). Replacing \( x \) by \( xt \), we find that

\[
xd_1(t)d_2(y) + d_1(x)g(t)d_2(y) = xty \quad \text{for all } x, t \in \mathcal{N}, y \in Z(\mathcal{N}).
\]

Since \( d_1(t)d_2(y) = ty \) for all \( t \in \mathcal{N}, y \in Z(\mathcal{N}) \) the last expression becomes \( d_1(x)g(\mathcal{N})d_2(y) = \{0\} \) for all \( x \in \mathcal{N}, y \in Z(\mathcal{N}) \). Since \( g \) is an automorphism, by 3-primeness of \( \mathcal{N} \), we conclude that \( d_2(Z(\mathcal{N})) = \{0\} \). In this case, for \( y \in Z(\mathcal{N}) \), our hypothesis gives \( 2xy = 0 \) for all \( x \in \mathcal{N}, y \in Z(\mathcal{N}) \) which implies that \( (2x)\mathcal{N}y = \{0\} \) for all \( x \in \mathcal{N}, y \in Z(\mathcal{N}) \), by 3-primeness and 2-torsion freeness of \( \mathcal{N} \), we conclude that \( Z(\mathcal{N}) = \{0\} \). \( \Box \)

The following example shows that the 3-primeness is necessary in the hypotheses of the above theorems.

**Example 3.3.** Let \( \mathcal{S} \) be a nonabelian near-ring. Define the sets \( \mathcal{N} \) by:

\[
\mathcal{N} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta \in \mathcal{S} \right\}.
\]

It is obvious that \( \mathcal{N} \) is a near-ring not 3-prime. Next, we define the maps \( d_1, d_2 : \mathcal{N} \rightarrow \mathcal{N} \) by:

\[
d_1 \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d_2 \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is easy to see that \( d_1 \) and \( d_1 \) are two skew derivations with \((g = id_{\mathcal{N}})\) such that:

1. \([d_1(x), d_2(y)] = [x, y]\)
2. \(d_1(x) \circ d_2(y) = x \circ y \) for all \( x, y \in \mathcal{N} \).

However, \( \mathcal{N} \) is not a commutative ring.
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